The Five Element Basis Theorem Revisited

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Basis Problems for Linear Orders

Definition

For linear orders L and M, define $L \leq M$ to mean that M contains an isomorphic copy of L.

Definition

If $\mathcal{C}$ is a class of linear orders, a *basis for* $\mathcal{C}$ is a class $\mathcal{B} \subseteq \mathcal{C}$ such that for all $C \in \mathcal{C}$ there is $B \in \mathcal{B}$ such that $B \leq C$.

For any linear order L, let L^* denote the converse linear order obtained by reversing the order. For the class of countably infinite linear orders, the set $\{\omega, \omega^*\}$ is a basis. This fact easily follows from Ramsey's theorem.

This talk will focus on basis problems for the class of uncountable linear orders.

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This talk will focus on basis problems for the class of uncountable linear orders.

Minimal Linear Orders

Definition

An uncountable linear order L is *minimal* if every uncountable subset of L contains an isomorphic copy of L, or equivalently, whenever $M \leq L$ and M is uncountable, then $L \leq M$.

Minimal linear orders are good candidates for basis elements.

Lemma

The linear orders ω_1 and ω_1^* are minimal. In fact, any uncountable subset of ω_1 is isomorphic to ω_1, and similarly with ω_1^*.

Corollary

The set $\{\omega_1, \omega_1^*\}$ forms a basis for the class of uncountable well-ordered or conversely well-ordered sets.

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Near and Far Linear Orders

Definition

Uncountable linear orders L and M are *near* if there exists an uncountable linear order N such that $N \leq L$ and $N \leq M$, and otherwise L and M are *far*.

Note that if $\mathcal{F}$ is a pairwise far family of elements of a class $\mathcal{C}$ of uncountable linear orders, then any basis for $\mathcal{C}$ has size at least $|\mathcal{F}|$.

Theorem (Essentially Sierpiński (1950))

There is a family $\{A_i : i < (2^\omega)^+\}$ of sets of reals, each of size 2^ω, such that for all $i < j$, A_i and A_j have no isomorphic suborder of size 2^ω.

Corollary

Assuming CH, any basis for the uncountable separable linear orders has size at least ω_2.

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ω_1-Dense Linear Orders

Definition

An uncountable linear order L is *ω_1-dense* if for all $x < y$ in L, $\{z \in L : x < z < y\}$ has size ω_1.

Proposition

If L is an uncountable linear order such that $\omega_1 \not\leq L$ and $\omega_1^* \not\leq L$, then L contains an ω_1-dense suborder without endpoints.

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ω_1-Dense Sets of Reals

Theorem

Any ω_1-dense separable linear order is isomorphic to a set of reals.

Since separable linear orders cannot contain uncountable well-ordered or conversely well-ordered subsets, we get:

Corollary

The family of all ω_1-dense sets of reals is a basis for the uncountable separable linear orders.

The linear order obtained from $\mathbb{R}$ by splitting each real into two elements is uncountable and separable but not isomorphic to a set of reals.

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Baumgartner's Axiom

Theorem (Baumgartner (1973))

Assuming CH, any two ω_1-dense sets of reals can be made isomorphic by a c.c.c. forcing of size ω_1.

Using this theorem, starting in any model of CH we can define a finite support iteration of c.c.c. forcings which forces that $2^\omega = \omega_2$ and any two ω_1-dense sets of reals are isomorphic.

Definition

Baumgartner's axiom is the statement that any two ω_1-dense sets of reals are isomorphic.

Corollary

It is consistent that there is a basis for the class of uncountable separable linear orders of size one.

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Aronszajn Lines

Basis problems concerning the class of uncountable linear orders go back to Erdős and Rado with the following question.

Question (Erdős-Rado (1956))

Does the family consisting of ω_1, ω_1^*, and the family of all sets of reals constitute a basis for the class of uncountable linear orders?

This question was answered by Specker (unpublished), who noted that lexicographically ordered Aronszajn trees provide a counter-example.

Definition

An *Aronszajn line* (or *Specker line*) is an uncountable linear order which does not contain any uncountable well-ordered or conversely well-ordered suborder nor an uncountable separable suborder.

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Aronszajn Lines and Trees

Theorem

Any lexicographically ordered Aronszajn tree is an Aronszajn line.

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Any Aronszajn line is the lexicographical ordering of some Aronszajn tree.

Since any Aronszajn tree has size ω_1, we get:

Corollary

Any Aronszajn line has size ω_1.

Any uncountable suborder of an Aronszajn line is an Aronszajn line. Since by definition Aronszajn lines do not contain uncountable well-ordered or conversely well-ordered suborders, every Aronszajn line contains an ω_1-dense suborder.

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A Finite Basis For the Uncountable Linear Orders

Note that just by definition of Aronszajn lines, the following forms a basis for the class of uncountable linear orders: ω_1, ω_1^*, the ω_1-dense sets of reals, and the (ω_1-dense) Aronszajn lines.

Question (Baumgartner (1976))

Is is consistent that there exists a finite basis for the class of uncountable linear orders?

In light of the consistency of Baumgartner's axiom that any two ω_1-dense sets of reals are isomorphic, the question seems to reduce to the consistency of a finite basis for the Aronszajn lines.

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In contrast to Baumgartner's axiom, the following is a theorem of ZFC:

Theorem

There exist 2^{ω_1}-many pairwise non-isomorphic ω_1-dense Aronszajn lines.

However, the proof of this result produces Aronszajn lines which are near, that is, have isomorphic uncountable suborders. So this result does not put a restriction on the size of a basis for the Aronszajn lines.

Isomorphism Types of Aronszajn Lines

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Countryman Lines

Definition

A *Countryman line* is an uncountable linear order C such that the partial order $C \times C$ ordered component-wise is a union of countably many chains.

Note that any uncountable suborder of a Countryman line is also a Countryman line. Any Countryman line is an Aronszajn line, and hence has size ω_1. The converse C^* of a Countryman line C is also a Countryman line.

Theorem (Shelah (1976))

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At Least Two Basis Elements For Aronszajn Lines

Theorem

If C is a Countryman line, then there does not exist an uncountable linear order L such that $L \leq C$ and $L \leq C^*$ (that is, C and C^* are far).

Otherwise there are uncountable $L, M \subseteq C$ and an order reversing function $f : L \to M$. Then $f \subseteq C \times C$ is an uncountable antichain (that is, a pairwise incomparable set), contradicting that $C \times C$ is a countable union of chains.

Corollary

Any basis for the class of Aronszajn lines must contain at least two Countryman lines, and hence must have size at least two. Therefore, any basis for the uncountable linear orders must have size at least five.

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Shelah's Conjecture

Conjecture (Shelah (1976))

It is consistent that every Aronszajn line contains a Countryman line.

Theorem (Todorčević)

Assume MA_{ω_1} and any two normal Aronszajn trees are club isomorphic. Then there exists a Countryman line C such that C and C^* form a basis for the class of Countryman lines (any of the lines $C(\rho_0)$, $C(\rho_1)$, $C(\rho)$, or $C(\rho_3)$ defined from walks on ordinals work).

Corollary

Assuming MA_{ω_1}, any two normal Aronszajn trees are club isomorphic, and Baumgartner's axiom, the following are equivalent:

1. *Every Aronszajn line contains a Countryman line.*
2. *There exists a five element basis for the uncountable linear orders.*

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Negative Results

Lemma (Shelah(1976))

$\diamondsuit_{\omega_1}$ implies that there exists a pairwise far family of 2^{ω_1}-many Countryman lines, and hence any basis for the Aronszajn lines has size 2^{ω_1}.

Theorem (Moore (2007))

The principle $\mho$ implies that there exists an Aronszajn line with no Countryman suborder.

$\mho$ is a consequence of $\diamondsuit_{\omega_1}$, but is consistent with MA_{ω_1}.

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MA_{ω_1} does not imply that every Aronszajn line contains a Countryman line.

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Aronszajn Trees

An *Aronszajn tree* is a tree with height ω_1, countable levels, and no uncountable branch. An Aronszajn tree is *special* if it is a union of countably many antichains.

Given Aronszajn trees T and U, we say that T and U are *club isomorphic* if there exists a club $C \subseteq \omega_1$ such that $T \restriction C$ and $U \restriction C$ are isomorphic.

For elements x and y of an Aronszajn tree T, let $x \wedge y$ denote the *meet*, or greatest lower bound, of x and y in T.

We assume going forward that all our trees are downwards closed subtrees of ${}^{<\omega_1}\omega$. For any such tree T and incomparable $x, y \in T$, let $\Delta(x, y)$ be the least $\alpha \in \mathrm{dom}(x) \cap \mathrm{dom}(y)$ such that $x(\alpha) \neq y(\alpha)$. Then $x \wedge y = x \restriction \Delta(x, y) = y \restriction \Delta(x, y)$.

Coherent Aronszajn Trees

Definition

An Aronszajn tree T (which is a downwards closed subtree of ${}^{<\omega_1}\omega$) is *coherent* if for all $x, y \in T$, the set

$$\{\alpha \in \mathrm{dom}(x) \cap \mathrm{dom}(y) : x(\alpha) \neq y(\alpha)\}$$

is finite.

Theorem (Kunen (1980))

Coherent Aronszajn trees exist.

Theorem (Todorčević)

If T is a special coherent Aronszajn tree, then the lexicographical ordering of T is a Countryman line.

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The Coloring Axiom For Trees

Definition (Abraham and Shelah (1985))

For an Aronszajn tree T, the *coloring axiom for T* (CAT(T)) is the statement that whenever $K \subseteq T$, there exists an uncountable antichain $A \subseteq T$ such that all meets of elements of A are in K or all meets are not in K (abbreviated $\wedge(A) \subseteq K$ or $\wedge(A) \subseteq T \setminus K$). The *coloring axiom for trees* (CAT) is the statement that for every Aronszajn tree T, CAT(T) holds.

Theorem

Assume MA_{ω_1} and any two normal Aronszajn trees are club isomorphic. Then CAT holds iff CAT(T) holds for some normal Aronszajn tree T.

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CAT iff Shelah's Conjecture

Theorem (Abraham and Shelah (1985))

Assume MA_{ω_1} and any two normal Aronszajn trees are club isomorphic. Then the following are equivalent:

1. *CAT.*
2. *Every Aronszajn line contains a Countryman line.*

Corollary

Assume MA_{ω_1}, Baumgartner's axiom, and any two normal Aronszajn trees are club isomorphic. Then the following are equivalent:

1. *There exists a coherent Aronszajn tree T such that $CAT(T)$ holds.*
2. *There exists a five element basis for the uncountable linear orders.*

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2. *There exists a five element basis for the uncountable linear orders.*

Moore's Five Element Basis Theorem

Theorem (Moore (2006))

Assume PFA. Let T be a binary, uniform, and coherent Aronszajn tree. Then CAT(T) holds.

PFA also implies:

- MA_{ω_1};
- Baumgartner's axiom;
- Any two normal Aronszajn trees are club isomorphic.

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PFA has considerable large cardinal consistency strength. Any known proof of the consistency of PFA uses a supercompact cardinal.

In contrast, starting with a model of GCH, one can build a generic extension satisfying the other “ingredients" of the five element basis theorem without using any large cardinals at all:

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Bounded Proper Forcing Axiom

BPFA is a restricted version of PFA which asserts that for any proper forcing $\mathbb{P}$ and for any family $\{A_i : i < \omega_1\}$ of maximal antichains of $\mathbb{P}$ each of size ω_1, there is a filter on $\mathbb{P}$ which meets each maximal antichain.

Roughly speaking, BPFA is equivalent to the statement that any Σ_1 sentence involving parameters in $H(\omega_2)$ which can be forced by a proper forcing is already true.

BPFA implies the other ingredients of the five element basis theorem: MA_{ω_1}, BA, and club isomorphisms of Aronszajn trees.

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Aronszajn Tree Saturation

Significant progress on these questions was made by König, Larson, Moore, and Veličković (2008). One of the most important ideas introduced in their work is *Aronszajn tree saturation*.

Definition

A *subtree* of an Aronszajn tree T is an uncountable downwards closed subset of T. Subtrees U and W of T are *almost disjoint* if $U \cap W$ is countable.

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An ω_1-tree T is *saturated* if any pairwise almost disjoint family of subtrees of T has size at most ω_1.

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The Property φ

Aronszajn tree saturation implies that there does not exist a Kurepa tree, and hence that ω_2 is an inaccessible cardinal in L. In fact, Aronszajn tree saturation is equiconsistent with an inaccessible cardinal.

König, Larson, Moore, and Veličković (2008) introduced a strengthening of Aronszajn tree saturation denoted by φ.

The property φ asserts the existence of "certificates" witnessing that a given Aronszajn tree is saturated so that the tree remains saturated in any ω_1-preserving generic extension. They were able to prove that φ is consistent assuming that there exists a Mahlo cardinal.

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Aronszajn Tree Saturation and Shelah's Conjecture

Theorem (König, Larson, Moore, and Veličković (2008))

Assume BPFA and φ. Then there exists a five element basis for the uncountable linear orders.

The authors were also able to use the concept of Aronszajn tree saturation to substantially reduce the consistency strength of a five element basis from a supercompact cardinal to something strictly between an inaccessible and a Mahlo cardinal. The exact large cardinal assumption is a technical statement involving multiple reflecting cardinals.

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Main Results

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Forcing an Instance of CAT

Both of the above theorems follow from:

Theorem (K. and Moore (2025))

Let T be a special coherent Aronszajn tree which is binary and uniform. Let $K \subseteq T$. Then there exists a proper forcing which adds an uncountable antichain A such that either $\wedge(A) \subseteq K$ or $\wedge(A) \subseteq T \setminus K$.

The proof involves defining by recursion a class-sized forcing iteration of attempts to add an antichain as above. At any given stage, if the attempt fails then a certain kind of subtree of a product tree of T is introduced by some proper forcing. The main point is to show that this recursion eventually terminates, which is proved using a new preservation theorem for iterated forcing.

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New Preservation Theorem For Iterated Forcing

Theorem (K. and Moore (2025))

Let T be an ω_1-tree. Suppose that $\langle \mathbb{P}_i, \dot{\mathbb{Q}}_i : i < \delta \rangle$ is a countable support forcing iteration of proper forcings, where δ is a limit ordinal. If U is a subtree of T in $V^{\mathbb{P}_\delta}$, then for some $\beta < \delta$ there exists a subtree W of T in $V^{\mathbb{P}_\beta}$ such that $W \subseteq U$.

This theorem is a natural generalization of two classic preservation theorems about trees due to Abraham-Shelah and Miyamoto: the preservation of Suslin trees and the preservation of not adding new uncountable branches by countable support forcing iterations of proper forcings.

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Open Problems

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The non-existence of a Kurepa tree implies that ω_2 is inaccessible in L.

Question

If there exists a Kurepa tree, does it follow that there is an Aronszajn line with no Countryman suborder?

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