NOT ADDING NEW COFINAL BRANCHES IN COUNTABLE SUPPORT FORCING ITERATIONS

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1. Introduction

In these notes we will provide a detailed proof of the following preservation theorem.

Theorem 1.1. *Let T be an ω_1-tree. Suppose that $\langle \mathbb{P}_i, \dot{\mathbb{Q}}_j : i \leq \beta,\ j < \beta \rangle$ is a countable support forcing iteration of proper forcings such for all $j < \beta$, $\Vdash_j \dot{\mathbb{Q}}_j$ does not add new cofinal branches to T. Then $\mathbb{P}_\beta$ does not add new cofinal branches to T.*

Theorem 1.1 was originally proven by Shelah in Chapter III Section 8 of [1].

In particular, if T is Aronszajn, then T being Aronszajn is preserved by such a forcing iteration. And if T is not a Kurepa tree, then T remains not being a Kurepa tree after such a forcing iteration.

The proof of Theorem 1.1 will be by induction on β, with the base case and the successor case being immediate. So we will focus on the case when β is a limit ordinal.

We will use the following standard facts from the theory of forcing iterations.

Lemma 1.2. *Let $\langle \mathbb{P}_i, \dot{\mathbb{Q}}_j : i \leq \beta,\ j < \beta \rangle$ be a countable support forcing iteration of proper forcings. Let λ be a large enough regular cardinal and N a countable elementary substructure of $H(\lambda)$ containing as a member the forcing iteration. Let $\alpha \in N \cap \beta$. Suppose that $\dot{p}$ is a $\mathbb{P}_\alpha$-name for a condition in $\mathbb{P}_\beta$, $q \in \mathbb{P}_\alpha$, and $q \Vdash_\alpha \dot{p} \in N$. Also, assume that q is $(N, \mathbb{P}_\alpha)$-generic and $q \Vdash_\alpha \dot{p} \restriction \alpha \in \dot{G}_\alpha$. Then there exists $q^+ \in \mathbb{P}_\beta$ such that $q^+ \restriction \alpha = q$, q^+ is $(N, \mathbb{P}_\beta)$-generic, and $q^+ \Vdash_\beta \dot{p} \in \dot{G}_\beta$.*

For a proof, see Lemma 31.17 of [2]. A variation of this lemma is proven as 3.3H Induction Lemma in Chapter III Section 3 of [1], although it is stated in a way which assumes that the name $\dot{p}$ is a member of N, rather than that q forces that $\dot{p} \in N$.

Lemma 1.3. *Let $\langle \mathbb{P}_i, \dot{\mathbb{Q}}_j : i \leq \beta,\ j < \beta \rangle$ be a countable support forcing iteration, where β is a limit ordinal. Let $p \in \mathbb{P}_\beta$. Consider a generic filter G_β on $\mathbb{P}_\beta$. Then $p \in G_\beta$ iff there are cofinally many $\alpha < \beta$ such that $p \restriction \alpha \in G_\beta \restriction \alpha$.*

Proof sketch. The forward direction of the equivalence is immediate. For the reverse direction, suppose for a contradiction that there is some condition $q \in \mathbb{P}_\beta$ which forces that p is a counter-example (and in particular, p and q are incompatible in $\mathbb{P}_\beta$). Now build a condition $t \in \mathbb{P}_\beta$ by choosing, for each $\gamma < \beta$, a name $\mathbb{P}_\gamma$-name $t(\gamma)$ which is forced to equal some lower bound of $p(\gamma)$ and $q(\gamma)$ in $\dot{\mathbb{Q}}_\gamma$ if such a

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lower bound exists, and otherwise equals $q(\alpha)$. Now prove that $t \leq_\beta p, q$, which is a contradiction. $\square$

Comment: We can avoid using Lemma 1.2 if we assume that our forcing iterations are separative.

2. Not Adding Branches

The proof of Theorem 1.1 uses a combination of Lemma 1.2 and the next lemma.

Lemma 2.1. *Let $\langle \mathbb{P}_i, \dot{\mathbb{Q}}_j : i \leq \beta,\ j < \beta \rangle$ be a countable support forcing iteration of proper forcings. Let T be an ω_1-tree and assume that for all $\alpha < \beta$, $\mathbb{P}_\alpha$ does not add new cofinal branches to T. Let $\alpha < \beta$. Assume that $p_0 \in \mathbb{P}_\beta$, $p_0 \Vdash_\beta \dot{b}$ is a cofinal branch of T, and D a dense open subset of $\mathbb{P}_\beta$. Then there exist $\mathbb{P}_\alpha$-names $\dot{p}_{0,0}$ and $\dot{p}_{0,1}$ for members of $\mathbb{P}_\beta$ such that $\mathbb{P}_\alpha$ forces: if $p_0 \restriction \alpha \in \dot{G}_\alpha$, then the following statements hold:*

1. *$\dot{p}_{0,0}$ and $\dot{p}_{0,1}$ both extend p_0 and are in D;*
2. *$\dot{p}_{0,0} \restriction \alpha$ and $\dot{p}_{0,1} \restriction \alpha$ are in $\dot{G}_\alpha$;*
3. *either $\dot{p}_{0,0}$ and $\dot{p}_{0,1}$ both force (in $\mathbb{P}_\beta$ over V) that $\dot{b} \in V$, or else there exist incomparable x_0 and x_1 on the same level of T such that $\dot{p}_{0,0} \Vdash_\beta^V x_0 \in \dot{b}$ and $\dot{p}_{0,1} \Vdash_\beta^V x_1 \in \dot{b}$.*

Proof. It suffices to prove that every condition in $\mathbb{P}_\alpha$ has an extension which forces the existence of $\dot{p}_{0,0}$ and $\dot{p}_{0,1}$ as described. So let $r \in \mathbb{P}_\alpha$. By extending r further if necessary, we may assume without loss of generality that r decides whether or not $\dot{p}_0 \restriction \alpha \in \dot{G}_\alpha$, and in the case that it does, that $r \leq p_0 \restriction \alpha$. If r decides the above statement negatively, then any two conditions in $\mathbb{P}_\beta$ witness the required properties vacuously, so assume that $r \leq p_0 \restriction \alpha$. Now extend $r \cup p_0 \restriction [\alpha, \beta)$ to $u \in D$. Note that $u \leq p_0$, and hence u forces (in $\mathbb{P}_\beta$ over V) that $\dot{b}$ is a cofinal branch of T.

We claim that $u \restriction \alpha$ is the desired extension of r. So let G_α be a generic filter for $\mathbb{P}_\alpha$ containing $u \restriction \alpha$, and we work in $V[G_\alpha]$. As usual, let $\mathbb{P}_\beta / G_\alpha$ be the suborder of $\mathbb{P}_\beta$ consisting of all $q \in \mathbb{P}_\beta$ such that $q \restriction \alpha \in G_\alpha$, and recall that $\mathbb{P}_\beta$ is forcing equivalent to $\mathbb{P}_\alpha * (\mathbb{P}_\beta / \dot{G}_\alpha)$.

Since $u \in \mathbb{P}_\beta / G_\alpha$, find $v \leq u$ in $\mathbb{P}_\beta / G_\alpha$ which decides over $V[G_\alpha]$ whether or not $\dot{b}$ is in $V[G_\alpha]$. If the decision is positive, then find $w \leq v \restriction \alpha$ in G_α which forces (in $\mathbb{P}_\alpha$ over V) that v forces (in $\mathbb{P}_\beta / \dot{G}_\alpha$ over $V[\dot{G}_\alpha]$) that $\dot{b} \in V[\dot{G}_\alpha]$. Now let $p^* := w \cup v \restriction [\alpha, \beta)$, and note that $p^* \Vdash_\beta^V \dot{b} \in V[\dot{G}_\alpha]$. Since $\mathbb{P}_\alpha$ does not add new cofinal branches of T, $p^* \Vdash_\beta^V \dot{b} \in V$. Also $p^* \restriction \alpha = w \in G_\alpha$, and since $p^* \leq u$, $p^* \in D$. So we can let $p_{0,0} = p_{0,1} = p^*$ and we are done.

Finally, assume that v forces (in $\mathbb{P}_\beta / \dot{G}_\alpha$ over $V[G_\alpha]$) that $\dot{b} \notin V[G_\alpha]$. Then the set c of all $x \in T$ such that v forces (in $\mathbb{P}_\beta / G_\alpha$ over $V[G_\alpha]$) that $x \in T$ must be countable, for otherwise this set is a cofinal branch of T in $V[G_\alpha]$ and hence forced by v to equal $\dot{b}$. Fix $\gamma < \omega_1$ greater than the height of all elements in c. Then there must exist extensions q_0 and q_1 of v and distinct x_0 and x_1 in T of height γ such that q_0 and q_1 force (in $\mathbb{P}_\beta / G_\alpha$ over $V[G_\alpha]$), respectively, that $x_0 \in \dot{b}$ and $x_1 \in \dot{b}$. Since $q_0 \restriction \alpha$ and $q_1 \restriction \alpha$ are in G_α, we can find $s \in G_\alpha$ below both of these conditions which force that information about q_0 and q_1. Now let $p_{0,1} := s \cup q_0 \restriction [\alpha, \beta)$ and $p_{0,1} := s \cup q_1 \restriction [\alpha, \beta)$. Then similar to the arguments above, $p_{0,0}$ and $p_{0,1}$ are as required. $\square$

The proof of Theorem 1.1 now follows from the next proposition.

Proposition 2.2. *Let $\langle \mathbb{P}_i, \dot{\mathbb{Q}}_j : i \leq \beta,\ j < \beta \rangle$ be a countable support forcing iteration of proper forcings, where β is a limit ordinal. Let T be a normal ω_1-tree and assume that for all $\alpha < \beta$, $\mathbb{P}_\alpha$ does not add new cofinal branches to T. Then $\mathbb{P}_\beta$ does not add new cofinal branches to T.*

Proof. Suppose for a contradiction that $p \in \mathbb{P}_\beta$ and p forces that $\dot{b}$ is a cofinal branch of T which is not in V. We will find a condition $r \leq p$ which forces that $\dot{b}$ is countable giving a contradiction. Fix a large enough regular cardinal λ and fix a countable $N \prec H(\lambda)$ containing the above parameters. Let $\delta = N \cap \omega_1$. Enumerate the dense open subsets of $\mathbb{P}_\beta$ in N as $\langle D_n : n < \omega \rangle$, and let $\langle a_n : n < \omega \rangle$ enumerate T_δ. Fix an increasing sequence $\langle \alpha_n : n < \omega \rangle$ of ordinals in $N \cap \beta$ which are cofinal in $\sup(N \cap \beta)$, where $\alpha_0 = 0$.

We define by induction two sequences $\langle q_n : n < \omega \rangle$ and $\langle \dot{p}_i : i < \omega \rangle$ satisfying:

(1) for all i, $\dot{p}_i$ is a $\mathbb{P}_{\alpha_i}$-name for a condition in $\mathbb{P}_\beta$, and $\dot{p}_0$ is a $\mathbb{P}_0$-name for p;
(2) for all i, q_i is an $(N, \mathbb{P}_{\alpha_i})$-generic condition, and q_i forces in $\mathbb{P}_{\alpha_i}$ that $\dot{p}_i \in N$ and $\dot{p}_i \restriction \alpha_i \in \dot{G}_{\alpha_i}$;
(3) for all i, $q_{i+1} \restriction \alpha_i = q_i$, and q_{i+1} forces in $\mathbb{P}_{\alpha_{i+1}}$ that:
  (a) $\dot{p}_{i+1} \leq_\beta \dot{p}_i$;
  (b) $\dot{p}_{i+1} \in D_i$;
  (c) $\dot{p}_{i+1} \Vdash^V_\beta a_i \notin \dot{b}$.

Begin by letting q_0 be the empty condition in $\mathbb{P}_0$ and $\dot{p}_0$ a $\mathbb{P}_0$-name for p. Note that all of the requirements on q_0 and $\dot{p}_0$ hold vacuously. Now let $i < \omega$ and assume that $\dot{p}_i$ and q_i are defined as described. Applying Lemma 1.2 to $\dot{p}_i \restriction \alpha_{i+1}$ and q_i, find $q_{i+1} \in \mathbb{P}_{\alpha_{i+1}}$ such that $q_{i+1} \restriction \alpha_i = q_i$, q_{i+1} is $(N, \mathbb{P}_{\alpha_{i+1}})$-generic, and $q_{i+1} \Vdash_{\alpha_{i+1}} \dot{p}_i \restriction \alpha_{i+1} \in \dot{G}_{\alpha_{i+1}}$.

To define $\dot{p}_{i+1}$, consider a generic filter $G_{\alpha_{i+1}}$ which contains q_{i+1}. Let $G_{\alpha_i} = G_{\alpha_{i+1}} \restriction \alpha_i$. Let $p_i = \dot{p}_i^{G_{\alpha_i}}$, which is in N. So $p_i \restriction \alpha_{i+1} \in G_{\alpha_{i+1}}$. Also, $p_i \leq_\beta p$, so p_i forces (in $\mathbb{P}_\beta$ over V) that $\dot{b}$ is a cofinal branch of T not in V.

Applying Lemma 2.1 in V to p_i and D_i, we conclude that there exist $\mathbb{P}_{\alpha_{i+1}}$-names $\dot{p}_{i,0}$ and $\dot{p}_{i,1}$ for members of $\mathbb{P}_\beta$ such that $\mathbb{P}_{\alpha_{i+1}}$ forces: if $\dot{p}_i \restriction \alpha_{i+1} \in \dot{G}_{\alpha_{i+1}}$, then the following statements hold:

(1) $\dot{p}_{i,0}$ and $\dot{p}_{i,1}$ both extend p_i and are in D_i;
(2) $\dot{p}_{i,0} \restriction \alpha_{i+1}$ and $\dot{p}_{i,1} \restriction \alpha_{i+1}$ are in $\dot{G}_{\alpha_{i+1}}$;
(3) either $\dot{p}_{i,0}$ and $\dot{p}_{i,1}$ both force (in $\mathbb{P}_\beta$ over V) that $\dot{b} \in V$, or else there exist incomparable x_0 and x_1 on the same level of T such that $\dot{p}_{i,0} \Vdash^V_\beta x_0 \in \dot{b}$ and $\dot{p}_{i,1} \Vdash^V_\beta x_1 \in \dot{b}$.

Since $p_i \in N$ (as well as the other parameters mentioned above), by elementarity we may assume that the names $\dot{p}_{i,0}$ and $\dot{p}_{i,1}$ are in N.

Now let us go back to $V[G_{\alpha_{i+1}}]$. Let $p_{i,0} = \dot{p}_{i,0}^{G_{\alpha_{i+1}}}$ and $p_{i,1} = \dot{p}_{i,1}^{G_{\alpha_{i+1}}}$. Note that since $p_i \restriction \alpha_{i+1} \in G_{\alpha_{i+1}}$, $p_{i,0} \restriction \alpha_{i+1}$ and $p_{i,1} \restriction \alpha_{i+1}$ are also in $G_{\alpha_{i+1}}$ by (2) above. Also, $p_{i,0}$ and $p_{i,1}$ are below p_i and hence below p, so they force that $\dot{b} \notin V$, so the second alternative of (3) holds. Since q_{i+1} is $(N, \mathbb{P}_{\alpha_{i+1}})$-generic, $N[G_{\alpha_{i+1}}] \cap V = N$. So $p_{i,0}$ and $p_{i,1}$ are in N. Let x_0 and x_1 be as in (3) above, where by elementarity we may assume that they are in N. Since $x_0 \neq x_1$ are on the same level of T, fix

$k < 2$ least such that $x_k \not<_T a_i$. Since $p_{i,k} \Vdash_\beta^V x_k \in \dot{b}$, and $x_k \not<_T a$, $p_{i,k} \Vdash_\beta^V a_i \notin \dot{b}$. Finally, let $p_{i+1} = p_{i,k}$.

Since the generic filter was chosen arbitrarily containing q_{i+1}, we can fix a $\mathbb{P}_{\alpha_{i+1}}$-name $\dot{p}_{i+1}$ for a member of $\mathbb{P}_\beta$ which q_{i+1} forces has the properties described above. Now it is routine to check that q_{i+1} and $\dot{p}_{i+1}$ are as required.

This completes the construction. Let $q = \bigcup_n q_n$, which is in $\mathbb{P}_\beta$. We will argue that q forces that $\dot{b}$ is a subset of N, and hence is countable, and that q and p are compatible. So taking any $r \leq p, q$, we are done.

Let G_β be a generic filter which contains q, and let $G_{\alpha_i} = G_\beta \restriction \alpha_i$ for all $i < \omega$. Let $p_i = \dot{p}_i^{G_{\alpha_i}}$ for all i and let $b = \dot{b}^{G_\beta}$. For all $i < \omega$, since $q \restriction \alpha_i$ forces that $\dot{p}_i \restriction \alpha_i \in \dot{G}_{\alpha_i}$, $p_i \restriction \alpha_i \in G_{\alpha_i}$. For all $i < j < \omega$, $p_j \restriction \alpha_j \leq_{\alpha_j} p_i \restriction \alpha_j$, so $p_i \restriction \alpha_j \in G_{\alpha_j}$. Using Lemma 1.3, since $\mathrm{dom}(p_i) \subseteq N \cap \beta = \sup_j \alpha_j$, it follows that $p_i \in G_\beta$. By the choice of p_{i+1}, $a_i \notin b$. So $b \subseteq T \restriction \delta \subseteq N$. As G_β was arbitrary, q forces all of the above statements. So $q \Vdash_\beta \dot{b} \subseteq N$. And since q forces that $p \in \dot{G}_\beta$, q and p are compatible. $\square$

References

[1] S. Shelah. *Proper and Improper Forcing.* Perspectives in Mathematical Logic. Springer-Verlag, Berlin, second edition, 1998.

[2] T. Jech. *Set Theory. The Third Millennium Edition, revised and expanded.* Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2003.

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