PRODUCTS OF SUSLIN TREES

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In these notes we will prove the following result about products of ω_1-Suslin trees.

Main Theorem. *Suppose that $\{S_i : i \in I\}$ is a collection of normal ω_1-Suslin trees such that for any distinct $i_0, \ldots, i_{n-1}$ in I, $S_{i_0} \otimes \cdots \otimes S_{i_{n-1}}$ is Suslin. Then:*

1. *the finite support product of the posets in $\{S_i : i \in I\}$ is c.c.c.;*
2. *the countable support product of the posets in $\{S_i : i \in I\}$ is countably distributive, proper, and assuming CH, ω_2-c.c.*

In the above, $S_{i_0} \otimes \cdots \otimes S_{i_{n-1}}$ refers to the tree consisting of all n-tuples in the product $S_{i_0} \times \cdots \times S_{i_{n-1}}$ whose elements have the same height in their respective tree, ordered componentwise. This tree is a dense subset of the product, so the poset $S_{i_0} \times \cdots \times S_{i_{n-1}}$ is c.c.c. iff $S_{i_0} \otimes \cdots \otimes S_{i_{n-1}}$ is Suslin.

We note that (2) can be improved to showing that the countable support product is $< \omega_1$-proper and Dee-complete.

Statement (1) of the theorem follows immediately from the following general observation.

Proposition. *Suppose that $\{\mathbb{Q}_i : i \in I\}$ is a collection of c.c.c. posets such that the product of any finitely many of the posets is c.c.c. Then the finite support product of $\{\mathbb{Q}_i : i \in I\}$ is c.c.c.*

Proof. Let $\mathbb{P}$ denote the finite support product, and consider a collection $\{p_i : i < \omega_1\}$ of conditions in $\mathbb{P}$. By the Δ-system lemma, fix a finite set $r \subseteq I$ and an uncountable set $X \subseteq \omega_1$ such that for all $i < j$ in X, $\mathrm{dom}(p_i) \cap \mathrm{dom}(p_j) = r$. Since $\prod\{\mathbb{Q}_i : i \in r\}$ is c.c.c. by assumption, fix $i < j$ in I such that $p_i \restriction r$ and $p_j \restriction r$ are compatible in this product, and let $q \leq p_i \restriction r, p_j \restriction r$. Then

$$q \cup (p_i \restriction (I \setminus r)) \cup (p_j \restriction (I \setminus r))$$

is a condition in $\mathbb{P}$ below both p_i and p_j. $\square$

The proof of (2) is a generalization of the following theorem of [1].

Theorem. *Suppose that T is a normal free Suslin tree. Then the countable support product of any number of copies of T is countably distributive, proper, and assuming CH, ω_2-c.c.*

We will use the following standard lemma (see Lemma 2.1 of [1]).

Lemma. *Assume that S is a Suslin tree. Let θ be a large enough regular cardinal and let M be a countable elementary substructure of $H(\theta)$ with $S \in M$. Suppose that $X \in M$ is a subset of S, $x \in S_{M \cap \omega_1}$, and there exists some $z \in X$ with $x \leq_S z$. Then there exists some $y <_S x$ with $y \in X$.*

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Fix a collection of normal ω_1-Suslin trees $\{S_i : i \in I\}$ such that the product of any finitely many of the trees is Suslin. Let $\mathbb{P}$ denote the countable support product of the posets in this collection.

Lemma. *Let θ be a large enough regular cardinal, $M \prec H(\theta)$ countable, and assume that $\{S_i : i \in I\}$ is in M. Let $\delta := M \cap \omega_1$. Suppose that $D \in M$ is a dense open subset of $\mathbb{P}$, $p \in M \cap \mathbb{P}$, $i_0, \ldots, i_{n-1}$ are distinct elements in $\mathrm{dom}(p)$, and $z_0, \ldots, z_{n-1}$ are elements of $(S_{i_0})_\delta, \ldots, (S_{i_{n-1}})_\delta$ respectively such that $p(i_k) <_{S_{i_k}} z_k$ for all $k < n$. Then there exists some $r \leq p$ in $M \cap D$ such that for all $k < n$, $r(i_k) <_{S_{i_k}} z_k$.*

Proof. By extending p further if necessary, we may assume without loss of generality that $p(i_0), \ldots, p(i_{n-1})$ all have the same height in their respective trees. Define a set X contained in the Suslin tree $S_{i_0} \otimes \cdots \otimes S_{i_{n-1}}$ as the set of all n-tuples $(y_0, \ldots, y_{n-1})$ for which there exists some $t \in D$ such that $t \leq p$ and $t(i_k) = y_k$ for all $k < n$. Observe that $X \in M$.

Define $q \leq p$ with the same domain as p by letting $q(i_k) = z_k$ for all $k < n$, and $q(j) = p(j)$ for any other j. Now extend q to $t \in D$. Note that $(t(i_0), \ldots, t(i_{n-1}))$ is in X as witnessed by t. By the above lemma, there exists some $(y_k : k < n)$ in $X \cap M$ which is below $(z_k : k < n)$ in the product tree. Fix a witness $r \in D \cap M$ with $r \leq p$ such that $r(i_k) = y_k$ for all $k < n$. Then r is as required. □

Proof of Statement (2) of Main Theorem. The proof that $\mathbb{P}$ is ω_2-c.c. assuming CH is a routine application of the Δ-system lemma for ω_2-sized collections of countable sets.

Let θ be a large enough regular cardinal and consider a countable elementary substructure $M \prec H(\theta)$ with $\mathbb{P} \in M$. Let $\delta := M \cap \omega_1$. We will show that for all $p \in M \cap \mathbb{P}$, there exists some $q \leq p$ such that for every dense open subset D of $\mathbb{P}$ in M, there exists some $s \in M \cap D$ with $q \leq s$. By standard arguments, it follows that $\mathbb{P}$ is proper and countably distributive.

Let $\langle D_n : n < \omega \rangle$ enumerate all dense open subsets of $\mathbb{P}$ in M and let $\langle i_k : k < \omega \rangle$ enumerate $M \cap I$. Consider $p \in M \cap \mathbb{P}$. By induction we define:

(1) a descending sequence of conditions $\langle p_n : n < \omega \rangle$;
(2) a sequence $\langle z_n : n < \omega \rangle$ of elements of $\bigcup \{(S_i)_\delta : i \in I\}$.

The following inductive hypotheses will be satisfied for all n:

(a) $p_{n+1} \in D_n$;
(b) $i_n \in \mathrm{dom}(p_{n+1})$;
(c) for all $m \geq n+1$, $p_m(i_n) <_{S_{i_n}} z_n$.

Let $p_0 := p$. Now consider $n < \omega$ and assume that p_n is defined together with z_k for all $k < n$. Let p_n' be equal to p_n if $i_n \in \mathrm{dom}(p_n)$, and otherwise let $p_n' := p_n \cup \{(i_n, x_{\mathrm{root}})\}$ where x_{root} is the root of S_{i_n}. Note that for all $k < n$, $p_n'(i_k) = p_n(i_k) <_R z_k$. Since S_{i_n} is normal, we can fix some $z_n \in (S_{i_n})_\delta$ such that $p_n'(i_n) <_{S_{i_n}} z_n$. Applying the previous lemma, fix $p_{n+1} \leq p_n'$ in $M \cap D_n$ such that for all $k \leq n$, $p_{n+1}(i_k) <_{S_{i_k}} z_k$.

This completes the construction. Define q as follows. Let the domain of q be equal to $M \cap \kappa$, and for each $n < \omega$ define $q(i_n) = z_n$. Then q is a condition and $q \leq p_n$ for all $n < \omega$. Clearly, q is as required. □

Corollary. *There exists a c.c.c. product forcing which destroys all ω_1-Suslin trees in the ground model. There exists a countably distributive and proper product forcing,* *which is ω_2-c.c. assuming CH, which destroys all ω_1-Suslin trees in the ground model.*

Proof. In either case, apply Zorn's lemma to find a collection $\{S_i : i \in I\}$ of normal Suslin trees which is maximal with respect to the property that any finite product of distinct trees in the collection is Suslin. So if T is a Suslin tree not in the collection, then for some finitely many distinct $i_0, \ldots, i_{n-1}$ in I, $S_{i_0} \otimes \cdots \otimes S_{i_{n-1}} \otimes T$ is not Suslin, which implies that $S_{i_0} \otimes \cdots \otimes S_{i_{n-1}}$ forces that T is not Suslin.

Now both finite support and countable support products add a cofinal branch to S_i for all $i \in I$. And any Suslin tree not in the collection is not Suslin in the generic extension due to the presence of generic filters on all finite products of trees in the collection. $\square$

References

[1] J. Krueger, *A Rigid Kurepa Tree From a Free Suslin Tree*, preprint (2023).

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