KUREPA TREES ARE NOT RETRACTS

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Recall that for any infinite cardinal κ, a κ^+-*Kurepa tree* is a tree with height κ^+, levels of size less than κ^+, and at least κ^{++}-many cofinal branches. In the case $\kappa = \omega$, an ω_1-Kurepa tree is called a *Kurepa tree*. We consider the space $[T]$ of all cofinal branches of a Kurepa tree T in the cone topology. For any Kurepa tree T, there exists a Kurepa tree U which is a downwards closed binary normal subtree of ${}^{<\omega_1}2$ such that $[T]$ and $[U]$ are homeomorphic.

Recall that if Y is a subspace of a topological space X, then Y is a *retract* of X if there exists a continuous function $f : X \to Y$ such that $f(y) = y$ for all $y \in Y$. The following theorem is due to Lücke and Schlicht.

Theorem ([LS20]). *Let κ be an infinite cardinal. If T is a κ^+-Kurepa tree which is a downwards closed subtree of ${}^{<\kappa^+}(\kappa^+)$ such that $[T]$ has no isolated points, then $[T]$ is a not retract of ${}^{\kappa^+}(\kappa^+)$*

In this note, we show that in the special case $\kappa = \omega$, we can remove the assumption of not having isolated points. The proof is basically an adjustment of the proof of the above theorem. We state it slightly differently due to the fact stated in the first paragraph above.

Theorem. *Suppose that T is a Kurepa tree which is a downwards closed subtree of ${}^{<\omega_1}2$. Then $[T]$ is not a retract of ${}^{\omega_1}2$.*

On the other hand, Lücke and Schlicht proved that if $V = L$, then for any cardinal κ with uncountable cofinality, there exists a κ^+-Kurepa tree which is a downwards closed subtree of ${}^{<\kappa^+}(\kappa^+)$ which is a retract of ${}^{<\kappa^+}(\kappa^+)$ ([LS20]).

Lemma. *Suppose that T is a tree with height and size ω_1 which has at least ω_2-many cofinal branches. Then there exists a downwards closed subtree U of T such that for all $x \in U$, there exist ω_2-many cofinal branches of U which include x.*

Proof. Let U be the set of $x \in T$ such that x is a member of ω_2-many cofinal branches of T. Note that U is downwards closed. We claim that every member of U is included in ω_2-many cofinal branches of U. Let $x \in U$. Fix an enumeration $\langle b_i : i < \omega_2 \rangle$ of distinct cofinal branches of T each of which includes x. Suppose for a contradiction that x is included in no more than ω_1-many cofinal branches of U. Then there is some $\delta < \omega_2$ such that for all $\delta \leq i < \omega_2$, $b_i \notin [U]$. For all $\delta \leq i < \omega_2$, fix some $y_i \in T$ such that $y_i \in b_i$ and $y_i \notin U$. Since T has size ω_1, we can find a set $Z \subseteq \omega_2$ with size ω_2 and some $y \in T$ such that for all $i \in Z$, $y_i = y$. Note that $y \notin U$. Then y is included in b_i for all $i \in Z$ and $|Z| = \omega_2$, so $y \in U$, which is a contradiction. $\square$

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Proposition. *Let T be a tree with height and size ω_1 which has at least ω_2-many cofinal branches. Assume that there are continuous functions $F : [T] \to 2^{\omega_1}$ and $G : 2^{\omega_1} \to [T]$ such that G is surjective and $G \circ F$ is the identity function. Then CH holds and for some $\alpha < \omega_1$, for all $\alpha \leq \theta < \omega_1$, T_θ has size ω_1.*

Proof. Suppose for a contradiction that F and G are as described. Note that F is injective. Applying the lemma, fix an uncountable downwards closed subtree U of T such that every every member of U belongs to ω_2-many cofinal branches of U.

Claim 1: For all $d \in [T]$ there are uncountably many $\delta < \omega_1$ such that:

(1) for all $b \in [T]$, $b \restriction \delta = d \restriction \delta$ implies $F(b) \restriction \delta = F(d) \restriction \delta$;
(2) for all $g \in 2^{\omega_1}$, $g \restriction \delta = F(d) \restriction \delta$ implies $G(g) \restriction \delta = d \restriction \delta$.

Proof: Let $\zeta < \omega_1$. Fix $d \in [T]$ and define by induction an increasing sequence $\langle \delta_n : n < \omega \rangle$ of countable ordinals. Let $\delta_0 = \zeta$. Now let $n < \omega$ and assume that δ_n is defined. *Case 1:* n is even. Applying the fact that F is continuous at d, find a countable ordinal $\delta_{n+1} > \delta_n$ such that for all $b \in [T]$, if $b \restriction \delta_{n+1} = d \restriction \delta_{n+1}$, then $F(b) \restriction \delta_n = F(d) \restriction \delta_n$. *Case 2:* n is odd. Applying the fact that G is continuous at $F(d)$, find a countable ordinal $\delta_{n+1} > \delta_n$ such that for all $g \in 2^{\omega_1}$, if $g \restriction \delta_{n+1} = F(d) \restriction \delta_{n+1}$ then $G(g) \restriction \delta_n = G(F(d)) \restriction \delta_n$.

Let $\delta = \sup\{\delta_n : n < \omega\}$. We prove (1) and (2). (1) Consider any $b \in [T]$. Suppose that $F(b) \restriction \delta \neq F(d) \restriction \delta$. Then for some large enough even $n < \omega$, $F(b) \restriction \delta_n \neq F(d) \restriction \delta_n$. By case 1, $b \restriction \delta_{n+1} \neq d \restriction \delta_{n+1}$, and hence $b \restriction \delta \neq d \restriction \delta$. (2) Consider any $g \in 2^{\omega_1}$. Suppose that $G(g) \restriction \delta \neq d \restriction \delta$. Then for some large enough odd $n < \omega$, $G(g) \restriction \delta_n \neq d \restriction \delta_n$. But $G(F(d)) = d$, so $G(g) \restriction \delta_n \neq G(F(d)) \restriction \delta_n$. By case 2, $g \restriction \delta_{n+1} \neq F(d) \restriction \delta_{n+1}$. So $g \restriction \delta \neq F(d) \restriction \delta$. This completes the proof of the claim.

Next we build by induction sequences $\langle u_s : s \in 2^{<\omega} \rangle$, $\langle v_s : s \in 2^{<\omega} \rangle$, $\langle \delta_s : s \in 2^{<\omega} \rangle$, and $\langle h_s : s \in 2^{<\omega} \rangle$ satisfying:

(1) for all $s \in 2^{<\omega}$:
  (a) $u_s \in U$, $v_s \in 2^{<\omega_1}$, $\delta_s < \omega_1$, and $h_s \in [U]$;
  (b) $\mathrm{ht}_T(u_s) = \mathrm{dom}(v_s) = \delta_s$;
  (c) $u_s = h_s \restriction \delta_s$ and $v_s = F(h_s) \restriction \delta_s$;
(2) for all $s \subsetneq t$ in $2^{<\omega}$, $u_s <_T u_t$ and $v_s \subsetneq v_t$;
(3) for all $s \in 2^{<\omega}$, letting $\zeta = \min\{\delta_{s^\frown 0}, \delta_{s^\frown 1}\}$, $u_{s^\frown 0} \restriction \zeta \neq u_{s^\frown 1} \restriction \zeta$ and $v_{s^\frown 0} \restriction \zeta \neq v_{s^\frown 1} \restriction \zeta$;
(4) for all $s \in 2^{<\omega}$:
  (a) for all $b \in [T]$, $b \restriction \delta_s = h_s \restriction \delta_s$ implies $F(b) \restriction \delta_s = F(h_s) \restriction \delta_s$;
  (b) for all $g \in 2^{\omega_1}$, $g \restriction \delta_s = F(h_s) \restriction \delta_s$ implies $G(g) \restriction \delta_s = h_s \restriction \delta_s$.

For the base case, let $u_\emptyset$ be any element of U with height 0, let $v_\emptyset$ be the emptyset, let $\delta_\emptyset = 0$, and let $h_\emptyset$ be any cofinal branch of U. For the inductive step, let $s \in 2^{<\omega}$ be given and assume that u_s, v_s, δ_s, and h_s are defined as required. Since $u_s \in U$, by the choice of U we can pick distinct cofinal branches $h_{s^\frown 0}$ and $h_{s^\frown 1}$ of U both containing u_s. As F is injective, fix $\zeta > \delta_s$ such that $h_{s^\frown 0} \restriction \zeta \neq h_{s^\frown 1} \restriction \zeta$ and $F(h_{s^\frown 0}) \restriction \zeta \neq F(h_{s^\frown 1}) \restriction \zeta$. Applying claim 1 separately to each of $h_{s^\frown 0}$ and $h_{s^\frown 1}$, fix countable ordinals $\delta_{s^\frown 0}$ and $\delta_{s^\frown 1}$ greater than ζ satisfying that for each $i < 2$:

- for all $b \in [T]$, $b \restriction \delta_{s^\frown i} = h_{s^\frown i} \restriction \delta_{s^\frown i}$ implies $F(b) \restriction \delta_{s^\frown i} = F(h_{s^\frown i}) \restriction \delta_{s^\frown i}$;
- for all $g \in 2^{\omega_1}$, $g \restriction \delta_{s^\frown i} = F(h_{s^\frown i}) \restriction \delta_{s^\frown i}$ implies $G(g) \restriction \delta_{s^\frown i} = h_{s^\frown i} \restriction \delta_{s^\frown i}$.

For each $i < 2$, let $u_{s^\frown i} = h_{s^\frown i} \restriction \delta_{s^\frown i}$ and let $v_{s^\frown i} = F(h_{s^\frown i}) \restriction \delta_{s^\frown i}$.

For checking that these objects satisfy (1)-(4), all of these properties are immediate by definition except for the second part of (2). So let $i < 2$ and we show that $v_s \subsetneq v_{s^\frown i}$. It suffices to show that $v_s \subseteq F(h_{s^\frown i})$. Since $u_s \subseteq h_{s^\frown i}$, $h_{s^\frown i} \restriction \delta_s = u_s = h_s \restriction \delta_s$. By (4a), $F(h_{s^\frown i}) \restriction \delta_s = F(h_s) \restriction \delta_s = v_s$.

Now let $\theta < \omega_1$ be greater than $\sup\{\delta_s : s \in 2^{<\omega}\}$. We claim that T_θ has size at least 2^ω. For each function $w \in 2^\omega$, fix a function $f_w \in 2^{\omega_1}$ such that $\bigcup\{v_{w\restriction n} : n < \omega\} \subseteq f_w$, and define $d_w = G(f_w)$. We claim that for all $n < \omega$, $d_w \restriction \delta_{w\restriction n} = u_{w\restriction n}$. Let $n < \omega$. Then $f_w \restriction \delta_{w\restriction n} = v_{w\restriction n} = F(h_{w\restriction n}) \restriction \delta_{w\restriction n}$. By (4b), $d_w \restriction \delta_{w\restriction n} = G(f_w) \restriction \delta_{w\restriction n} = h_{w\restriction n} \restriction \delta_{w\restriction n} = u_{w\restriction n}$.

We show that for all distinct $w, z \in 2^\omega$, d_w and d_z split before level θ, which implies that T_θ has size at least 2^ω. This also implies CH since T has size ω_1. So let $w, z \in 2^\omega$ be distinct, and let n be least such that $w(n) \neq z(n)$. Let $s = w \restriction n$. Without loss of generality, assume that $\delta_{w\restriction(n+1)} = \min\{\delta_{s^\frown 0}, \delta_{s^\frown 1}\}$. Fix $i < 2$ such that $w \restriction (n+1) = s^\frown i$. By the previous paragraph, $d_w \restriction \delta_{s^\frown i} = u_{s^\frown i}$ and $d_z \restriction \delta_{s^\frown i} = (d_z \restriction \delta_{s^\frown(1-i)}) \restriction \delta_{s^\frown i} = u_{s^\frown(1-i)} \restriction \delta_{s^\frown i}$. By (3), $u_{s^\frown i} \neq u_{s^\frown(1-i)} \restriction \delta_{s^\frown i}$. So $d_w \restriction \delta_{s^\frown i} \neq d_z \restriction \delta_{s^\frown i}$. □

Corollary. *Assume that T is a Kurepa tree which is a downwards closed subtree of $2^{<\omega_1}$. Then $[T]$ is not a retract of 2^{ω_1}.*

Proof. Apply the above proposition letting F be the identity function on $[T]$. □

References

[LS20] P. Lücke and P. Schlicht, *Descriptive properties of higher Kurepa trees*, Research Trends in Contemporary Logic, College Publications, 2020, to appear.

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