SEPARATING WEAK PARTIAL SQUARE PRINCIPLES

JOHN KRUEGER AND ERNEST SCHIMMERLING

Abstract. We introduce the weak partial square principles $\square^p_{\lambda,\kappa}$ and $\square^p_{\lambda,<\kappa}$, which combine the ideas of a weak square sequence and a partial square sequence. We construct models in which weak partial square principles fail. The main result of the paper is that $\square_{\lambda,\kappa}$ does not imply $\square_{\lambda,<\kappa}$.

Let λ be a regular uncountable cardinal and let $\kappa \leq \lambda$ be a nonzero cardinal. Recall the square principle $\square_{\lambda,\kappa}$ (respectively $\square_{\lambda,<\kappa}$), which asserts the existence of a sequence $\langle \mathcal{C}_\alpha : \alpha \leq \lambda^+,\ \alpha \text{ limit} \rangle$ satisfying that every $\mathcal{C}_\alpha$ is a nonempty family of no more than κ many (respectively less than κ many) club subsets of α, each with order type at most λ, such that for all $c \in \mathcal{C}_\alpha$ and $\gamma \in \lim(c)$, $c \cap \gamma \in \mathcal{C}_\gamma$.

The principle $\square_{\lambda,1}$ is the same as the square principle $\square_\lambda$, and $\square_{\lambda,\lambda}$ is the same as the weak square principle $\square^*_\lambda$; both of these principles are due to Jensen [3]. The other principles were introduced by Schimmerling [7]. Jensen [2] proved that for nonzero cardinals $\mu < \kappa \leq \lambda$, $\square_{\lambda,\kappa}$ does not imply $\square_{\lambda,\mu}$.

Now consider an uncountable cardinal λ and a regular cardinal $\mu \leq \lambda$. A set $S \subseteq \lambda^+ \cap \mathrm{cof}(\mu)$ is said to *carry a partial square* if there exists a sequence $\langle c_\alpha : \alpha \in S \rangle$ such that each c_α is a club subset of α with order type μ, and if γ is a common limit point of c_α and c_β, then $c_\alpha \cap \gamma = c_\beta \cap \gamma$. Shelah [8] proved that if λ is regular, then there are densely many stationary subsets of $\lambda^+ \cap \mathrm{cof}(< \lambda)$ which carry a partial square. As a consequence, consistency results on partial squares usually focus on stationary subsets of $\lambda^+ \cap \mathrm{cof}(\lambda)$.

In this paper we introduce principles which combine the weak square principles of Jensen and Schimmerling with the idea of a partial square sequence. Let λ be an uncountable cardinal, let $\mu \leq \lambda$ be regular, and let $\kappa \leq \lambda$ be a nonzero cardinal. A set $S \subseteq \lambda^+ \cap \mathrm{cof}(\mu)$ is said to carry a *κ-weak partial square* (respectively *$<\kappa$-weak partial square*) if there exists a sequence $\langle c_\alpha : \alpha \in S \rangle$ satisfying:

(1) for all $\alpha \in S$, c_α is a club subset of α with order type μ;
(2) for all $\beta < \lambda^+$, the set $\{c_\alpha \cap \beta : \alpha \in S,\ \beta \in \lim(c_\alpha)\}$ has size less than or equal to κ (respectively less than κ).

For a regular uncountable cardinal λ, let $\square^p_{\lambda,\kappa}$ (respectively $\square^p_{\lambda,<\kappa}$) be the statement that there exists a stationary subset of $\lambda^+ \cap \mathrm{cof}(\lambda)$ which carries a κ-weak partial square (respectively $<\kappa$-weak partial square).

The goal of the paper is to construct models in which weak partial square principles fail, and to separate the different weak partial square principles. For this purpose we prove in Sections 1 and 3 two propositions which show that certain kinds of forcing posets cannot add threads to sequences of families of clubs. In

Date: April 2012.
2010 *Mathematical Subject Classification*: 03E05, 03E55.
Key words and phrases: weak partial square sequence.

Section 2 we show that if $\mu > \lambda$ is a Mahlo cardinal, then the Lévy collapse $\textsc{Coll}(\lambda, <\mu)$ forces $\neg\Box_{\lambda,<\lambda}$. We also construct a model in which the weak partial square principle $\Box^p_{\lambda,<\lambda}$ fails. In Section 4 we prove that $\Box^p_{\lambda,\kappa}$ does not imply $\Box^p_{\lambda,<\kappa}$.

1. Threads and Closed Forcings

In this section we prove a proposition about threads and closed forcing which will be used in all the consistency results of the paper.

Let δ be a limit ordinal, and let $\vec{\mathcal{C}} = \langle \mathcal{C}_\alpha : \alpha < \delta,\ \alpha \text{ limit} \rangle$ be a seqence such that each $\mathcal{C}_\alpha$ is a family of club subsets of α. A set $c \subseteq \delta$ is a *thread of* $\vec{\mathcal{C}}$ if for all $\beta \in \lim(c)$, $c \cap \beta \in \mathcal{C}_\beta$.

Proposition 1.1. *Let λ be a regular uncountable cardinal. Assume $\vec{\mathcal{C}} = \langle \mathcal{C}_\alpha : \alpha < \lambda^+,\ \alpha \text{ limit} \rangle$ is a sequence such that each $\mathcal{C}_\alpha$ is a family of fewer than λ many club subsets of α. Let $\mathbb{P}$ be a λ-closed forcing poset. Then $\mathbb{P}$ forces that any thread of $\vec{\mathcal{C}}$ is in the ground model.*

Proof. Consider the function from $\lambda^+ \cap \mathrm{cof}(\omega)$ into λ which maps α to $|\mathcal{C}_\alpha|$. Fix a stationary set $S \subseteq \lambda^+ \cap \mathrm{cof}(\omega)$ and a cardinal $\mu < \lambda$ such that for all $\alpha \in S$, $|\mathcal{C}_\alpha| = \mu$.

Suppose for a contradiction that p is a condition in the λ-closed forcing poset $\mathbb{P}$ which forces that $\dot{d}$ is a thread of $\vec{\mathcal{C}}$ which is not in the ground model. Then

$$\forall q \le p\ \exists r, s \le q\ \exists \gamma\ (r \Vdash \gamma \notin \dot{d}\ \wedge\ s \Vdash \gamma \in \dot{d}).$$

For if not, then there is $q \le p$ such that for any $\gamma < \lambda^+$, any two conditions below q which decide the statement "$\gamma \in \dot{d}$" decide it in the same way. This implies that q decides the statement "$\gamma \in \dot{d}$" for all $\gamma < \lambda^+$. But then q decides the entire set $\dot{d}$, and therefore forces that $\dot{d}$ is in the ground model, which is a contradiction.

Consider the tree ${}^{\le\mu}2$ of all binary sequences of length less than or equal to μ, ordered by letting $\sigma \trianglelefteq \pi$ if σ is an initial segment of π. We will define a map $\sigma \mapsto q_\sigma$ from ${}^{\le\mu}2$ into $\mathbb{P}$, by induction on the length of σ, as well as a set of ordinals $\langle \gamma_\sigma : \sigma \in {}^{<\mu}2 \rangle$, satisfying:

1. $\sigma \trianglelefteq \pi \Rightarrow q_\pi \le q_\sigma$;
2. $q_{\sigma ^\frown 0} \Vdash \gamma_\sigma \notin \dot{d}$ and $q_{\sigma ^\frown 1} \Vdash \gamma_\sigma \in \dot{d}$, for all $\sigma \in {}^{<\mu}2$.

Let $q_{\langle\ \rangle} = p$. Now let $\xi < \mu$ and suppose that q_π is defined for all $\pi \in {}^{\le\xi}2$. Consider σ in ${}^{\xi}2$. Since $q_\sigma \le p$, we can choose an ordinal $\gamma_\sigma < \lambda^+$ and conditions $q_{\sigma ^\frown 0}, q_{\sigma ^\frown 1} \le q_\sigma$ such that $q_{\sigma ^\frown 0} \Vdash \gamma_\sigma \notin \dot{d}$ and $q_{\sigma ^\frown 1} \Vdash \gamma_\sigma \in \dot{d}$.

Now let $\delta \le \mu$ be a limit ordinal, and suppose q_π is defined for all $\pi \in {}^{<\delta}2$. Consider σ in ${}^{\delta}2$. Then $\langle q_{\sigma \restriction i} : i < \delta \rangle$ is a descending sequence of conditions in $\mathbb{P}$. Since $\delta < \lambda$ and $\mathbb{P}$ is λ-closed, we can fix a lower bound q_σ of this sequence. This completes the construction.

Now the set of conditions $\{q_\sigma : \sigma \in {}^{\mu}2\}$ has size 2^μ, and $2^\mu \ge \mu^+$. Fix a subset $\{q_i : i < \mu^+\}$ of this set of conditions of size μ^+. Consider $i < j < \mu^+$. Let $q_i = q_\sigma$ and $q_j = q_\tau$, where $\sigma \ne \tau$. Let α be least such that $\sigma(\alpha) \ne \tau(\alpha)$, and let $\pi = \sigma \restriction \alpha = \tau \restriction \alpha$. Then $q_{\pi ^\frown 0} \Vdash \gamma_\pi \notin \dot{d}$ and $q_{\pi ^\frown 1} \Vdash \gamma_\pi \in \dot{d}$. Each of the conditions q_i, q_j is below one of $q_{\pi ^\frown 0}$, $q_{\pi ^\frown 1}$. So if we let $\gamma_{i,j} = \gamma_\pi$, then q_i and q_j disagree about the statement "$\dot{\gamma}_{i,j} \in \dot{d}$".

Recall that S is a stationary subset of $\lambda^+ \cap \mathrm{cof}(\omega)$, and for all α in S, $|\mathcal{C}_\alpha| = \mu$. Fix a regular cardinal θ large enough so that λ^+, $\mathbb{P}$, and $\dot{d}$ are in $H(\theta)$. Since S is stationary and $\mu^+ \le \lambda$, we can fix a set N satisfying:

(1) $N \prec (H(\theta), \in, \vec{\mathcal{C}}, \mathbb{P}, \dot{d})$;
(2) $|N| = \lambda$;
(3) $N \cap \lambda^+ \in S$;
(4) $\{q_i : i < \mu^+\} \cup \{\gamma_{i,j} : i < j < \mu^+\} \subseteq N$.

Let $\delta = N \cap \lambda^+$. Then $\delta \in S$, so $\mathrm{cf}(\delta) = \omega$. Fix an increasing sequence $\langle \delta_n : n < \omega \rangle$ which is cofinal in δ.

For each $i < \mu^+$ we define a descending sequence $\langle q^i_n : n < \omega \rangle$. Let $q^i_0 = q_i$, which is in N. Given q^i_n in N, apply the elementarity of N to fix $q^i_{n+1} \leq q^i_n$ in N satisfying that there exists an ordinal $\zeta > \delta_n$ in N such that $q^i_{n+1} \Vdash \zeta \in \dot{d}$. Since $\mathbb{P}$ is λ-closed, we can fix r_i which is a lower bound of $\langle q^i_n : n < \omega \rangle$. Note that by construction, r_i forces that δ is a limit point of $\dot{d}$.

This defines a family of conditions $\{r_i : i < \mu^+\}$. Each r_i forces that δ is a limit point of $\dot{d}$. But $\dot{d}$ is forced to be a thread of $\vec{\mathcal{C}}$, so r_i forces that $\dot{d} \cap \delta \in \mathcal{C}_\delta$. In particular, r_i forces that $\dot{d} \cap \delta$ is in the ground model. Extend r_i to s_i which decides $\dot{d} \cap \delta$ as e_i. Then $s_i \Vdash e_i \in \mathcal{C}_\delta$, so in fact, $e_i \in \mathcal{C}_\delta$.

Consider distinct $i < j < \mu^+$. Recall that q_i, q_j disagree about the statement "$\gamma_{i,j} \in \dot{d}$". But $\gamma_{i,j} < \delta$, and s_i, s_j are below q_i, q_j and decide $\dot{d} \cap \delta$ as e_i, e_j. So $\gamma_{i,j}$ is in the symmetric difference of e_i, e_j, hence e_i, e_j are distinct. Therefore $\{e_i : i < \mu^+\}$ is a family of μ^+ many elements of the set $\mathcal{C}_\delta$. But this contradicts the fact that $|\mathcal{C}_\delta| = \mu$. $\square$

2. The Failure of Weak Partial Square Principles

Consider a regular uncountable cardinal λ and a Mahlo cardinal $\mu > \lambda$. Solovay proved that the Lévy collapse $\mathrm{Coll}(\lambda, {<}\,\mu)$ forces that $\mu = \lambda^+$ and $\neg\square_\lambda$. In fact, it is known that the Lévy collapse $\mathrm{Coll}(\lambda, {<}\mu)$ forces $\neg\square_{\lambda,\kappa}$ for all cardinals $\kappa < \lambda$ ([2]). In the first theorem of this section we prove that the Lévy collapse $\mathrm{Coll}(\lambda, {<}\,\mu)$ forces the failure of $\square_{\lambda,<\lambda}$. In the second theorem we show how to construct a model in which the weak partial square principle $\square^{p}_{\lambda,<\lambda}$ fails.

Theorem 2.1. *Let λ be a regular uncountable cardinal and let $\mu > \lambda$ be a Mahlo cardinal. Then $\mathrm{Coll}(\lambda, {<}\mu)$ forces that $\mu = \lambda^+$ and $\neg\square_{\lambda,<\lambda}$.*

Proof. Suppose for a contradiction that p is a condition in $\mathrm{Coll}(\lambda, {<}\,\mu)$ which forces that $\vec{\mathcal{C}}$ is a $\square_{\lambda,<\lambda}$-sequence. For each limit ordinal $\alpha < \mu$ choose a sequence $\langle \dot{c}^\alpha_i : i < \lambda \rangle$ of nice names for subsets of α such that

$$p \Vdash \vec{\mathcal{C}}(\alpha) = \{\dot{c}^\alpha_i : i < \lambda\}$$

(we assume that the listing is with repetitions, as $\vec{\mathcal{C}}(\alpha)$ is forced to have size less than λ).

Since the Lévy collapse is μ-c.c., for each α we can choose an ordinal $\mu_\alpha < \mu$ such that for all $i < \lambda$, $\dot{c}^\alpha_i$ is a $\mathrm{Coll}(\lambda, {<}\,\mu_\alpha)$-name. By the Mahloness of μ, we can fix a strongly inaccessible cardinal $\nu < \mu$ greater than λ which is closed under the function $\alpha \mapsto \mu_\alpha$. Then for all $\alpha < \nu$ and $i < \lambda$, $\dot{c}^\alpha_i$ is a $\mathrm{Coll}(\lambda, {<}\nu)$-name.

Let G be a generic filter on $\mathrm{Coll}(\lambda, {<}\mu)$ which contains p. Factor $\mathrm{Coll}(\lambda, {<}\mu)$ as $\mathrm{Coll}(\lambda, {<}\,\nu) \times \mathrm{Coll}(\lambda, [\nu, \mu))$. Let $G_\nu = G \cap \mathrm{Coll}(\lambda, {<}\,\nu)$ and $G^\nu = G \cap \mathrm{Coll}(\lambda, [\nu, \mu))$. Then in $V[G_\nu]$, $\nu = \lambda^+$.

For each limit ordinal $\alpha < \nu$ and $i < \lambda$, let $c^\alpha_i = (\dot{c}^\alpha_i)^{G_\nu} = (\dot{c}^\alpha_i)^G$. Let $\mathcal{C}_\alpha = \{c^\alpha_i : i < \lambda\}$. Then the sequence $\langle \mathcal{C}_\alpha : \alpha < \nu,\ \alpha \text{ limit} \rangle$ is in $V[G_\nu]$. Moreover, for each limit ordinal $\alpha < \nu$, $\mathcal{C}_\alpha$ is a family of fewer than λ many club subsets of α in $V[G]$, and hence also in $V[G_\nu]$.

We apply Proposition 1.1 in the model $V[G_\nu]$ to conclude that the λ-closed forcing poset $\textsc{Coll}(\lambda, [\nu, \mu))$ cannot add a new thread of the sequence $\langle \mathcal{C}_\alpha : \alpha < \nu,\ \alpha \text{ limit}\rangle$. Let $\vec{\mathcal{C}} = \dot{\vec{\mathcal{C}}}^G$. Note that for all $\alpha < \nu$, $\vec{\mathcal{C}}(\alpha) = \mathcal{C}_\alpha$. Fix any c in $\vec{\mathcal{C}}(\nu)$. Then since $\vec{\mathcal{C}}$ is a $\Box_{\lambda,<\lambda}$-sequence, for all $\gamma \in \lim(c)$, $c \cap \gamma \in \mathcal{C}_\gamma$. So c is a thread of the sequence $\langle \mathcal{C}_\alpha : \alpha < \nu\rangle$, and hence is in $V[G_\nu]$. But c is a club subset of ν of order type at most λ, which contradicts that $\nu = \lambda^+$ in $V[G_\nu]$. $\Box$

Our motivation for this theorem was to determine the consistency strength of the failure of $\Box_{\lambda,<\lambda}$ when λ is strongly inaccessible. It turns out that the consistency strength is exactly a Mahlo cardinal. For if λ is strongly inaccessible and $\mu > \lambda$ is a Mahlo cardinal, then in a generic extension by the Lévy collapse $\textsc{Coll}(\lambda, <\mu)$, λ is still strongly inaccessible, $\mu = \lambda^+$, and $\neg\Box_{\lambda,<\lambda}$.

We now construct a model in which the weak partial square principle $\Box^p_{\lambda,<\lambda}$ fails. The forcing construction is similar to the one we developed in [4] to create a model in which there is no stationary subset of $\kappa^+ \cap \operatorname{cof}(\kappa)$ which carries a partial square.

We will use the following theorem of Magidor [5].

Theorem 2.2. *Let λ be a regular uncountable cardinal and let $\mu > \lambda$ be strongly inaccessible. Let $\nu < \mu$ be a strongly inaccessible cardinal greater than λ. Let $\dot{\mathbb{Q}}$ be a $\textsc{Coll}(\lambda, <\nu)$-name for a λ-closed forcing poset of size less than μ. Then $\textsc{Coll}(\lambda, <\mu)$ factors as $\textsc{Coll}(\lambda, <\nu) * \dot{\mathbb{Q}} * \dot{\mathbb{S}}$, for some $\dot{\mathbb{S}}$ which is forced to be λ-closed.*

Theorem 2.3. *Let λ be a regular uncountable cardinal, and let $\mu > \lambda$ be a measurable cardinal. Assume $2^\mu = \mu^+$. Then there exists a λ-closed forcing poset which collapses μ to become λ^+ and forces $\neg\Box^p_{\lambda,<\lambda}$.*

Proof. The forcing poset we define will be of the form $\textsc{Coll}(\lambda, <\mu) * \mathbb{P}$, where $\mathbb{P}$ is a $\leq\lambda$-support forcing iteration of length μ^+ which kills the stationarity of all subsets of $\mu \cap \operatorname{cof}(\lambda)$ which carry a $<\lambda$-weak partial square.

Let G be a generic filter on $\textsc{Coll}(\lambda, <\mu)$ over V. Then in $V[G]$, cardinals and cofinalities less than or equal to λ are preserved, $\mu = \lambda^+$, $2^\lambda = \mu$, and $2^\mu = \mu^+$.

Define by induction in $V[G]$ a $\leq\lambda$-support forcing iteration

$$\langle \mathbb{P}_i, \dot{\mathbb{Q}}_j : i \leq \lambda^{++},\ j < \lambda^{++}\rangle.$$

We will assume that each $\mathbb{P}_i$ is λ^+-distributive; this assumption will be justified after the definition is complete. Each factor forcing $\dot{\mathbb{Q}}_i$ will be forced to be λ-closed. Since the iteration is with $\leq\lambda$-support, it follows that each $\mathbb{P}_i$ is λ-closed.

Since the iteration is with $\leq\lambda$-support, it suffices to describe the factor forcings $\dot{\mathbb{Q}}_\alpha$ for $\alpha < \lambda^{++}$. So fix $\alpha < \lambda^{++}$, and assume that $\mathbb{P}_\alpha$ is defined. Since $\mathbb{P}_\alpha$ is λ^+-distributive, $\mathbb{P}_\alpha$ forces that $\mu = \lambda^+$. We fix a particular nice $\mathbb{P}_\alpha$-name $\dot{S}_\alpha$ for a subset of $\lambda^+ \cap \operatorname{cof}(\lambda)$ which carries a $<\lambda$-weak partial square. Let $\dot{\mathbb{Q}}_\alpha$ be a name for the forcing poset which adds a club subset of λ^+ using closed bounded subsets of λ^+ which are disjoint from $\dot{S}_\alpha$, ordered by end-extension. Let $\mathbb{P}_{\alpha+1} = \mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha$.

This completes the construction, assuming that each $\mathbb{P}_\alpha$ is λ^+-distributive. By a standard argument, the assumption that each $\mathbb{P}_\alpha$ is λ^+-distributive and λ-closed implies that for each α, there are densely many conditions p in $\mathbb{P}_\alpha$ satisfying that for all $\beta \in \operatorname{dom}(p)$, $p(\beta)$ is the canonical $\mathbb{P}_\beta$-name of a particular set in the ground model. The density of this set together with $2^\lambda = \lambda^+$ implies that each $\mathbb{P}_\alpha$ is λ^{++}-c.c.

Since $\mathbb{P}_{\lambda^{++}}$ is λ^{++}-c.c., any nice $\mathbb{P}_{\lambda^{++}}$-name for a subset of $\lambda^+ \cap \mathrm{cof}(\lambda)$ which carries a $<\lambda$-weak partial square is actually a nice $\mathbb{P}_\alpha$-name for a subset of $\lambda^+ \cap \mathrm{cof}(\lambda)$ which carries a $<\lambda$-weak partial square, for some $\alpha < \lambda^{++}$. Since $2^{(\lambda^+)} = \lambda^{++}$, we can use a suitable bookkeeping function to select the names $\dot{S}_\alpha$ in such a way that any nice $\mathbb{P}_{\lambda^{++}}$-name for a set carrying a $<\lambda$-weak partial square is equal to $\dot{S}_\alpha$ for some $\alpha < \lambda^{++}$. But $\dot{\mathbb{Q}}_\alpha$ adds a club set which is disjoint from $\dot{S}_\alpha$, forcing that $\dot{S}_\alpha$ is nonstationary. It follows that $\mathbb{P}_{\lambda^{++}}$ forces that there is no stationary subset of $\lambda^+ \cap \mathrm{cof}(\lambda)$ which carries a $<\lambda$-weak partial square sequence.

It remains to prove that each iterate is λ^+-distributive. Note that if $\mathbb{P}_\alpha$ is λ^+-distributive for all $\alpha < \lambda^{++}$, then $\mathbb{P}_{\lambda^{++}}$ is λ^+-distributive. For by the λ^{++}-c.c. of $\mathbb{P}_{\lambda^{++}}$, any nice name for a function $\dot{f} : \lambda^+ \to On$ is actually a $\mathbb{P}_\alpha$-name for some $\alpha < \lambda^{++}$, and therefore is forced to be in the ground model. So let $\alpha < \lambda^{++}$ be given, and assume that for all $\beta < \alpha$, $\mathbb{P}_\beta$ is λ^+-distributive. Let $\mathcal{D}$ be a family of less than λ^+ many dense open subsets of $\mathbb{P}_\alpha$, and let p be a condition in $\mathbb{P}_\alpha$. We will prove that there exists $q \leq p$ in $\bigcap \mathcal{D}$.

Fix an elementary embedding $j : V \to M$ in V such that the critical point of j is μ and $M^\mu \subseteq M$. The closure of M implies that there is a name $\dot{\mathbb{P}}_\alpha$ for $\mathbb{P}_\alpha$ in M. By Theorem 2.2, in M the Lévy collapse $j(\textsc{Coll}(\lambda, <\mu)) = \textsc{Coll}(\lambda, <j(\mu))$ factors as $\textsc{Coll}(\lambda, <\mu) * \dot{\mathbb{P}}_\alpha * \dot{\mathbb{S}}$, where $\dot{\mathbb{S}}$ is forced to be λ-closed.

Let $H_\alpha * I$ be a generic filter for $\mathbb{P}_\alpha * \dot{\mathbb{S}}$ over $V[G]$ such that $p \in H_\alpha$. Since $j[G] = G \subseteq G * H_\alpha * I$, in $V[G*H_\alpha*I]$ we can extend j to $j : V[G] \to M[G*H_\alpha*I]$ so that $j(G) = G * H_\alpha * I$. Note that $j(\mathcal{D}) = j[\mathcal{D}]$ since $\mathcal{D}$ has size less than μ. So by the elementarity of j, we will be done if we can prove that there exists $q \leq j(p)$ in $\bigcap j[\mathcal{D}]$. Since $p \in H_\alpha$ and H_α meets each dense open set in $\mathcal{D}$, it suffices to find q in $j(\mathbb{P}_\alpha)$ such that $q \leq j(u)$ for all $u \in H_\alpha$.

We construct the condition q in $M[G * H_\alpha * I]$ using the set of conditions $j[H_\alpha]$. Recall that $j(\dot{u}^G) = j(\dot{u})^{G*H_\alpha*I}$, so that $j \restriction \mathbb{P}_\alpha$ is in $M[G * H_\alpha * I]$. Also in this model, $j(\mu) = \lambda^+$ and $j(\mu) > (\lambda^{++})^{M[G]}$. The domain of q will be $j[\alpha]$; since $\alpha < (\lambda^{++})^{M[G]}$, $j[\alpha]$ is a subset of $j(\alpha)$ of cardinality at most λ, so the domain of q is of the appropriate form. Note that for all u in H_α, $\mathrm{dom}(j(u)) = j(\mathrm{dom}(u)) = j[\mathrm{dom}(u)]$, which is a subset of $j[\alpha]$. For all $\beta < \alpha$, let $q(j(\beta))$ be a $j(\mathbb{P}_\beta)$-name such that

$$j(\mathbb{P}_\beta) \Vdash q(j(\beta)) = \bigcup\{j(u)(j(\beta)) : u \in H_\alpha\} \cup \{\mu\}.$$

We prove that q is a condition below $j[H_\alpha]$.

For $\beta \leq \alpha$, let $H_\beta = \{u \restriction \beta : u \in H_\alpha\}$. Then H_β is a generic filter on $\mathbb{P}_\beta$ over $V[G]$. Using the fact that the critical point of j is μ and each $u(\beta)$ is a name for a bounded subset of μ, it is not hard to show that if $q \restriction j(\beta)$ is a condition, then $q \restriction j(\beta)$ forces $j(u)(j(\beta)) = u(\beta)$ for all $u \in H_\alpha$. Therefore $q \restriction j(\beta)$ forces that $q(j(\beta)) = \bigcup\{u(\beta) : u \in H_\alpha\} \cup \{\mu\}$.

Suppose for a contradiction that q is not a condition in $\mathbb{P}_\alpha$. Then there is a least ordinal $\beta < \alpha$ such that $q \restriction j(\beta)$ is a condition in $j(\mathbb{P}_\beta)$, but $q \restriction j(\beta)$ does not force that $q(j(\beta))$ is in $j(\dot{\mathbb{Q}}_\beta)$. Fix a condition $t \leq q \restriction j(\beta)$ which forces that $q(j(\beta))$ is not in $j(\dot{\mathbb{Q}}_\beta)$. By definition of q, $q \restriction j(\beta) \leq j(v)$ for all $v \in H_\beta$. Let h_β be a generic filter on $j(\mathbb{P}_\beta)$ over $V[G * H_\alpha * I]$ such that $t \in h_\beta$. Then $j[H_\beta] \subseteq h_\beta$, so in $V[G * H_\alpha * I * h_\beta]$ we can extend j to $j : V[G * H_\beta] \to M[G * H_\alpha * I * h_\beta]$.

For each u in H_α, $u(\beta)$ is a closed bounded subset of λ^+ disjoint from S_β. So $j(u)(j(\beta)) = j(u(\beta)) = u(\beta)$. Also the set $\{u(\beta) : u \in H_\alpha\}$ is a generic filter on $\mathbb{Q}_\beta$ over $V[G * H_\beta]$. Therefore $q(j(\beta)) \cap \mu$ is a club subset of μ which is disjoint from $S_\beta = j(S_\beta) \cap \mu$. Since $q(j(\beta))$ is not a condition in $j(\mathbb{Q}_\beta)$, it follows that μ is in $j(S_\beta)$.

Fix a $<\lambda$-weak partial square sequence $\langle c_\xi : \xi \in S_\beta \rangle$ in $V[G * H_\beta]$. Define $\vec{\mathcal{C}}$ by letting $\vec{\mathcal{C}}(\gamma) = \{c_\gamma\}$ if $\gamma \in S_\beta$, and $\vec{\mathcal{C}}(\gamma) = \{c_\xi \cap \gamma : \xi \in S_\beta, \gamma \in \lim(c_\xi)\}$ otherwise. Then $\vec{\mathcal{C}}$ satisfies the assumptions of Proposition 1.1 and is in the model $M[G * H_\beta]$.

Since μ is in $j(S_\beta)$, let c_μ be the club subset of μ which is indexed at μ on the sequence $j(\langle c_\xi : \xi \in S_\beta \rangle)$. Since $j(\vec{\mathcal{C}}) \restriction \mu = \vec{\mathcal{C}}$, the club c_μ is a thread of the sequence $\vec{\mathcal{C}}$ in $M[G * H_\alpha * I * h_\beta]$. But $M[G * H_\alpha * I * h_\beta]$ is a generic extension of $M[G * H_\beta]$ by the λ-closed forcing $(\mathbb{P}_\alpha/\mathbb{P}_\beta) * \dot{\mathbb{S}} * j(\mathbb{P}_\beta)$. By Proposition 1.1, c_μ is in $M[G * H_\beta]$. By the induction hypothesis, $\mathbb{P}_\beta$ is λ^+-distributive, and therefore $\mu = \lambda^+$ in $M[G * H_\beta]$. But c_μ has order type λ and is cofinal in μ, which is a contradiction. $\square$

Note that in the model just produced, $\lambda^{<\lambda} = \lambda$, which implies weak square $\square_{\lambda,\lambda}$. Thus we see that $\square_{\lambda,\lambda}$ does not imply $\square^p_{\lambda,<\lambda}$.

Using the methods of [4], it is possible to reduce the large cardinal assumption of measurability to greatly Mahlo in the last theorem.

3. Threads and Adding a Weak Square Sequence

We now turn towards the goal of separating the square principles $\square_{\lambda,\kappa}$ and $\square^p_{\lambda,<\kappa}$, where λ is a regular uncountable cardinal and $\kappa < \lambda$ is a nonzero cardinal. We need a version of Proposition 1.1 which applies to the forcing poset for adding a weak square sequence and then threading it; this version is Proposition 3.2 below. One complication is that we would like to force with some additional forcing poset prior to threading the weak square sequence. The proof of Proposition 3.2 is a variation of arguments given in Section 7 of [1], which in turn rely on ideas of Jensen [2] which were used to separate weak square principles.

Let λ be a regular uncountable cardinal and let $\kappa \leq \lambda$ be a nonzero cardinal. We review the forcing posets which add a $\square_{\lambda,\kappa}$-sequence and then thread it.

Let $\mathbb{P}$ be the forcing poset whose conditions are sequences $\langle \mathcal{C}_\alpha : \alpha \leq \beta,\ \alpha \text{ limit} \rangle$, where $\beta < \lambda^+$ is a limit ordinal, satisfying that for all limit ordinals $\alpha \leq \beta$:

(1) $\mathcal{C}_\alpha$ is a nonempty family of club subsets of α;
(2) for all $c \in \mathcal{C}_\alpha$, $\mathrm{ot}(c) \leq \lambda$;
(3) $|\mathcal{C}_\alpha| \leq \kappa$;
(4) for all $c \in \mathcal{C}_\alpha$ and $\gamma \in \lim(c)$, $c \cap \gamma \in \mathcal{C}_\gamma$.

For conditions p and q, let $q \leq p$ if $\mathrm{dom}(p) \leq \mathrm{dom}(q)$ and $q \restriction \mathrm{dom}(p) = p$. The forcing poset $\mathbb{P}$ is $(\lambda + 1)$-strategically closed. If $2^\lambda = \lambda^+$, then $\mathbb{P}$ has size λ^+.

Let G be a generic filter on $\mathbb{P}$ over V. In $V[G]$ let $\vec{\mathcal{C}} = \bigcup G$, which is a $\square_{\lambda,\kappa}$-sequence. Rather than forcing with the threading forcing for $\vec{\mathcal{C}}$ over $V[G]$, we force over a larger generic extension. So let $\mathbb{Q}$ be a λ-closed forcing poset in $V[G]$, and let H be a generic filter on $\mathbb{Q}$ over $V[G]$. Note that in $V[G * H]$ the ordinal $(\lambda^+)^V$ may or may not still be a cardinal, but since $\mathbb{Q}$ is λ-closed its cofinality is at least λ.

We define in $V[G*H]$ a forcing poset $\mathbb{R}$ which adds a thread of $\vec{\mathcal{C}}$. The conditions of $\mathbb{R}$ are the closed bounded subsets c of $(\lambda^+)^V$ with order type less than λ such that for all $\gamma \in \lim(c)$, $c \cap \gamma \in \mathcal{C}_\gamma$. For conditions c and d, let $d \leq c$ if d is an end-extension of c. (Note that since $\mathbb{Q}$ is λ-closed, $\mathbb{R}$ is actually in $V[G]$, so the iteration $\mathbb{Q} * \mathbb{R}$ is forcing equivalent to the product $\mathbb{Q} \times \mathbb{R}$.)

This completes the description of $\mathbb{P} * \dot{\mathbb{Q}} * \dot{\mathbb{R}}$. Next we define a λ-closed dense subset of $\mathbb{P} * \dot{\mathbb{Q}} * \dot{\mathbb{R}}$. Since $\mathbb{P}$ is $(\lambda+1)$-strategically closed and $\mathbb{Q}$ is λ-closed, every condition in $\mathbb{R}$ is in V. A condition $p*q*r$ in $\mathbb{P}*\dot{\mathbb{Q}}*\dot{\mathbb{R}}$ is said to be *flat* if p decides r as some set c such that $\max(\mathrm{dom}(p)) = \max(c)$.

The set of flat conditions is dense. Indeed, given a condition $p*q*r$, extend $p*q$ to $s*t$ which decides r as c. Further extending s if necessary, we may assume that $\max(\mathrm{dom}(s)) > \max(c)$. Now let u be the canonical name for $c \cup \{\max(\mathrm{dom}(s))\}$. Then $s*t*u \leq p*q*r$ and $s*t*u$ is flat.

If $p*q*r$ is a flat condition, we will often abuse notation and identify r with the set from V which it denotes.

Let us see that the dense set of flat conditions is λ-closed. So let $\delta < \lambda$ be a limit ordinal and let $\langle p_i * q_i * r_i : i < \delta \rangle$ be a descending sequence of flat conditions. Let $\xi_i = \max(\mathrm{dom}(p_i)) = \max(r_i)$ for all $i < \delta$. Let $\xi = \sup_i \xi_i$. Since $\dot{\mathbb{Q}}$ is forced to be λ-closed, let q be a $\mathbb{P}$-name for a lower bound of $\langle q_i : i < \delta\rangle$. Let $r = (\bigcup_i r_i) \cup \{\xi\}$. Let $p = (\bigcup_i p_i) \cup \{\langle \xi, \{r\}\rangle\}$. Then $p*q*r$ is a flat condition which is below $p_i*q_i*r_i$ for all $i < \delta$.

The next lemma, which is used in the proof of Proposition 3.2, describes a more general context in which we will construct lower bounds. The proof is straightforward.

Lemma 3.1. *Let $\delta < \lambda$ be a limit ordinal. Suppose that $\langle p_i * q_i : i < \delta\rangle$ is a descending sequence of conditions in $\mathbb{P} * \dot{\mathbb{Q}}$, $\langle r_\pi : \pi \in {}^{<\delta}2\rangle$ is a sequence of names for conditions in $\mathbb{R}$, and $\langle \xi_i : i < \delta\rangle$ is a sequence of ordinals satisfying:*

1. *for all π in ${}^{<\delta}2$, $p_{\mathrm{dom}(\pi)} * q_{\mathrm{dom}(\pi)} * r_\pi$ is flat;*
2. *for all π in ${}^{<\delta}2$, $\max(\mathrm{dom}(p_{\mathrm{dom}(\pi)})) = \max(r_\pi) = \xi_{\mathrm{dom}(\pi)}$;*
3. *for $\pi \trianglelefteq \tau$ in ${}^{<\delta}2$, $p_{\mathrm{dom}(\tau)} * q_{\mathrm{dom}(\tau)} * r_\tau \leq p_{\mathrm{dom}(\pi)} * q_{\mathrm{dom}(\pi)} * r_\pi$.*

Let $\xi = \sup_i \xi_i$. Let A be a nonempty subset of ${}^{\delta}2$ of size less than or equal to κ. For each π in A, let

$$r_\pi = \bigcup\{r_{\pi\restriction i} : i < \delta\} \cup \{\xi\}.$$

Define

$$p_\delta = \bigcup\{p_i : i < \delta\} \cup \{\langle \xi, \{r_\pi : \pi \in A\}\rangle\}.$$

Let q_δ be any $\mathbb{P}$-name for a lower bound of $\langle q_i : i < \delta\rangle$. Then for all $\pi \in A$, $p_\delta * q_\delta * r_\pi$ is a flat condition and is a lower bound of the sequence $\langle p_i * q_i * r_{\pi\restriction i} : i < \delta\rangle$.

Proposition 3.2. *Let λ be a regular uncountable cardinal and let $\kappa < \lambda$ be a nonzero cardinal. Suppose $\mathbb{P} * \dot{\mathbb{Q}} * \dot{\mathbb{R}}$ satisfies:*

1. *$\mathbb{P}$ is the forcing poset for adding a $\square_{\lambda,\kappa}$-sequence;*
2. *$\mathbb{P}$ forces that $\dot{\mathbb{Q}}$ is λ-closed;*
3. *$\mathbb{P} * \dot{\mathbb{Q}}$ forces that $\dot{\mathbb{R}}$ is the forcing poset for adding a thread of the $\square_{\lambda,\kappa}$-sequence which was added by $\mathbb{P}$.*

In $V^{\mathbb{P}*\dot{\mathbb{Q}}}$ assume that $\vec{\mathcal{D}} = \langle \mathcal{D}_\alpha : \alpha < (\lambda^+)^V,\ \alpha \text{ limit } \rangle$ is a sequence such that for all α, $\mathcal{D}_\alpha$ is a family of club subsets of α of size less than κ. Then any thread of $\vec{\mathcal{D}}$ in $V^{\mathbb{P}*\dot{\mathbb{Q}}*\dot{\mathbb{R}}}$ lies in $V^{\mathbb{P}*\dot{\mathbb{Q}}}$.

Proof. Suppose for a contradiction that $p * q * r$ is a condition in $\mathbb{P} * \dot{\mathbb{Q}} * \dot{\mathbb{R}}$ which forces that $\dot{d}$ is a thread of $\vec{\mathcal{D}}$ which does not lie in $V^{\mathbb{P}*\dot{\mathbb{Q}}}$. Note that since $\dot{d}$ threads $\vec{\mathcal{D}}$, every proper initial segment of $\dot{d}$ is in $V^{\mathbb{P}*\dot{\mathbb{Q}}}$. It follows that for all $\beta < \lambda^+$, $p * q * r$ forces that $\dot{d} \setminus \beta$ is not in $V^{\mathbb{P}*\dot{\mathbb{Q}}}$.

We make the following claim: for all $p' * q' * r' \leq p * q * r$ and $\xi < \lambda^+$, there are flat conditions $t * u * v_0$ and $t * u * v_1$ below $p' * q' * r'$ and $\gamma > \xi$ such that $t*u*v_0 \Vdash \gamma \notin \dot{d}$ and $t*u*v_1 \Vdash \gamma \in \dot{d}$. Suppose not. Then there is $p'*q'*r' \leq p*q*r$ and $\xi < \lambda^+$ such that for any $\gamma > \xi$, any two flat conditions of the form $t * u * v_0$ and $t * u * v_1$ below $p' * q' * r'$ which decide the statement "$\gamma \in \dot{d}$" decide it in the same way. We will prove $p' * q'$ forces that r' decides the set $\dot{d} \setminus (\xi + 1)$. It follows that $p' * q' * r'$ forces that $\dot{d} \setminus (\xi + 1)$ is in $V^{\mathbb{P}*\dot{\mathbb{Q}}}$, which is a contradiction.

Let $G * H$ be a generic filter on $\mathbb{P} * \dot{\mathbb{Q}}$ which contains $p' * q'$. If r' does not decide $\dot{d} \setminus (\xi+1)$, then there are $v_0, v_1 \leq r'$ and $\gamma > \xi$ such that $v_0 \Vdash \gamma \notin \dot{d}$ and $v_1 \Vdash \gamma \in \dot{d}$. Now find $t*u \leq p'*q'$ in $G*H$ which forces this fact about v_0 and v_1, decides v_0, v_1, and γ, and satisfies that $\mathrm{dom}(t) > \max(v_0), \max(v_1)$. Let $w_0 = v_0 \cup \{\max(\mathrm{dom}(t))\}$ and $w_1 = v_1 \cup \{\max(\mathrm{dom}(t))\}$. Then $t*u*w_0$ and $t*u*w_1$ are flat conditions below $p' * q' * r'$ and decide "$\gamma \in \dot{d}$" in different ways, which contradicts our assumptions. This completes the proof of the claim.

Now $\mathbb{P}$ is $(\lambda + 1)$-strategically closed and forces that $\dot{\mathbb{Q}}$ is λ-closed, so $\mathbb{P} * \dot{\mathbb{Q}}$ is λ-strategically closed, and therefore does not add any new sets of ordinals of size less than λ. In particular, in $V^{\mathbb{P}*\dot{\mathbb{Q}}}$ the ordinal $(\lambda^+)^V$ has cofinality at least λ. Fix a winning strategy σ for Player II which witnesses that $\mathbb{P}*\dot{\mathbb{Q}}$ is λ-strategically closed.

Let ν be the least infinite cardinal such that $2^\nu \geq \kappa$. Note that if κ is finite, then $\nu = \omega$. If κ is infinite, then ν is the least cardinal such that $2^\nu \geq \kappa$. In either case, $\nu < \lambda$ and for all $\xi < \nu$, $2^\xi < \lambda$.

Consider the tree ${}^{\leq\nu}2$ of all binary sequences of length less than or equal to ν, ordered by letting $\pi \trianglelefteq \tau$ if π is an initial segment of τ. We define by induction sequences $\langle p_i * q_i : i \leq \nu \rangle$, $\langle r_\pi : \pi \in {}^{\leq\nu}2 \rangle$, $\langle \xi_i : i \leq \nu \rangle$, $\langle \gamma_\pi : \pi \in {}^{<\nu}2 \rangle$, and $\langle \zeta_i : i < \nu \rangle$ satisfying:

(1) $\langle p_i * q_i : i \leq \nu \rangle$ is a descending sequence of conditions in $\mathbb{P} * \dot{\mathbb{Q}}$;
(2) for all π in ${}^{\leq\nu}2$, $p_{\mathrm{dom}(\pi)} * q_{\mathrm{dom}(\pi)} * r_\pi$ is flat and $\max(\mathrm{dom}(p_{\mathrm{dom}(\pi)})) = \max(r_\pi) = \xi_{\mathrm{dom}(\pi)}$.
(3) if $\pi \trianglelefteq \tau$ in ${}^{\leq\nu}2$, then $p_{\mathrm{dom}(\tau)} * q_{\mathrm{dom}(\tau)} * r_\tau \leq p_{\mathrm{dom}(\pi)} * q_{\mathrm{dom}(\pi)} * r_\pi$;
(4) $\langle \zeta_i : i < \nu \rangle$ is an increasing sequence of ordinals in λ^+;
(5) for all π in ${}^{<\nu}2$, $\zeta_{\mathrm{dom}(\pi)} < \gamma_\pi < \zeta_{\mathrm{dom}(\pi)+1}$;
(6) for all π in ${}^{<\nu}2$, $p_{\mathrm{dom}(\pi)+1} * q_{\mathrm{dom}(\pi)+1} * r_{\pi^\frown 0} \Vdash \gamma_\pi \notin \dot{d}$ and $p_{\mathrm{dom}(\pi)+1} * q_{\mathrm{dom}(\pi)+1} * r_{\pi^\frown 1} \Vdash \gamma_\pi \in \dot{d}$.

Let $p_0 * q_0 * r_{\langle\,\rangle}$ be an extension of $p*q*r$ which is flat. Let $\xi_0 = \max(\mathrm{dom}(p_0)) = \max(r_{\langle\,\rangle})$. Let $\zeta_0 = 0$.

Now let $\beta < \nu$ be given, and assume that we have defined all the objects as required for $i \leq \beta$ and π in ${}^{\leq\beta}2$. Enumerate ${}^{\beta}2$ as $\langle \pi_i : i < 2^\beta \rangle$. Recall that $2^\beta < \lambda$.

Using the winning strategy σ, we define by induction a descending sequence of conditions $\langle p_i^\beta * q_i^\beta : i \leq 2^\beta \rangle$. We also define sequences $\langle \gamma_\pi : \pi \in {}^\beta 2\rangle$ and $\langle r'_\pi : \pi \in {}^{\beta+1}2\rangle$. Let $p_0^\beta * q_0^\beta = p_\beta * q_\beta$.

Assume that $\xi < 2^\beta$ and $p_i^\beta * q_i^\beta$ is defined for all $i < \xi$. Let $s * t = \sigma(\langle p_i^\beta * q_i^\beta : i < \xi\rangle)$. Fix $p_\xi^\beta * q_\xi^\beta$, $r'_{\pi_\xi {}^\frown 0}$, $r'_{\pi_\xi {}^\frown 1}$, and $\gamma_{\pi_\xi} > \zeta_\beta$ such that $p_\xi^\beta * q_\xi^\beta * r'_{\pi_\xi {}^\frown 0}$ and $p_\xi^\beta * q_\xi^\beta * r'_{\pi_\xi {}^\frown 1}$ are flat conditions below $s * t * r_{\pi_\xi}$, $p_\xi^\beta * q_\xi^\beta * r'_{\pi_\xi {}^\frown 0} \Vdash \gamma_{\pi_\xi} \notin \dot{d}$, and $p_\xi^\beta * q_\xi^\beta * r'_{\pi_\xi {}^\frown 1} \Vdash \gamma_{\pi_\xi} \in \dot{d}$.

This completes the definition of the sequence $\langle p_i^\beta * q_i^\beta : i < 2^\beta\rangle$. Now let $y * z = \sigma(\langle p_i^\beta * q_i^\beta : i < 2^\beta\rangle)$. Let $\xi_{\beta+1}$ be some ordinal larger than $\max(\mathrm{dom}(y))$; then $\xi_{\beta+1}$ is larger than $\max(r'_{\pi_i {}^\frown k})$ for all $i < 2^\beta$ and $k = 0, 1$. Extend y to $p_{\beta+1}$ such that $\max(\mathrm{dom}(p_{\beta+1})) = \xi_{\beta+1}$. Let $q_{\beta+1} = z$. For $i < 2^\beta$ and $k = 0, 1$, let $r_{\pi_i {}^\frown k} = r'_{\pi_i {}^\frown k} \cup \{\xi_{\beta+1}\}$. Let $\zeta_{\beta+1}$ be some ordinal in λ^+ larger than γ_{π_i} for all $i < 2^\beta$. This handles the successor stage.

If $\nu = \omega$, then there are no limit stages less than ν to take care of. So assume $\nu > \omega$. Then $\kappa > \omega$ and ν is the least cardinal such that $2^\nu \geq \kappa$.

Let $\delta < \nu$ be a limit ordinal, and assume that the objects are defined as required for all $i < \delta$ and all $\pi \in {}^{<\delta}2$. Then in particular, $\langle p_i * q_i : i < \delta\rangle$, $\langle r_\pi : \pi \in {}^{<\delta}2\rangle$, and $\langle \xi_i : i < \delta\rangle$ satisfy the assumptions of Lemma 3.1.

Let $\xi_\delta = \sup_{i<\delta} \xi_i$. We apply Lemma 3.1 letting $A = {}^\delta 2$, which has size less than κ. For π in ${}^\delta 2$, let

$$r_\pi = \bigcup \{r_{\pi \restriction i} : i < \delta\} \cup \{\xi_\delta\}.$$

Define

$$p_\delta = \bigcup \{p_i : i < \delta\} \cup \{\langle \xi_\delta, \{r_\pi : \pi \in {}^\delta 2\}\rangle\}.$$

Let q_δ be any $\mathbb{P}$-name for a lower bound of $\langle q_i : i < \delta\rangle$. Then by Lemma 3.1, for all $\pi \in {}^\delta 2$, $p_\delta * q_\delta * r_\pi$ is flat and is a lower bound of the sequence $\langle p_i * q_i * r_{\pi\restriction i} : i < \delta\rangle$. Also $\xi_\delta = \max(\mathrm{dom}(p_\delta)) = \max(r_\pi)$ for all $\pi \in {}^\delta 2$. Let $\zeta_\delta = \sup_{i<\delta} \zeta_i$. This completes the limit stage of the construction.

Finally, assume all the objects are defined as required for all $i < \nu$ and $\pi \in {}^{<\nu}2$. Since $2^\nu \geq \kappa$, we may fix a set $A \subseteq {}^\nu 2$ of size κ. Moreover, assume that every π in A satisfies that the set $\{i < \nu : \pi(i) = 1\}$ is cofinal in ν; this is possible since there are 2^ν many such functions in ${}^\nu 2$. Apply Lemma 3.1 again to define $p_\nu * q_\nu$, ξ_ν, and r_π for all $\pi \in A$. Let $\zeta = \sup_{i<\nu} \zeta_i$.

Let $G * H$ be a generic filter on $\mathbb{P} * \dot{\mathbb{Q}}$ which contains the condition $p_\nu * q_\nu$. Working in $V[G * H]$, for each $\pi \in A$ extend r_π to s_π which decides $\dot{d} \cap \zeta$ as a set c_π. For each π in A, by (5), (6), and the fact that there are cofinally many $i < \nu$ with $\pi(i) = 1$, s_π forces that ζ is a limit point of $\dot{d}$ and hence that $\dot{d} \cap \zeta \in \mathcal{D}_\zeta$. It follows that c_π is in $\mathcal{D}_\zeta$ for all π in A. But for distinct π and τ in A, $c_\pi \neq c_\tau$ by (6). Hence $\mathcal{D}_\zeta$ has size at least κ, which contradicts our assumptions about $\vec{\mathcal{D}}$. $\square$

4. Separating Weak Partial Square Principles

In Section 2 we separated the principles $\square_{\lambda,\lambda}$ and $\square^p_{\lambda,<\lambda}$. In this section we show how to separate the principles $\square_{\lambda,\kappa}$ and $\square^p_{\lambda,<\kappa}$, where $\kappa < \lambda$.

Theorem 4.1. *Let λ be a regular uncountable cardinal, $\kappa < \lambda$ a nonzero cardinal, and $\mu > \lambda$ a measurable cardinal. Assume $2^\mu = \mu^+$. Then there exists a forcing* *poset which collapses μ to become λ^+, preserves cardinals and cofinalities less than or equal to λ, and forces $\square_{\lambda,\kappa}$ and $\neg\square^p_{\lambda,<\kappa}$.*

Proof. The forcing poset we define will be of the form $\textsc{Coll}(\lambda,<\mu)*\mathbb{P}*\mathbb{P}_{\mu^+}$, where $\mathbb{P}$ is the forcing poset described in the previous section for adding a $\square_{\lambda,\kappa}$-sequence, and $\mathbb{P}_{\mu^+}$ is a $\leq\lambda$-support forcing iteration of length μ^+ which kills the stationarity of all subsets of $\mu\cap\mathrm{cof}(\lambda)$ which carry a $<\kappa$-weak partial square sequence.

Let G be a generic filter on $\textsc{Coll}(\lambda,<\mu)$ over V. Then in $V[G]$, cardinals and cofinalities less than or equal to λ are preserved, $\mu=\lambda^+$, $2^\lambda=\mu$, and $2^\mu=\mu^+$. In $V[G]$, the forcing poset $\mathbb{P}$ for adding a $\square_{\lambda,\kappa}$-sequence is $(\lambda+1)$-strategically closed and has size λ^+. Let H be a generic filter on $\mathbb{P}$ over $V[G]$. Then in $V[G*H]$, $\mu=\lambda^+$, $2^\lambda=\lambda^+$, and $2^\mu=\mu^+$.

We would like to iterate forcing in $V[G*H]$ to kill the stationarity of subsets of $\lambda^+\cap\mathrm{cof}(\lambda)$ which carry a $<\kappa$-weak partial square. Define by induction a $\leq\lambda$-support forcing iteration

$$\langle\mathbb{P}_i,\dot{\mathbb{Q}}_j : i\leq\lambda^{++},\ j<\lambda^{++}\rangle.$$

We will assume that each $\mathbb{P}_i$ is λ^+-distributive; this assumption will be justified after the definition is complete. Each factor forcing $\dot{\mathbb{Q}}_j$ is forced to be λ-closed, which implies that each iteration $\mathbb{P}_i$ is λ-closed.

Assume that $\mathbb{P}_\alpha$ is defined as required. Since $\mathbb{P}_\alpha$ is λ^+-distributive, it forces that $\mu=\lambda^+$. We consider a particular nice name $\dot{S}_\alpha$ for a subset of $\lambda^+\cap\mathrm{cof}(\lambda)$ which carries a $<\kappa$-weak partial square. Let $\dot{\mathbb{Q}}_\alpha$ be a $\mathbb{P}_\alpha$-name for the forcing poset which adds a club subset of λ^+ using closed bounded subsets of λ^+ which are disjoint from $\dot{S}_\alpha$, ordered by end-extension. Let $\mathbb{P}_{\alpha+1}=\mathbb{P}_\alpha*\dot{\mathbb{Q}}_\alpha$.

This completes the construction, assuming that each $\mathbb{P}_\alpha$ is λ^+-distributive. As in the proof of Theorem 2.3, each $\mathbb{P}_\alpha$ is λ^{++}-c.c. By using a suitable bookkeeping function, we can arrange that every $\mathbb{P}_{\lambda^{++}}$-name for a subset of $\lambda^+\cap\mathrm{cof}(\lambda)$ which carries a $<\kappa$-weak partial square is equal to $\dot{S}_\alpha$ for some $\alpha<\lambda^{++}$. It follows that $\mathbb{P}_{\lambda^{++}}$ forces $\square_{\lambda,\kappa}$ and $\neg\square^p_{\lambda,<\kappa}$.

It remains to prove that each iteration is λ^+-distributive. As in the proof of Theorem 2.3, it suffices to show that $\mathbb{P}_\alpha$ is λ^+-distributive for all $\alpha<\lambda^{++}$. So let $\alpha<\lambda^{++}$ be given, and assume that for all $\beta<\alpha$, $\mathbb{P}_\beta$ is λ^+-distributive. We prove that $\mathbb{P}_\alpha$ is λ^+-distributive. Let $\mathcal{D}$ be a family of less than λ^+ many dense open subsets of $\mathbb{P}_\alpha$, and let p be a condition in $\mathbb{P}_\alpha$. We will prove that there exists $q\leq p$ in $\bigcap\mathcal{D}$.

In $V[G]$ let $\dot{\mathbb{R}}$ be a $\mathbb{P}*\dot{\mathbb{P}}_\alpha$-name for the forcing poset which threads the $\square_{\lambda,\kappa}$-sequence which was introduced by $\mathbb{P}$. By the material in Section 3, $\mathbb{P}*\dot{\mathbb{P}}_\alpha*\dot{\mathbb{R}}$ has a λ-closed dense subset.

Fix an elementary embedding $j:V\to M$ in V such that the critical point of j is μ and $M^\mu\subseteq M$. The closure of M implies that there is a name $\dot{\mathbb{P}}*\dot{\mathbb{P}}_\alpha*\dot{\mathbb{R}}$ for $\mathbb{P}*\dot{\mathbb{P}}_\alpha*\dot{\mathbb{R}}$ in M. So by Theorem 2.2, in M the Lévy collapse $j(\textsc{Coll}(\lambda,<\mu))=\textsc{Coll}(\lambda,<j(\mu))$ factors as $\textsc{Coll}(\lambda,<\mu)*\dot{\mathbb{P}}*\dot{\mathbb{P}}_\alpha*\dot{\mathbb{R}}*\dot{\mathbb{S}}$, where $\dot{\mathbb{S}}$ is forced to be λ-closed.

Let $I_\alpha*J*K$ be a generic filter on $\mathbb{P}_\alpha*\dot{\mathbb{R}}*\dot{\mathbb{S}}$ over $V[G*H]$ such that $p\in I_\alpha$. Since $j[G]=G\subseteq G*H*I_\alpha*J*K$, in $V[G*H*I_\alpha*J*K]$ we can extend j to $j:V[G]\to M[G*H*I_\alpha*J*K]$.

We would like to extend j to have domain $V[G*H]$. Let $\vec{\mathcal{C}}=\bigcup H$, which is a $\square_{\lambda,\kappa}$-sequence in $M[G*H]$. Let $c=\bigcup J$, which is a club subset of μ of order type

λ which threads the sequence $\vec{\mathcal{C}}$. Then the sequence

$$t = \vec{\mathcal{C}} \cup \{\langle \mu, \{c\}\rangle\}$$

is a condition in $j(\mathbb{P})$. Moreover, for all $s \in H$, $t \leq s = j(s)$. Fix a generic filter h on $j(\mathbb{P})$ over $V[G*H*I_\alpha*J*K]$ which contains the condition t. Then $j[H] \subseteq h$. So in $V[G*H*I_\alpha*J*K*h]$ we can extend j to $j : V[G*H] \to M[G*H*I_\alpha*J*K*h]$.

Note that $j(\mathcal{D}) = j[\mathcal{D}]$ since $\mathcal{D}$ has size less than μ. By the elementarity of j, we will be done if we can prove there exists $q \leq j(p)$ in $\bigcap j[\mathcal{D}]$. Since $p \in I_\alpha$ and I_α meets each dense open set in $\mathcal{D}$, it suffices to find q in $j(\mathbb{P}_\alpha)$ such that $q \leq j(u)$ for all $u \in I_\alpha$.

We construct the condition q in $M[G * H * I_\alpha * J * K * h]$. As in the proof of Theorem 2.3, we let the domain of q equal $j[\alpha]$, and for all $\beta < \alpha$, let $q(j(\beta))$ be a $j(\mathbb{P}_\beta)$-name such that

$$j(\mathbb{P}_\beta) \Vdash q(j(\beta)) = \bigcup\{j(u)(j(\beta)) : u \in I_\alpha\} \cup \{\mu\}.$$

We prove that q is a condition below $j[I_\alpha]$.

Suppose for a contradiction that q is not a condition in $j(\mathbb{P}_\alpha)$. Then there is a least ordinal $\beta < \alpha$ such that $q \restriction j(\beta)$ is a condition in $j(\mathbb{P}_\beta)$, but $q \restriction j(\beta)$ does not force that $q(j(\beta))$ is in $j(\dot{\mathbb{Q}}_\beta)$. Fix a condition $t \leq q \restriction j(\beta)$ which forces that $q(j(\beta))$ is not in $j(\dot{\mathbb{Q}}_\beta)$. Let $I_\beta = \{u \restriction \beta : u \in I_\alpha\}$. Then I_β is a generic filter on $\mathbb{P}_\beta$ over $V[G*H]$. Also $q \restriction j(\beta) \leq j(v)$ for all $v \in I_\beta$. Let i_β be a generic filter on $j(\mathbb{P}_\beta)$ over $V[G*H*I_\alpha*J*K*h]$ such that $t \in i_\beta$. Then $j[I_\beta] \subseteq i_\beta$, so in $V[G*H*I_\alpha*J*K*h*i_\beta]$ we can extend j to $j : V[G * H * I_\beta] \to M[G * H * I_\alpha * J * K * h * i_\beta]$.

As in the proof of Theorem 2.3, $q(j(\beta)) \cap \mu$ is a club subset of μ which is disjoint from $S_\beta = j(S_\beta) \cap \mu$. Since $q(j(\beta))$ is not a condition in $j(\mathbb{Q}_\beta)$, it follows that μ is in $j(S_\beta)$. Fix a $<\kappa$-weak partial square sequence $\langle d_\xi : \xi \in S_\beta\rangle$ in $V[G*H*I_\beta]$. Define $\vec{\mathcal{D}}$ by letting $\vec{\mathcal{D}}(\gamma) = \{d_\gamma\}$ if $\gamma \in S_\beta$, and $\vec{\mathcal{D}}(\gamma) = \{c_\xi \cap \gamma : \xi \in S_\beta, \gamma \in \lim(c_\xi)\}$ otherwise. Then $\vec{\mathcal{D}}$ satisfies the assumptions of Proposition 3.2 and is in the model $M[G * H * I_\beta]$.

Since μ is in $j(S_\beta)$, let d_μ be the club indexed at μ on the sequence $j(\langle d_\xi : \xi \in S_\beta\rangle)$. Since $j(\vec{\mathcal{D}}) \restriction \mu = \vec{\mathcal{D}}$, clearly d_μ is a thread of the sequence $\vec{\mathcal{D}}$ in $M[G * H * I_\alpha * J * K * h * i_\beta]$. Also d_μ has order type λ.

Since $j(\mathbb{P})$ is $(\lambda + 1)$-strategically closed and $j(\mathbb{P}_\beta)$ is $j(\lambda^+)$-distributive by the induction hypothesis, d_μ is in $M[G * H * I_\alpha * J * K]$. But $M[G * H * I_\alpha * J * K]$ is a generic extension of $M[G * H * I_\alpha * J]$ by the λ-closed forcing $\mathbb{S}$. So by Proposition 1.1, d_μ is in $M[G * H * I_\alpha * J]$. Now we apply Proposition 3.2 to $\mathbb{P} * \mathbb{P}_\alpha * \mathbb{R}$, $\vec{\mathcal{D}}$, and d_μ to conclude that d_μ is in $M[G * H * I_\alpha]$.

Now $\mathbb{P}_\alpha = \mathbb{P}_\beta * (\mathbb{P}_\alpha/\mathbb{P}_\beta)$, where $\mathbb{P}_\alpha/\mathbb{P}_\beta$ is λ-closed. Factor $I_\alpha = I_\beta * (I_\alpha/I_\beta)$, so that $M[G * H * I_\alpha] = M[G * H * I_\beta * (I_\alpha/I_\beta)]$. Since $\vec{\mathcal{D}}$ is in $M[G * H * I_\beta]$, another application of Proposition 1.1 gives that d_μ is in $M[G * H * I_\beta]$. But d_μ is a club subset of μ of order type λ, so μ is not a cardinal in $M[G * H * I_\beta]$. But by the induction hypothesis, $\mathbb{P}_\beta$ is λ^+-distributive, and therefore $\mu = \lambda^+$ in $M[G * H * I_\beta]$. This contradiction shows that q is in $j(\mathbb{P}_\alpha)$, and q is below $j[I_\alpha]$. $\square$

Again by the methods of [4], the assumption of measurability can be reduced to greatly Mahlo.

Comment: In [6] Mitchell defined a variation of Shelah's approachability ideal $I[\aleph_2]$, and constructed a model in which this ideal is just the nonstationary ideal when restricted to ordinals of cofinality $\aleph_1$. Although phrased in a slightly different language, this property of the ideal is equivalent to the failure of the weak partial square principle $\square^p_{\aleph_1,\aleph_1}$.

References

[1] J. Cummings, M. Foreman, and M. Magidor. Squares, scales, and stationary reflection. *J. Math. Log.*, 1(1):35–98, 2001.

[2] R. Jensen. Some remarks on $\square$ below zero-pistol. Circulated notes.

[3] R. Jensen. The fine structure of the constructible hierarchy. *Ann. Math. Logic*, 4:229–308, 1972.

[4] J. Krueger and E. Schimmerling. An equiconsistency result on partial squares. *J. Math. Log.*, 11(1):1–31, 2011.

[5] M. Magidor. Reflecting stationary sets. *J. Symbolic Logic*, 47(4):755–771, 1982.

[6] W. Mitchell. A weak variation of Shelah's $I[\omega_2]$. *J. Symbolic Logic*, 69(1):94–100, 2004.

[7] E. Schimmerling. Combinatorial principles in the core model for one woodin cardinal. *Ann. Pure Appl. Logic*, 74(2):153–201, 1995.

[8] S. Shelah. Reflecting stationary sets and successors of singular cardinals. *Arch. Math. Logic*, 31:25–53, 1991.

Department of Mathematics, University of North Texas
E-mail address: jkrueger@unt.edu

Department of Mathematical Sciences, Carnegie Mellon University
E-mail address: eschimme@andrew.cmu.edu