INDESTRUCTIBLE GUESSING MODELS AND THE CONTINUUM

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ABSTRACT. We introduce a stronger version of an ω_1-guessing model, which we call an indestructibly ω_1-guessing model. The principle IGMP states that there are stationarily many indestructibly ω_1-guessing models. This principle, which follows from PFA, captures many of the consequences of PFA, including the Suslin hypothesis and the singular cardinal hypothesis. We prove that IGMP is consistent with the continuum being arbitrarily large.

The idea of an ω_1-guessing model was introduced by Viale-Weiss [10], who showed that the combinatorial principle $\mathsf{ISP}(\omega_2)$ of Weiss [11] is equivalent to the existence of stationarily many ω_1-guessing models in $P_{\omega_2}(H(\theta))$, for all cardinals $\theta \geq \omega_2$. They showed that PFA implies $\mathsf{ISP}(\omega_2)$, and in turn $\mathsf{ISP}(\omega_2)$ implies many of the consequences of PFA, including the failure of square principles. In [10] it was asked whether $\mathsf{ISP}(\omega_2)$ determines the value of the continuum. We answered this question negatively in [5], by showing that $\mathsf{ISP}(\omega_2)$ is consistent with 2^ω having any value of uncountable cofinality greater than ω_1.

In this paper we introduce a stronger kind of guessing model, which we call an *indestructibly ω_1-guessing model*. We also introduce a new principle, denoted by IGMP, which asserts the existence of stationarily many indestructibly ω_1-guessing models in $P_{\omega_2}(H(\theta))$, for any cardinal $\theta \geq \omega_2$. As before, PFA implies IGMP, but IGMP captures more of the consequences of PFA than does $\mathsf{ISP}(\omega_2)$, including the Suslin hypothesis. As with the principle $\mathsf{ISP}(\omega_2)$, a natural question is whether the stronger principle IGMP determines the value of the continuum. The main result of this paper is that IGMP is consistent with 2^ω being equal to any $\lambda \geq \omega_2$ with cofinality at least ω_2.

In Section 1, we review material which will be necessary for reading the paper. In Section 2, we discuss ω_1-guessing models, and give new proofs of several previously known consequences of $\mathsf{ISP}(\omega_2)$. In Section 3, we introduce indestructibly ω_1-guessing models, the principle IGMP, and derive some consequences of IGMP, including Suslin's hypothesis and SCH.

In Section 5, we review the ideas of strong genericity and the strongly proper collapse. We also prove a new theorem about the preservation of strong properness

Date: October 2015; revised December 2016.

2010 *Mathematics Subject Classification:* 03E05, 03E35, 03E40, 03E65.

Key words and phrases. Indestructible guessing model, IGMP, approximation property, strongly proper.

Part of this work was completed in June 2015 under a collaboration sponsored by the first author's Simons Foundation grant 318467.

The research of the second author was partially supported by NSF grant DMS-1464859.

after proper forcing. In Sections 6 and 7, we carefully develop a finite support iteration of specializing forcings, and prove that a certain quotient of such an iteration has the ω_1-approximation property. In Section 8, we prove our consistency result, that IGMP is consistent with the continuum being greater than ω_2.

1. Preliminaries

We review some background material which will be necessary for understanding the paper. We assume that the reader is already familiar with forcing, iterated forcing, and proper forcing.

If κ is a regular cardinal and X is a set, $P_\kappa(X)$ denotes the set $\{a \subseteq X : |a| < \kappa\}$. The reader should be familiar with the basic definitions and facts regarding club and stationary subsets of $P_\kappa(X)$.

A *tree* is a strict partial ordering $(T, <_T)$ such that for any $t \in T$, the set $\{u \in T : u <_T t\}$ is well-ordered by $<_T$. We write $u \leq_T v$ to mean that either $u <_T v$ or $u = v$. We sometimes say that T is a tree without explicitly mentioning its partial order, which is always denoted by $<_T$. For an ordinal α, T_α is the set of $t \in T$ such that the set $\{u \in T : u <_T t\}$, ordered by $<_T$, has order type α. The set T_α is called *level α of T*. The *height of T* is the least δ such that T_δ is empty. Let $T \restriction \beta := \bigcup\{T_\alpha : \alpha < \beta\}$. A set b is a *branch of T* if it is a maximal linearly ordered subset of T.

For a regular uncountable cardinal κ, a *κ-Aronszajn tree* is a tree of height κ, all of whose levels have size less than κ, which has no branch of length κ. A *weak κ-Kurepa tree* is a tree of height and size κ which has more than κ many branches of length κ.

Let T be a tree. A *specializing function for T* is a function $f : T \to \omega$ such that for all $x, y \in T$, if $x <_T y$ then $f(x) \neq f(y)$. Note that if T has a specializing function, then T has no branches of length ω_1. On the other hand, suppose that T is a tree which has no branches of length ω_1. Define $P(T)$ as the forcing poset, ordered by reverse inclusion, whose conditions are finite functions p from a subset of T into ω, such that for all $x, y \in \mathrm{dom}(p)$, if $x <_T y$ in $\mathrm{dom}(p)$, then $p(x) \neq p(y)$. Then $P(T)$ is ω_1-c.c., and if G is a V-generic filter for $P(T)$, then $\bigcup G$ is a specializing function for T ([2]).

We will frequently use the product lemma. This result says that if $\mathbb{P}$ and $\mathbb{Q}$ are forcing posets, then the V-generic filters for $\mathbb{P} \times \mathbb{Q}$ are exactly those filters of the form $G \times H$, where G is a V-generic filter for $\mathbb{P}$, and H is a $V[G]$-generic filter for $\mathbb{Q}$. Moreover, in that case H is a V-generic filter for $\mathbb{Q}$, G is a $V[H]$-generic filter for $\mathbb{P}$, and $V[G \times H] = V[G][H] = V[H][G]$.

Let $\mathbb{P}$ and $\mathbb{Q}$ be forcing posets, where $\mathbb{P}$ is a suborder of $\mathbb{Q}$. We say that $\mathbb{P}$ is a *regular suborder* of $\mathbb{Q}$ if (a) for all p and q in $\mathbb{P}$, if p and q are incompatible in $\mathbb{P}$, then p and q are incompatible in $\mathbb{Q}$, and (b) if A is a maximal antichain of $\mathbb{P}$, then A is predense in $\mathbb{Q}$.

Let $\mathbb{P}$ be a regular suborder of $\mathbb{Q}$, and let G be a V-generic filter on $\mathbb{P}$. In $V[G]$, let $\mathbb{Q}/G$ be the forcing poset consisting of conditions $q \in \mathbb{Q}$ such that for all $s \in G$, q and s are compatible in $\mathbb{Q}$, with the same ordering as $\mathbb{Q}$. Then $\mathbb{P} * (\mathbb{Q}/\dot{G}_{\mathbb{P}})$ is forcing equivalent to $\mathbb{Q}$. Moreover:

Lemma 1.1. *Let $\mathbb{P}$ be a regular suborder of $\mathbb{Q}$. Suppose that G is a V-generic filter on $\mathbb{P}$, and H is a $V[G]$-generic filter on $\mathbb{Q}/G$. Then H is a V-generic filter on $\mathbb{Q}$, $H \cap \mathbb{P} = G$, and $V[G][H] = V[H]$.*

Conversely, if H is a V-generic filter on $\mathbb{Q}$, then $H \cap \mathbb{P}$ is a V-generic filter on $\mathbb{P}$, H is a $V[H \cap \mathbb{P}]$-generic filter on $\mathbb{Q}/(H \cap \mathbb{P})$, and $V[H] = V[H \cap \mathbb{P}][H]$.

Proof. See [5, Lemma 1.6] □

Lemma 1.2. *Let $\mathbb{P}$ be a regular suborder of $\mathbb{Q}$. Then for all $q \in \mathbb{Q}$, there is $s \in \mathbb{P}$ such that for all $t \leq s$ in $\mathbb{P}$, q and t are compatible in $\mathbb{Q}$. Moreover, this property of s is equivalent to s forcing in $\mathbb{P}$ that q is in $\mathbb{Q}/\dot{G}_{\mathbb{P}}$.*

Proof. See [5, Lemmas 1.1, 1.3]. □

Lemma 1.3. *Let $\mathbb{P}$ and $\mathbb{Q}$ be forcing posets, and assume that $\mathbb{P}$ is a regular suborder of $\mathbb{Q}$. If D is a dense subset of $\mathbb{Q}$, then $\mathbb{P}$ forces that $D \cap (\mathbb{Q}/\dot{G}_{\mathbb{P}})$ is a dense subset of $\mathbb{Q}/\dot{G}_{\mathbb{P}}$.*

Proof. See [5, Lemma 1.5]. □

Lemma 1.4. *Let $\mathbb{P}$ and $\mathbb{Q}$ be forcing posets, and assume that $\mathbb{P}$ is a regular suborder of $\mathbb{Q}$. Let G be a V-generic filter on $\mathbb{P}$. Suppose that $s \in G$ and $p \in \mathbb{Q}/G$. Then s and p are compatible in $\mathbb{Q}/G$.*

Proof. If not, then there is $t \in G$ such that t forces in $\mathbb{P}$ that (a) p is in $\mathbb{Q}/\dot{G}_{\mathbb{P}}$, and (b) p and s are incompatible in $\mathbb{Q}/\dot{G}_{\mathbb{P}}$. Fix $u \in G$ with $u \leq s, t$.

As $p \in \mathbb{Q}/G$ and $u \in G$, fix $v \leq u, p$ in $\mathbb{Q}$. By Lemma 1.2, fix $w \in \mathbb{P}$ such that for all $z \leq w$ in $\mathbb{P}$, v and z are compatible in $\mathbb{Q}$. Then in particular, v and w are compatible in $\mathbb{Q}$, and since $v \leq u$, u and w are compatible in $\mathbb{Q}$. As $\mathbb{P}$ is a regular suborder of $\mathbb{Q}$ and u and w are in $\mathbb{P}$, it follows that u and w are compatible in $\mathbb{P}$. Fix $y \leq w, u$ in $\mathbb{P}$.

Since $y \leq w$, every extension of y in $\mathbb{P}$ is compatible with v in $\mathbb{Q}$. By Lemma 1.2, y forces in $\mathbb{P}$ that $v \in \mathbb{Q}/\dot{G}_{\mathbb{P}}$. Now $v \leq u, p$ in $\mathbb{Q}$. Since $u \leq s$, we have that $v \leq s, p$ in $\mathbb{Q}$. But y forces that $v \in \mathbb{Q}/\dot{G}_{\mathbb{P}}$, so y forces that s and p are compatible in $\mathbb{Q}/\dot{G}_{\mathbb{P}}$. This contradicts the choice of t and the fact that $y \leq t$. □

Let $\mathbb{P}$ and $\mathbb{Q}$ be forcing posets. A function $f : \mathbb{P} \to \mathbb{Q}$ is a *regular embedding* if (a) for all $p, q \in \mathbb{P}$, if $q \leq p$ in $\mathbb{P}$, then $f(q) \leq f(p)$ in $\mathbb{Q}$; (b) for all $p, q \in \mathbb{P}$, if p and q are incompatible in $\mathbb{P}$, then $f(p)$ and $f(q)$ are incompatible in $\mathbb{Q}$; (c) if A is a maximal antichain of $\mathbb{P}$, then $f[A]$ is predense in $\mathbb{Q}$. Note that if $f : \mathbb{P} \to \mathbb{Q}$ is a regular embedding, then $f[\mathbb{P}]$ is a regular suborder of $\mathbb{Q}$. A function $f : \mathbb{P} \to \mathbb{Q}$ is a *dense embedding* if it satisfies (a) and (b) above, and $f[\mathbb{P}]$ is dense in $\mathbb{Q}$. Note that any dense embedding is a regular embedding.

Assume that $j : V \to M$ is an elementary embedding with critical point κ, living in some outer model W of V. Let $\mathbb{P}$ be a forcing poset in V which is κ-c.c. We claim that $j \restriction \mathbb{P}$ is a regular embedding of $\mathbb{P}$ into $j(\mathbb{P})$. Namely, the preservation of the order and incompatibility of conditions from $\mathbb{P}$ to $j(\mathbb{P})$ follows from the elementarity of j. And if A is a maximal antichain of $\mathbb{P}$, then by elementarity M models that $j(A)$ is a maximal antichain of $j(\mathbb{P})$. By upwards absoluteness, $j(A)$ is a maximal antichain of $j(\mathbb{P})$. But since $\mathbb{P}$ is κ-c.c., $|A| < \kappa$, and therefore $j(A) = j[A]$. So $j[A]$ is a maximal antichain of $j(\mathbb{P})$.

A function $\pi : \mathbb{Q} \to \mathbb{P}$, where $\mathbb{P}$ and $\mathbb{Q}$ are forcing posets, is called a *projection mapping* if (a) $\pi(1_{\mathbb{Q}}) = 1_{\mathbb{P}}$, (b) $q \leq p$ in $\mathbb{Q}$ implies that $\pi(q) \leq \pi(p)$ in $\mathbb{P}$, and (c) whenever $p \leq \pi(q)$ in $\mathbb{P}$, then there is some $r \leq q$ in $\mathbb{Q}$ such that $\pi(r) \leq p$ in $\mathbb{P}$.

If $\pi : \mathbb{Q} \to \mathbb{P}$ is a projection mapping and G is a V-generic filter on $\mathbb{P}$, then in $V[G]$ we can define the forcing poset $\mathbb{Q}/G$ whose conditions are those $q \in \mathbb{Q}$ such that $\pi(q) \in G$, with the same ordering as $\mathbb{Q}$. Then $\mathbb{Q}$ is forcing equivalent to $\mathbb{P} * (\mathbb{Q}/\dot{G}_{\mathbb{P}})$.

Lemma 1.5. *Let $\mathbb{P}$ be a suborder of $\mathbb{Q}$. Suppose that there exists a projection mapping $\pi : \mathbb{Q} \to \mathbb{P}$ satisfying that (i) $\pi(p) = p$ for all $p \in \mathbb{P}$, and (ii) $q \leq \pi(q)$ for all $q \in \mathbb{Q}$. Then $\mathbb{P}$ is a regular suborder of $\mathbb{Q}$. Moreover, if G is a V-generic filter on $\mathbb{P}$, then the two kinds of quotients $\mathbb{Q}/G$ in $V[G]$ are the same.*

Proof. Suppose that p and q are in $\mathbb{P}$, and p and q are compatible in $\mathbb{Q}$. Let $s \leq p, q$ in $\mathbb{Q}$. Then $\pi(s) \leq \pi(p) = p$ and $\pi(s) \leq \pi(q) = q$. So p and q are compatible in $\mathbb{P}$.

Let A be a maximal antichain of $\mathbb{P}$, and we will show that A is predense in $\mathbb{Q}$. Let $q \in \mathbb{Q}$, and we will show that q is compatible with some condition in A. Then $\pi(q) \in \mathbb{P}$, so as A is a maximal antichain of $\mathbb{P}$, we can fix $s \in A$ and $r \in \mathbb{P}$ such that $r \leq \pi(q), s$. Since π is a projection mapping, we can fix $t \leq q$ in $\mathbb{Q}$ such that $\pi(t) \leq r$. But then $t \leq q$, and $t \leq \pi(t) \leq r \leq s$, so q is compatible with s.

Let G be a V-generic filter on $\mathbb{P}$. We will prove that for all $q \in \mathbb{Q}$, q is compatible in $\mathbb{Q}$ with every condition in G iff $\pi(q) \in G$. Assume that q is compatible with every condition in G. Since G is a V-generic filter on $\mathbb{P}$, to show that $\pi(q) \in G$ it suffices to show that $\pi(q)$ is compatible in $\mathbb{P}$ with every condition in G. So let $s \in G$, and we will show that $\pi(q)$ and s are compatible in $\mathbb{P}$. By assumption, s and q are compatible in $\mathbb{Q}$, so fix $t \leq q, s$. Then $\pi(t) \leq \pi(q)$ and $\pi(t) \leq \pi(s) = s$. Hence $\pi(q)$ and s are compatible in $\mathbb{P}$.

Conversely, assume that $\pi(q) \in G$, and we will show that q is compatible in $\mathbb{Q}$ with every condition in G. Fix $s \in G$. Then there is $t \in G$ with $t \leq \pi(q), s$. Since π is a projection mapping, there is $u \leq q$ such that $\pi(u) \leq t$. Then $u \leq q$ and $u \leq \pi(u) \leq t \leq s$, so q and s are compatible in $\mathbb{Q}$. $\square$

A pair (V, W) of transitive sets or classes with $V \subseteq W$ is said to have the ω_1-*covering property* if for every set $a \in W$ which is a subset of $V \cap On$ and which W models is countable, there is a set of ordinals $b \in V$ which V models is countable such that $a \subseteq b$. A forcing poset $\mathbb{P}$ has the ω_1-*covering property* if $\mathbb{P}$ forces that $(V, V[\dot{G}_{\mathbb{P}}])$ has the ω_1-covering property.

For a set or class N, a set $d \subseteq N$ is said to be *countably approximated by* N if for any set a in N which N models is countable, $d \cap a \in N$. A pair (V, W) of transitive sets or classes with $V \subseteq W$ is said to have the ω_1-*approximation property* if whenever $d \in W$ is a bounded subset of $V \cap On$ which is countably approximated by V, we have that $d \in V$. A forcing poset $\mathbb{P}$ has the ω_1-*approximation property* if $\mathbb{P}$ forces that $(V, V[\dot{G}_{\mathbb{P}}])$ has the ω_1-approximation property.

Note that if $\mathbb{P}$ has the ω_1-approximation property, then $\mathbb{P}$ preserves ω_1.

Lemma 1.6. *Suppose that $\mathbb{P}$ has the ω_1-approximation property. Let T be a tree. Suppose that G is a V-generic filter on $\mathbb{P}$. Then any branch of T in $V[G]$ whose length has uncountable cofinality is in V.*

Proof. Without loss of generality, assume that the underlying set of T is an ordinal. Let b be a branch of T in $V[G]$ whose length has uncountable cofinality. We claim that b is countably approximated by V. Then $b \in V$ and we are done. Let a be a countable set in V. Since the length of b has uncountable cofinality, there is $y \in b$ such that $a \cap b \subseteq \{x \in T : x <_T y\}$. Then $a \cap b = a \cap \{x \in T : x <_T y\}$, which is in V. $\square$

Lemma 1.7. *Suppose that (V_0, V_1) and (V_1, V_2) are pairs of transitive sets or classes which model ZFC, have the same ordinals, and satisfy that $V_0 \subseteq V_1 \subseteq V_2$. Assume that both pairs have the ω_1-covering property and the ω_1-approximation property. Then (V_0, V_2) has the ω_1-covering property and the ω_1-approximation property.*

Proof. It is easy to check that (V_0, V_2) has the ω_1-covering property. Let d be a bounded subset of $V_0 \cap On$ in V_2 which is countably approximated by V_0. We will show that $d \in V_0$.

We claim that d is countably approximated by V_1. Let a be a countable set in V_1. Since (V_0, V_1) has the ω_1-covering property, we can fix a countable set b in V_0 such that $a \subseteq b$. Since d is countably approximated by V_0, $b \cap d$ is in V_0 and hence in V_1. Therefore $a \cap d = a \cap (b \cap d)$ is in V_1.

Since (V_1, V_2) has the ω_1-approximation property, $d \in V_1$. As d is countably approximated by V_0 and (V_0, V_1) has the ω_1-approximation property, $d \in V_0$. $\square$

It follow that if $\mathbb{P}$ has the ω_1-covering property and the ω_1-approximation property, and $\mathbb{P}$ forces that $\dot{\mathbb{Q}}$ has the ω_1-covering property and the ω_1-approximation property, then the two step iteration $\mathbb{P} * \dot{\mathbb{Q}}$ has the ω_1-covering property and the ω_1-approximation property.

Lemma 1.8. *Suppose that $V \subseteq W_0 \subseteq W$ are transitive sets or classes, and (V, W) has the ω_1-approximation property. Then (V, W_0) has the ω_1-approximation property. It follows that if $\mathbb{P}$ is a regular suborder of $\mathbb{Q}$, and $\mathbb{Q}$ has the ω_1-approximation property, then $\mathbb{P}$ has the ω_1-approximation property.*

Proof. Straightforward. $\square$

A set N of size ω_1 is said to be *internally unbounded* if for any countable set $a \subseteq N$, there is a countable set $b \in N$ such that $a \subseteq b$. If $N \prec H(\theta)$ for some cardinal $\theta \geq \omega_2$, then easily N is internally unbounded iff there exists a $\subseteq$-increasing sequence $\langle N_i : i < \omega_1 \rangle$ of countable sets in N with union equal to N.

Finally, we will need to know some facts about the Y-c.c. property of forcing posets, which is a property introduced recently in [4]. The actual definition of being Y-c.c. is beyond the scope of this paper. In [4] it is proven that any Y-c.c. forcing poset is ω_1-c.c. and has the ω_1-approximation property, and any finite support iteration of Y-c.c. forcing posets is itself Y-c.c. Also, the forcing poset $P(T)$ defined earlier in this section, for adding a specializing function for a tree which has no branches of length ω_1, is Y-c.c.

2. Guessing Models and GMP

Guessing models were introduced by Viale-Weiss [10].

Definition 2.1. *Let N be a set. A set $d \subseteq N$ is said to be* N-guessed *if there exists $e \in N$ such that $d = e \cap N$. We say that N is* ω_1-guessing *if for any set $d \subseteq N \cap On$ with* $\sup(d) < \sup(N \cap On)$*, if d is countably approximated by N, then d is N-guessed.*

A typical situation which we will consider is that N is an elementary substructure of $H(\chi)$, for some cardinal $\chi \geq \omega_2$, and $|N| = \omega_1$. In the next section, we will also consider the case that N is an elementary substructure in an inner model over which the universe is a generic extension.

There is a useful characterization of being ω_1-guessing in terms of the approximation property.

Lemma 2.2. *Let N be an elementary substructure of $H(\chi)$, for some uncountable cardinal χ. Then the following are equivalent:*

(1) *N is ω_1-guessing;*
(2) *the pair $(\overline{N}, V)$ has the ω_1-approximation property, where $\overline{N}$ is the transitive collapse of N.*

Proof. See [5, Lemma 1.10]. □

Next we prove two technical lemmas about ω_1-guessing models.

Lemma 2.3. *Let N be in $P_{\omega_2}(H(\chi))$, where $\chi \geq \omega_2$ is a cardinal, and assume that $N \prec H(\chi)$ and N is ω_1-guessing. Then for any cardinal $\theta \in N$ with uncountable cofinality, $\mathrm{cf}(\sup(N \cap \theta)) = \omega_1$. In particular, $\omega_1 \subseteq N$, and hence $N \cap \omega_2 \in \omega_2$.*

Proof. Since N has size at most ω_1, $\mathrm{cf}(\sup(N \cap \theta)) \leq \omega_1$. Suppose for a contradiction that $\mathrm{cf}(\sup(N \cap \theta)) = \omega$. Fix a sequence $\langle \alpha_n : n < \omega \rangle$ of ordinals in $N \cap \theta$ which is increasing and cofinal in $\sup(N \cap \theta)$.

We claim that the set $\{\alpha_n : n < \omega\}$ is countably approximated by N. Let $a \in N$ be countable, and we will show that $a \cap \{\alpha_n : n < \omega\}$ is in N. By elementarity, $\sup(a \cap \theta) \in N \cap \theta$, so we can fix k such that $\sup(a \cap \theta) < \alpha_k$. Then $a \cap \{\alpha_n : n < \omega\} \subseteq a \cap \{\alpha_n : n < k\}$. So $a \cap \{\alpha_n : n < \omega\}$ is a finite subset of N, and hence is in N.

Since the set $\{\alpha_n : n < \omega\}$ is countably approximated by N, and is a bounded subset of $N \cap On$, there exists $e \in N$ such that $\{\alpha_n : n < \omega\} = N \cap e$. We claim that $e = \{\alpha_n : n < \omega\}$. This is a contradiction, for then $\sup(e) = \sup(N \cap \theta)$ would be in N by elementarity, which is impossible.

As $N \cap e \subseteq \theta$, by elementarity it follows that $e \subseteq \theta$. Also, for all $\beta \in N \cap \theta$, $N \cap e \cap \beta$ is finite; therefore by elementarity, $e \cap \beta$ must be finite. So N models that every proper initial segment of e is finite, and hence e is at most countable. Therefore $e \subseteq N$, so $e = N \cap e = \{\alpha_n : n < \omega\}$. □

Lemma 2.4. *Let N be in $P_{\omega_2}(H(\chi))$, where $\chi \geq \omega_2$ is a cardinal, and assume that $N \prec H(\chi)$. Let $T \in N$ be a tree with height and size ω_1. Suppose that W is an outer model of V with $\omega_1^V = \omega_1^W$, and N is ω_1-guessing in W. Then every branch of T in W with length ω_1 is in N.*

Proof. Since N is ω_1-guessing in W, it is easy to check that N is ω_1-guessing in V. So by Lemma 2.3, $\omega_1 \subseteq N$.

Without loss of generality, assume that the underlying set of T is ω_1. Let b be a branch of T of length ω_1 in W. Then b is a bounded subset of $N \cap On$. We claim that b is countably approximated by N in W. Let $a \in N$ be countable. Since b has length ω_1, we can fix $y \in b$ such that $a \cap b \subseteq \{x \in T : x <_T y\}$. Then $a \cap b = a \cap \{x \in T : x <_T y\}$. Now $N \prec H(\chi)$ in V, so the set $\{x \in T : x <_T y\}$ is in N, and hence the set $a \cap b$ is in N.

As N is ω_1-guessing in W, there is $e \in N$ such that $N \cap e = b$. Since $N \cap e \subseteq T$, it follows by elementarity that $e \subseteq T$. But the underlying set of T is ω_1, which is a subset of N. So $e \subseteq N$. Hence $e = e \cap N = b$. So $e = b$, and $b \in N$. □

Definition 2.5. *For a cardinal $\theta \geq \omega_2$, let $\mathsf{GMP}(\theta)$ be the statement that there exist stationarily many sets $N \in P_{\omega_2}(H(\theta))$ such that N is ω_1-guessing. Let GMP be the statement that $\mathsf{GMP}(\theta)$ holds, for all cardinals $\theta \geq \omega_2$.*[1]

It is easy to see that if $\omega_2 \leq \theta_0 < \theta_1$ are cardinals, $N \in P_{\omega_2}(H(\theta_1))$ is ω_1-guessing, $N \prec H(\theta)$, and $\theta_0 \in N$, then $N \cap H(\theta_0)$ is ω_1-guessing. In particular, $\mathsf{GMP}(\theta_1)$ implies $\mathsf{GMP}(\theta_0)$.

Weiss [11] introduced the principle $\mathsf{ISP}(\omega_2)$, and Viale-Weiss [10] showed that it is equivalent to what we are calling GMP. They proved that $\mathsf{ISP}(\omega_2)$ follows from PFA, and that $\mathsf{ISP}(\omega_2)$ implies some of the consequence of PFA, such as the failure of square principles.

The original formulation of $\mathsf{ISP}(\omega_2)$ involves completely different concepts than guessing models, namely ineffable branches in slender $P_\kappa(\lambda)$-lists. Some of the consequences of $\mathsf{ISP}(\omega_2)$ were derived in [10] and [11] using these different concepts. We offer new proofs of two of the most important consequences of $\mathsf{ISP}(\omega_2)$ using arguments involving guessing models, namely, the failure of the approachability property on ω_1, and the nonexistence of ω_2-Aronszajn trees. We also derive a new consequence, namely the nonexistence of weak ω_1-Kurepa trees.

For the original proof of the next result, see [10, Corollary 4.9].

Proposition 2.6. *$\mathsf{GMP}(\omega_3)$ implies $\neg\mathsf{AP}_{\omega_1}$.*

Proof. Suppose for a contradiction that the approachability property AP_{ω_1} holds. So there exists a sequence $\vec{a} = \langle a_i : i < \omega_2\rangle$ of countable subsets of ω_2, a club $C \subseteq \omega_2$, and a sequence $\vec{c} = \langle c_\alpha : \alpha \in C \cap \mathrm{cof}(\omega_1)\rangle$ such that for all $\alpha \in C \cap \mathrm{cof}(\omega_1)$, c_α is a club subset of α with order type ω_1, and for all $\beta < \alpha$, there is $i < \alpha$ such that $c_\alpha \cap \beta = a_i$.

Fix N in $P_{\omega_2}(H(\omega_3))$ such that $N \prec H(\omega_3)$, $\vec{a}$, C, and $\vec{c}$ are in N, and N is ω_1-guessing. Since $C \in N$, $N \cap \omega_2 \in C$. As $\omega_2 \in N$, it follows that $\mathrm{cf}(N \cap \omega_2) = \omega_1$ by Lemma 2.3. So $N \cap \omega_2 \in C \cap \mathrm{cof}(\omega_1)$.

Let $\alpha := N \cap \omega_2$. We claim that c_α is countably approximated by N. Let $a \in N$ be countable. Then for some $\beta < \alpha$, $c_\alpha \cap a \subseteq \beta$. Fix $i < \alpha$ such that $c_\alpha \cap \beta = a_i$. Then $i \in N$, and hence $c_\alpha \cap \beta = a_i \in N$. So $c_\alpha \cap a = c_\alpha \cap \beta \cap a = a_i \cap a$, which is in N since a and a_i are in N.

As N is ω_1-guessing, we can fix $e \in N$ such that $e \cap N = c_\alpha$. Since $N \cap e \subseteq \omega_2$, it follows that $e \subseteq \omega_2$ by elementarity. And as $N \cap e = c_\alpha$ is cofinal in $N \cap \omega_2$, e is cofinal in ω_2 by elementarity. Therefore e has order type ω_2. By elementarity, fix $\gamma \in N \cap e$ such that $e \cap \gamma$ has order type ω_1. Since $\omega_1 \subseteq N$, $e \cap \gamma \subseteq N$, so $e \cap \gamma = e \cap N \cap \gamma = c_\alpha \cap \gamma$. So $c_\alpha \cap \gamma$ has order type ω_1, which contradicts that c_α has order type ω_1 and $\gamma < \alpha$. □

The next result was originally proven in [11, Section 2].

Theorem 2.7. *$\mathsf{GMP}(\omega_3)$ implies that there does not exist an ω_2-Aronszajn tree.*

Proof. Let T be a tree of height ω_2, all of whose levels have cardinality less than ω_2. We will prove that there is a branch of T with order type ω_2. Without loss of generality, assume that T has underlying set ω_2, so that $T \in H(\omega_3)$. Fix N in $P_{\omega_2}(H(\omega_3))$ such that $N \prec H(\omega_3)$, $T \in N$, and N is ω_1-guessing.

[1] GMP stands for *guessing model principle.*

Let $\alpha := N \cap \omega_2$. For all $\beta < \alpha$, T_β is in N by elementarity, and since T_β has size at most ω_1, $T_\beta \subseteq N$. It follows that $T \restriction \alpha \subseteq N$.

Fix a node y on level α of T. Let $d := \{x \in T : x <_T y\}$. Then d is a bounded subset of $N \cap On$. We claim that d is countably approximated by N. Let $a \in N$ be countable. Since α has cofinality ω_1 by Lemma 2.3, there is $y^* <_T y$ such that $a \cap d = a \cap \{x \in T : x <_T y^*\}$. As a and y^* are in N, so is $a \cap d$.

Since N is ω_1-guessing, we can fix $e \in N$ such that $e \cap N = d$. We claim that e is a branch of T with length ω_2, which completes the proof. First, if x and y are in $e \cap N$, then x and y are in d, and since d is a branch, either $x \leq_T y$ or $y <_T x$. By elementarity, e is a chain in T. Similarly, if $x \in e \cap N$, $x_0 \in N \cap T$, and $x_0 <_T x$, then $x_0 \in d$ and hence $x_0 \in d \subseteq e$. By elementarity, e is closed downwards. If e does not have order type ω_2, then by elementarity there is $\beta < N \cap \omega_2 = \alpha$ such that $e \subseteq T \restriction \beta$. But the node in d at level β is in e, which is a contradiction. $\square$

Theorem 2.8. *GMP*(ω_2) *implies that there does not exist a weak ω_1-Kurepa tree.*

Proof. Suppose for a contradiction that T is a weak ω_1-Kurepa tree. Without loss of generality, assume that the underlying set of T is ω_1. Then $T \in H(\omega_2)$. Fix a set $N \in P_{\omega_2}(H(\omega_2))$ such that $N \prec H(\omega_2)$, $T \in N$, and N is ω_1-guessing. By Lemma 2.4, every branch of T with length ω_1 is in N. But this is impossible, since N has size ω_1 and T has more than ω_1 many branches of length ω_1. $\square$

Note that if CH holds, then there exists a weak ω_1-Kurepa tree, namely the tree of functions in $2^{<\omega_1}$. Hence GMP(ω_2) implies that CH fails. Viale-Weiss [10] asked whether GMP implies that 2^ω is equal to ω_2. This question was settled in [5], where we showed that GMP is consistent with 2^ω being equal to any given cardinal $\lambda \geq \omega_2$ of uncountable cofinality.

Let us make an additional observation about the model constructed in [5]. Recall that the *pseudo-intersection number* $\mathfrak{p}$ is the least size of a collection X of infinite subsets of ω, closed under finite intersections, for which there is no set b such that $b \setminus a$ is finite for all $a \in X$. Viale [9, Lemma 4.2] proved that under the assumption that $\omega_1 < \mathfrak{p}$, if $\chi \geq \omega_2$ is a regular cardinal, $N \in P_{\omega_2}(H(\chi))$, $N \prec H(\chi)$, and N is ω_1-guessing, then N is internally unbounded.

Viale [9, Remark 4.3] asked whether it is consistent that there are stationarily many ω_1-guessing models in a model where $\mathfrak{p} = \omega_1$. We point out that in the model constructed in [5], GMP holds and $\mathfrak{p} = \omega_1$, which settles this question. Namely, the model of [5] is obtained by forcing with a forcing poset of the form $\mathbb{P} * \mathrm{Add}(\omega, \lambda)$, where $\lambda > \omega_1$. But by [3, Section 11.3], any model obtained by forcing with $\mathrm{Add}(\omega, \lambda)$, where $\lambda \geq \omega_1$, satisfies that $\mathfrak{p} = \omega_1$.

3. Indestructibly Guessing Models and IGMP

We introduce a stronger form of ω_1-guessing, and with it a new principle.

Definition 3.1. *A set N is* indestructibly ω_1-guessing *if for any forcing poset $\mathbb{P}$ which preserves ω_1, $\mathbb{P}$ forces that N is ω_1-guessing.*

Definition 3.2. *For a cardinal $\theta \geq \omega_2$, let IGMP(θ) be the statement that there exist stationarily many sets $N \in P_{\omega_2}(H(\theta))$ such that N is indestructibly ω_1-guessing. Let IGMP be the statement that IGMP(θ) holds, for all cardinals $\theta \geq \omega_2$.*[2]

[2]IGMP stands for *indestructibly guessing model principle.*

It is easy to see that if $\omega_2 \leq \theta_0 < \theta_1$ are cardinals, $N \in P_{\omega_2}(H(\theta_1))$ is indestructibly ω_1-guessing, $N \prec H(\theta_1)$, and $\theta_0 \in N$, then $N \cap H(\theta_0)$ is indestructibly ω_1-guessing. In particular, $\mathsf{IGMP}(\theta_1)$ implies $\mathsf{IGMP}(\theta_0)$.

We will prove in Section 4 that PFA implies IGMP.[3]

It turns out that indestructibly ω_1-guessing models are internally unbounded. The proof is a variation of the proof of [9, Lemma 4.2] that if $\mathfrak{p} > \omega_1$ and N is ω_1-guessing, then N is internally unbounded.

Proposition 3.3. *Let $\theta \geq \omega_2$ be a cardinal. Suppose that $N \in P_{\omega_2}(H(\theta))$, $N \prec H(\theta)$, $\mathrm{cf}(\sup(N \cap \theta)) = \omega_1$, and N is indestructibly ω_1-guessing. Then N is internally unbounded.*

Proof. Suppose for a contradiction that x is a countable subset of N which is not covered by any set in $P_{\omega_1}(N) \cap N$. Without loss of generality, x is a set of ordinals. Since $\mathrm{cf}(N \cap On) = \mathrm{cf}(N \cap \theta) = \omega_1$, x is bounded in $\sup(N \cap On)$. Then easily $F = \{x \setminus y : y \in P_{\omega_1}(N) \cap N\}$ is a collection of infinite subsets of x which is closed under finite intersections.

Let $\mathbb{P}$ be the ω_1-c.c. Mathias forcing for adding a pseudointersection to F, that is, a subset of x which is almost contained modulo finite in every member of F (see, for example, [3, Theorem 7.7]). Let G be a V-generic filter on $\mathbb{P}$. Let b be the pseudointersection given by G. Then $b \notin V$.

We claim that b is countably approximated by N. Let a be a countable set in N. Then $a \in P_{\omega_1}(N) \cap N$, so $x \setminus a \in F$. Hence $b \setminus (x \setminus a) = b \cap a$ is finite. As $b \cap a$ is a finite subset of N, it is in N. Since N is indestructibly ω_1-guessing and $\mathbb{P}$ is ω_1-c.c., N is ω_1-guessing in $V[G]$. As b is a bounded subset of $N \cap On$ which is countably approximated by N, we can fix $e \in N$ such that $e \cap N = b$. Note that e is countable, for otherwise by elementarity $e \cap N$ would be uncountable, contradicting that $e \cap N = b$ and b is countable. Therefore $e \subseteq N$, so $e = e \cap N = b$. But this is impossible since $e \in V$ and $b \notin V$. □

Corollary 3.4. *Let $\theta \geq \omega_2$ be a cardinal. Then $\mathsf{IGMP}(\theta^+)$ implies that there are stationarily many $N \in P_{\omega_2}(H(\theta))$ such that N is internally unbounded and indestructibly ω_1-guessing.*

Proof. If $N \in P_{\omega_2}(H(\theta^+))$, $N \prec H(\theta^+)$, and N is ω_1-guessing, then $\sup(N \cap \theta)$ has cofinality ω_1 by Lemma 2.3. So if N is indestructibly ω_1-guessing, then by the comments after Definition 3.2 and Proposition 3.3, $N \cap H(\theta)$ is an elementary substructure of $H(\theta)$, is indestructibly ω_1-guessing, and is internally unbounded. □

Corollary 3.5. *IGMP implies SCH.*

Proof. By [9, Theorem 7.9], SCH holds provided that for all regular $\theta \geq \omega_2$, there are stationarily many $N \in P_{\omega_2}(H(\theta))$ such that N is internally unbounded and ω_1-guessing. This statement hold under IGMP by Corollary 3.4. □

Now we move towards proving that IGMP implies the Suslin hypothesis.

Theorem 3.6. *Assume $\mathsf{IGMP}(\omega_2)$. Let T be a tree with height and size ω_1. Assume that $\mathbb{P}$ is a forcing poset which preserves ω_1. Then $\mathbb{P}$ does not add any new branches of length ω_1 to T.*

[3] The original proof of [10, Section 4] that PFA implies $\mathsf{ISP}(\omega_2)$ implicitly shows that PFA implies IGMP, although they did not formulate this principle.

Proof. Without loss of generality, assume that the underlying set of T is ω_1. Then $T \in H(\omega_2)$. By $\mathsf{IGMP}(\omega_2)$, we can fix $N \in P_{\omega_2}(H(\omega_2))$ such that $N \prec H(\omega_2)$, $T \in N$, and N is indestructibly ω_1-guessing. By Lemma 2.4, every branch of T with length ω_1 in a generic extension by $\mathbb{P}$ is in N, and hence in V. $\square$

Let us say that a tree T is *nontrivial* if (1) for all $t \in T$, there are incomparable u, v in T with $t \leq_T u, v$, and (2) for all $t \in T$ and for all α less than the height of T, there is $u \in T$ with height at least α such that $t \leq_T u$.

Suppose that T is a nontrivial tree. Define $\mathbb{P}_T$ as the forcing poset whose conditions are nodes in T, ordered by $t \leq_{\mathbb{P}_T} s$ iff $s \leq_T t$. The assumption of T being nontrivial implies that the forcing poset $\mathbb{P}_T$ adds a branch of T which is not in the ground model with length equal to the height of T.

Theorem 3.7. *$\mathsf{IGMP}(\omega_2)$ implies that for any nontrivial tree T with height and size ω_1, $\mathbb{P}_T$ collapses ω_1.*

Proof. Let G be a V-generic filter on $\mathbb{P}_T$. Then in $V[G]$, G is a branch of T which is not in V. By Theorem 3.6, ω_1^V is not equal to $\omega_1^{V[G]}$. $\square$

We note that the conclusion of Theorem 3.7 was previously known to follow from PFA ([1, Section 7]). Namely, under PFA, every tree with height and size ω_1 is special (see Definition 4.1 below). And adding a new branch of length ω_1 to a special tree by forcing will collapse ω_1 (see Proposition 4.3 below).

Recall that if T is an ω_1-Suslin tree, then T is a tree with height and size ω_1, and $\mathbb{P}_T$ is ω_1-c.c. In particular, $\mathbb{P}_T$ preserves ω_1. Moreover, if there exists an ω_1-Suslin tree, then there exists a nontrivial ω_1-Suslin tree.

Corollary 3.8. *$\mathsf{IGMP}(\omega_2)$ implies that there does not exist an ω_1-Suslin tree, so the Suslin hypothesis holds.*

Proof. Immediate from Theorem 3.7. $\square$

On the other hand, the Suslin hypothesis is consistent with the principle GMP. For example, the model of GMP constructed in [5] is a generic extension by a forcing poset of the form $\mathbb{P} * \mathrm{Add}(\omega, \lambda)$. But Shelah [7] proved that Cohen forcing $\mathrm{Add}(\omega)$ adds an ω_1-Suslin tree, so there exists an ω_1-Suslin tree in this model.

Theorem 3.9. *Assume that $2^\omega \leq \omega_2$, and there exist cofinally many sets in $P_{\omega_2}(H(\omega_2))$ which are indestructibly ω_1-guessing. Suppose that W is a generic extension of V, and W contains a subset of ω_1 which is not in V. Then either W contains a real which is not in V, or ω_2^V is not a cardinal in W.*

Proof. Since $2^\omega \leq \omega_2$, $H(\omega_1)$ has size at most ω_2. So we can inductively define a $\subseteq$-increasing sequence $\langle N_i : i < \omega_2 \rangle$, whose union contains $H(\omega_1)$, such that for each $i < \omega_2$, N_i is in $P_{\omega_2}(H(\omega_2))$, $\omega_1 + 1 \subseteq N_i$, and N_i is indestructibly ω_1-guessing.

Suppose that $W \setminus V$ does not contain a real, and we will show that ω_2^V is not a cardinal in W. Since every subset of ω in W is in V, it follows that $\omega_1^V = \omega_1^W$, and $W \setminus V$ contains no bounded subset of ω_1. Fix b which is a subset of ω_1 in $W \setminus V$. Then every proper initial segment of b is in V, and hence in $H(\omega_1)^V$. So we can fix, for each $\alpha < \omega_1$, the least ordinal $i_\alpha < \omega_2^V$ such that $b \cap \alpha \in N_{i_\alpha}$. Note that the sequence $\langle i_\alpha : \alpha < \omega_1 \rangle$ is in W.

We claim that this sequence is unbounded in ω_2^V, which implies that ω_2^V is not a cardinal in W. Otherwise there is $\delta < \omega_2^V$ such that $i_\alpha < \delta$ for all $\alpha < \omega_1$. Since the sequence $\langle N_\alpha : \alpha < \omega_2 \rangle$ is $\subseteq$-increasing, it follows that for all $\alpha < \omega_1$, $b \cap \alpha \in N_\delta$. As $\omega_1 + 1 \subseteq N_\delta$, b is a bounded subset of $N_\delta \cap On$.

For any countable set $a \in N_\delta$, $a \cap b = a \cap (b \cap \alpha)$ for some $\alpha < \omega_1$, and hence $a \cap b \in N_\delta$. So b is countably approximated by N_δ in W. Since N_δ is indestructibly ω_1-guessing in V, N_δ is ω_1-guessing in W. So there is $e \in N_\delta$ such that $b = e \cap N_\delta$. But then e and N_δ are in V, and hence $b \in V$, which is a contradiction. □

We note that the conclusion of Theorem 3.9 was previously shown to follow from PFA by Todorčević [8, Theorem 2].

4. Trees and Guessing Models

In this section we review some ideas of Baumgartner and Viale-Weiss concerning trees and guessing models. The main result of this section is Corollary 4.5, which gives a sufficient condition under which IGMP holds. We also observe that PFA implies IGMP.

The next definition is due to Baumgartner [1, Section 7].

Definition 4.1. *Let T be a tree. We say that T is* special *if there exists a function $f : T \to \omega$ such that whenever s, t, and u are in T and $f(s) = f(t) = f(u)$, if $s <_T t$ and $s <_T u$, then t and u are comparable in T.*

Recall that for a tree T, a function $g : T \to \omega$ is a *specializing function* if for all $s, t \in T$, if $s <_T t$ then $g(s) \neq g(t)$. Clearly if T has a specializing function, then T has no branches of length ω_1. Baumgartner [1, Theorem 7.3] proved that for a tree T of height ω_1 which has no branches of length ω_1, T is special in the sense of Definition 4.1 iff T has a specializing function.

The next theorem appears as Theorem 7.5 of [1].

Theorem 4.2. *Assume that every tree of height and size ω_1 which has no branches of length ω_1 is special. Then every tree of height and size ω_1 which has at most ω_1 many branches of length ω_1 is special.*

Proposition 4.3. *Suppose that T is a tree with height ω_1 which is special. Then whenever W is an outer model of V with the same ω_1, any branch of T in W of length ω_1 is in V.*

Proof. The proof follows easily from ideas of Baumgartner [1, Section 7]. Fix a function $f : T \to \omega$ such that whenever s, t, u are in T and $f(s) = f(t) = f(u)$, if $s <_T t$ and $s <_T u$ then t and u are comparable. Let W be an outer model with $\omega_1^V = \omega_1^W$, and suppose that b is a branch of T with length ω_1 in W. We will show that $b \in V$.

Since $f \restriction b$ is a function from a set of size ω_1 into ω, we can fix $n < \omega$ such that the set $\{t \in b : f(t) = n\}$ has size ω_1. Fix s in b such that $f(s) = n$. Then the set

$$X := \{t \in b : s <_T t,\ f(t) = n\}$$

is uncountable. But X is a subset of the set

$$Y := \{t \in T : s <_T t,\ f(t) = n\},$$

and hence Y is uncountable. Note that Y is in V.

The set Y is an uncountable chain. For if t and u are in Y, then $s <_T t$, $s <_T u$, and $f(s) = f(t) = f(u) = n$. Since T is special, it follows that t and u are comparable. Let c be the downwards closure of Y. Then c is a branch of T in V

with length ω_1, and $c \in V$. There are cofinally many nodes above s in b which take value n under f, and any such node is in c. So $b = c$, and therefore $b \in V$. $\square$

We now establish a connection between special trees and indestructibly ω_1-guessing models. This connection involves constructing a tree from a guessing model; a similar construction was done previously in [10, Lemma 4.6].

Proposition 4.4. *Let $\theta \geq \omega_2$ be a cardinal. Suppose that N is in $P_{\omega_2}(H(\theta))$, $N \prec H(\theta)$, and N is internally unbounded and ω_1-guessing. Then there exists a tree T of height and size ω_1 which has ω_1 many branches of length ω_1 such that T being special implies that N is indestructibly ω_1-guessing.*

Proof. Since N is internally unbounded, we can fix a $\subseteq$-increasing sequence $\langle N_i : i < \omega_1 \rangle$ with union equal to N such that for all $i < \omega_1$, $N_i \in P_{\omega_1}(N) \cap N$.

Fix an uncountable ordinal δ in N, and we will define a tree T_δ. The desired tree T will then be the direct sum over all such trees T_δ. The underlying set of T_δ consists of all pairs in N of the form (i, f), where $i < \omega_1$ and $f : N_i \cap \delta \to 2$. For (i, f) and (j, g) in T_δ, let $(i, f) <_{T_\delta} (j, g)$ if $i < j$ and $f \subseteq g$. Note that T_δ is a tree with height and size ω_1.

We claim that T_δ has at most ω_1 many branches of length ω_1. Consider a branch b of length ω_1, and let $F_b := \bigcup\{f : \exists i \ (i, f) \in b\}$. Note that F_b is a function with domain equal to $N \cap \delta$, and $b = \{(i, F_b \restriction N_i) : i < \omega_1\}$. Since $b \subseteq T_\delta \subseteq N$, we have that for all $i < \omega_1$, $F_b \restriction N_i \in N$.

Let $A_b := \{\alpha \in N \cap \delta : F_b(\alpha) = 1\}$. We claim that A_b is N-guessed. As N is ω_1-guessing, it is enough to show that A_b is countably approximated by N. So let a be a countable set in N. Then for some $i < \omega_1$, $a \subseteq N_i$. Therefore $a \cap A_b = a \cap A_b \cap N_i$. Now $A_b \cap N_i = \{\alpha \in N_i \cap \delta : F_b(\alpha) = 1\}$, which is definable in N from the parameters N_i, δ, and $F_b \restriction N_i$. It follows that $A_b \cap N_i$ is in N. Since a is also in N, $a \cap A_b = a \cap A_b \cap N_i$ is in N. Since N is ω_1-guessing, we can fix e_b in N such that $A_b = e_b \cap N$.

Suppose that b and c are distinct branches of T with length ω_1. Then easily the functions F_b and F_c are different, and therefore the sets A_b and A_c are distinct. Since $A_b = e_b \cap N$ and $A_c = e_c \cap N$, it follows that e_b and e_c are distinct. Thus the map $b \mapsto e_b$ from the set of branches of T_δ with length ω_1 into N is injective. Since N has size ω_1, it follows that T_δ has no more than ω_1 many branches of length ω_1. This completes the analysis of T_δ.

Let T be the disjoint sum of the trees T_δ, for δ an uncountable ordinal in N. In other words, the underlying set of T consists of pairs of the form (δ, t), where δ is an uncountable ordinal in N and $t \in T_\delta$. And the order on T is given by letting $(\delta_1, t_1) <_T (\delta_2, t_t)$ iff $\delta_1 = \delta_2$ and $t_1 <_{T_\delta} t_2$. Then T is a tree of height and size ω_1 which has at most ω_1 many branches of length ω_1.

Suppose that T is special, and we will show that N is indestructibly ω_1-guessing. Let W be an outer model of V with $\omega_1^V = \omega_1^W$. Assume that d is a bounded subset of $N \cap On$ in W which is countably approximated by N. We will show that d is N-guessed. Fix an uncountable ordinal δ in N such that $d \subseteq \delta$.

Let $h : N \cap \delta \to 2$ be the characteristic function of d, in other words, $h(\alpha) = 1$ if $\alpha \in d$, and $h(\alpha) = 0$ otherwise. We claim that for all $i < \omega_1$, $(i, h \restriction N_i)$ is in T_δ. It suffices to show that $h \restriction N_i$ is in N. Since $N_i \in N$, N_i is countable, and d is countably approximated by N, it follows that $d \cap N_i \in N$. But $h \restriction N_i$ is the characteristic function of $d \cap N_i$, so $h \restriction N_i \in N$. Hence $(i, h \restriction N_i)$ is in T_δ.

It follows that $b := \{(i, h \restriction N_i) : i < \omega_1\}$ is a branch of T_δ, and hence of T, with length ω_1. Since T is special, any branch of T in W with length ω_1 is in V by Proposition 4.3. It follows that $b \in V$. Therefore $h \in V$, and hence $d \in V$. Since d is countably approximated by N in W, it is also countably approximated by N in V. As N is ω_1-guessing in V, d is N-guessed. □

Corollary 4.5. *Suppose that every tree of height and size ω_1 which has no branches of length ω_1 is special. Assume that for all sufficiently large regular cardinals $\theta \geq \omega_2$, there are stationarily many sets N in $P_{\omega_2}(H(\theta))$ such that N is internally unbounded and ω_1-guessing. Then IGMP holds.*

Proof. By Theorem 4.2, every tree of height and size ω_1 which has at most ω_1 many branches of length ω_1 is special. By the comments after Definition 3.2, it suffices to show that for all sufficiently large regular cardinals $\theta \geq \omega_2$, $\mathsf{IGMP}(\theta)$ holds. By assumption, for all sufficiently large regular cardinals $\theta \geq \omega_2$, there are stationarily many $N \in P_{\omega_2}(H(\theta))$ such that $N \prec H(\theta)$, N is internally unbounded, and N is ω_1-guessing. By Proposition 4.4, for any such N there exists a tree T with height and size ω_1 which has at most ω_1 many branches of length ω_1 such that if T is special then N is indestructibly ω_1-guessing. By our assumption about trees, T is indeed special, so N is indestructibly ω_1-guessing. □

Corollary 4.6. *PFA implies IGMP.*

Proof. By [10, Section 4], PFA implies that for all regular cardinals $\theta \geq \omega_2$, there are stationarily many N in $P_{\omega_2}(H(\lambda))$ which are internally unbounded and ω_1-guessing. By [2], MA, and hence PFA, implies that every tree of height and size ω_1 which has no branches of length ω_1 is special. The result now follows from Corollary 4.5. □

Corollary 3.4 and Proposition 4.4 suggest an alternative definition of indestructibly ω_1-guessing. Let us call an internally unbounded ω_1-guessing model a *special ω_1-guessing model* if some tree as described in the proof of Proposition 4.4 is special. The argument of Proposition 4.4 shows that in that case, N is ω_1-guessing in any outer model W with the same ω_1. This conclusion about N is apparently stronger than being indestructibly ω_1-guessing, since the latter property is restricted to outer models W which are generic extensions of V.

Thus we could formulate another principle which asserts that there exist stationarily many special ω_1-guessing models, and this principle clearly implies IGMP. Note that by the proof of Corollary 4.6, PFA implies this principle. We do not know whether the two principles are equivalent, so we leave this as an open question. They are equivalent if IGMP implies that every tree of height and size ω_1 with at most ω_1 many branches is special, but that is not known.

5. Strong Genericity and the Strongly Proper Collapse

We now turn to developing the forcing posets which will be used in the consistency result of Section 8. In this section we review the ideas of strong genericity and strong properness, prove a theorem about the preservation of strong properness by proper forcing, and discuss the strongly proper collapse. More details on these topics can be found in [5].

Definition 5.1. *Let $\mathbb{Q}$ be a forcing poset, $q \in \mathbb{Q}$, and N a set. Then q is a* strongly $(N, \mathbb{Q})$-generic condition *if for any set D which is a dense subset of the forcing poset $N \cap \mathbb{Q}$, D is predense in $\mathbb{Q}$ below q.*

If $\mathbb{Q}$ is understood from context, we say that q is a strongly N-generic condition.

Lemma 5.2. *Let $\mathbb{Q}$ be a forcing poset, $q \in \mathbb{Q}$, and N a set. Then q is strongly N-generic iff there exists a function $r \mapsto r \restriction N$, defined on conditions $r \leq q$, satisfying that $r \restriction N \in N \cap \mathbb{Q}$, and for all $v \leq r \restriction N$ in $N \cap \mathbb{Q}$, r and v are compatible.*

Proof. See [5, Lemma 2.2]. □

For a forcing poset $\mathbb{Q}$, let $\lambda_{\mathbb{Q}}$ denote the smallest cardinal λ such that $\mathbb{Q} \subseteq H(\lambda)$.

Definition 5.3. *A forcing poset $\mathbb{Q}$ is* strongly proper on a stationary set *if there are stationarily many N in $P_{\omega_1}(H(\lambda_{\mathbb{Q}}))$ such that whenever $p \in N \cap \mathbb{Q}$, there is $q \leq p$ which is a strongly N-generic condition.*

Standard arguments show that being strongly proper on a stationary set is equivalent to the property above, where we replace $\lambda_{\mathbb{Q}}$ with any cardinal $\theta \geq \lambda_{\mathbb{Q}}$.

Proposition 5.4. *If $\mathbb{Q}$ is strongly proper on a stationary set, then $\mathbb{Q}$ has the ω_1-covering property and the ω_1-approximation property.*

Proof. A condition which is strongly N-generic is also N-generic in the sense of proper forcing. By standard proper forcing arguments, $\mathbb{Q}$ has the ω_1-covering property. For a proof that $\mathbb{Q}$ has the ω_1-approximation property, see the comments after [5, Proposition 2.13]. □

Theorem 5.5. *Suppose that $\mathbb{Q}$ is strongly proper on a stationary set, and $\mathbb{P}$ is proper. Then $\mathbb{P}$ forces that $\mathbb{Q}$ is strongly proper on a stationary set.*

Proof. Fix θ such that $\mathbb{P}$ forces that θ is a cardinal and $\theta \geq \lambda_{\mathbb{Q}}$. Fix a $\mathbb{P}$-name $\dot{F}$ for a function from $(H(\theta)^{V[\dot{G}_{\mathbb{P}}]})^{<\omega}$ to $H(\theta)^{V[\dot{G}_{\mathbb{P}}]}$, and let $p \in \mathbb{P}$. We will find $u \leq p$, and a name $\dot{M}$ for a countable subset of $H(\theta)^{V[\dot{G}_{\mathbb{P}}]}$ which is closed under $\dot{F}$, such that u forces that for all $s \in \dot{M} \cap \mathbb{Q}$, there is $t \leq s$ which is strongly $(\dot{M}, \mathbb{Q})$-generic.

Let χ be a regular cardinal larger than $2^{|\mathbb{P}|}$ such that $\mathbb{P}$, $\mathbb{Q}$, θ, and $\dot{F}$ are in $H(\chi)$. Since $\mathbb{P}$ is proper and $\mathbb{Q}$ is strongly proper on a stationary set, we can fix N in $P_{\omega_1}(H(\chi))$ satisfying:

(1) $N \prec (H(\chi), \in, \mathbb{P}, p, \mathbb{Q}, \theta, \dot{F})$;
(2) for all $p_0 \in N \cap \mathbb{P}$, there is $q_0 \leq p_0$ which is $(N, \mathbb{P})$-generic;
(3) for all $s \in N \cap \mathbb{Q}$, there is $t \leq s$ which is strongly $(N, \mathbb{Q})$-generic.

Since $p \in N \cap \mathbb{P}$, by (2) we can fix $q \leq p$ which is $(N, \mathbb{P})$-generic. We claim that q forces that

$$\dot{M} := N[\dot{G}_{\mathbb{P}}] \cap H(\theta)^{V[\dot{G}_{\mathbb{P}}]}$$

is as required.

Since q is $(N, \mathbb{P})$-generic, q forces that $N[\dot{G}_{\mathbb{P}}] \cap V = N$. By (1), $\mathbb{P}$ forces that

$$N[\dot{G}_{\mathbb{P}}] \prec (H(\chi)^{V[\dot{G}_{\mathbb{P}}]}, \in, \dot{F}),$$

and therefore that $N[\dot{G}_{\mathbb{P}}]$ is closed under $\dot{F}$. Hence $\mathbb{P}$ forces that $\dot{M}$ is closed under $\dot{F}$.

Let $r \leq q$ and $\dot{s}$ be given such that r forces in $\mathbb{P}$ that $\dot{s} \in \dot{M} \cap \mathbb{Q}$. Then r forces that $\dot{s} \in N[\dot{G}_{\mathbb{P}}] \cap \mathbb{Q} \subseteq N[\dot{G}_{\mathbb{P}}] \cap V = N$. So we can fix $u \leq r$ and $s \in N$ such that u forces that $\dot{s} = \check{s}$.

By (3), let $t \leq s$ be strongly $(N, \mathbb{Q})$-generic. Then there exists a function $g : \{z \in \mathbb{Q} : z \leq t\} \to N \cap \mathbb{Q}$ satisfying that for all $z \leq t$ in $\mathbb{Q}$, if $w \leq g(z)$ is in $N \cap \mathbb{Q}$, then w and z are compatible in $\mathbb{Q}$. Note that by upwards absoluteness, g is forced to satisfy the same property in $V[\dot{G}_{\mathbb{P}}]$. But u forces that $N \cap \mathbb{Q} = N[\dot{G}_{\mathbb{P}}] \cap \mathbb{Q} = \dot{M} \cap \mathbb{Q}$. Therefore u forces that $g : \{z \in \mathbb{Q} : z \leq t\} \to \dot{M} \cap \mathbb{Q}$ and for all $z \leq t$ in $\mathbb{Q}$, whenever $w \leq g(z)$ is in $\dot{M} \cap \mathbb{Q}$, then w and z are compatible in $\mathbb{Q}$. In other words, u forces that t is strongly $(\dot{M}, \mathbb{Q})$-generic. $\square$

Assume that κ is a strongly inaccessible cardinal. In the proof of the consistency result in Section 8, we will use the forcing poset $\mathbb{P}$ from [5, Section 6], which is called a strongly proper collapse. The forcing poset $\mathbb{P}$ is strongly proper, κ-c.c., has size κ, and collapses κ to become ω_2. Roughly speaking, this forcing poset consists of finite adequate sets of countable elementary substructures, ordered by reverse inclusion. A more detailed description of this forcing poset is beyond the scope of this paper; see [5, Section 6] for more details.

We will need one more technical fact about the strongly proper collapse $\mathbb{P}$.

Proposition 5.6. *Let $\lambda \geq \kappa$ be a cardinal. Then $\mathbb{P} \times Add(\omega, \lambda)$ is strongly proper. Moreover, if $\mathbb{P}_0$ is any regular suborder of $\mathbb{P} \times Add(\omega, \lambda)$, then $\mathbb{P}_0$ forces that $(\mathbb{P} \times Add(\omega, \lambda))/\dot{G}_{\mathbb{P}_0}$ is strongly proper on a stationary set.*

Proof. This follows from Theorem 2.11 and Proposition 7.3 of [5]. $\square$

6. Special iterations

To obtain a model in which IGMP holds, we will use a finite support iteration $\mathbb{P}$ of forcings which specialize trees of height and size ω_1 which have no branches of length ω_1. It was proven recently in [4] that such an iteration has the ω_1-approximation property.

For our purposes, we will need to know that a certain quotient of such an iteration has the ω_1-approximation property. Specifically, we will have an elementary substructure N of size ω_1, and we will need to know that the regular suborder $N \cap \mathbb{P}$ forces that $\mathbb{P}/\dot{G}_{N \cap \mathbb{P}}$ has the ω_1-approximation property. Unlike the situation in [5], we do not have a general result which implies that such a quotient has the ω_1-approximation property. Instead, we will prove directly that $\mathbb{P}/\dot{G}_{N \cap \mathbb{P}}$ is forced by $N \cap \mathbb{P}$ to be forcing equivalent to a finite support iteration of specializing forcings.

Let $\mathrm{Fn}(\omega_1, \omega)$ denote the set of all finite functions whose domain is a subset of ω_1 and whose range is a subset of ω. Recall that if T is a tree with no branches of length ω_1, then $P(T)$ is the forcing poset described in Section 1 for adding a specializing function to T.

Definition 6.1. *For an ordinal λ, let $S(\lambda)$ denote the set of all functions p, whose domain is a finite subset of λ, such that for all $\alpha \in \mathrm{dom}(p)$, $p(\alpha) \in Fn(\omega_1, \omega)$. Define a partial order on $S(\lambda)$ by letting $q \leq p$ if $\mathrm{dom}(p) \subseteq \mathrm{dom}(q)$, and for all $\alpha \in \mathrm{dom}(p)$, $p(\alpha) \subseteq q(\alpha)$.*

Note that if W is an outer model of V with $\omega_1^V = \omega_1^W$, then $S(\lambda)^V = S(\lambda)^W$, and the order on $S(\lambda)$ is the same in V and W.

Instead of working directly with forcing iterations, we will use a simpler dense suborder.

Definition 6.2. *For an ordinal λ and a set $A \subseteq \lambda$, a sequence $\langle \mathbb{P}_i : i \leq \lambda \rangle$ is said to be a* special A-iteration *if there a exist sequence $\langle \dot{T}_i : i \in A \rangle$ such that the following statements are satisfied:*

1. *for all $i \leq \lambda$, $\mathbb{P}_i$ is a suborder of $S(i)$;*
2. *for all $i \in A$, $\dot{T}_i$ is a $\mathbb{P}_i$-name for a tree with underlying set ω_1 which has no branches of length ω_1;*
3. *$\mathbb{P}_0 = \{\emptyset\}$;*
4. *for all $i \in A$, a set $p \in S(i+1)$ is in $\mathbb{P}_{i+1}$ iff $p \restriction i \in \mathbb{P}_i$ and if $i \in \operatorname{dom}(p)$ then $p \restriction i \Vdash_{\mathbb{P}_i} p(i) \in P(\dot{T}_i)$;*
5. *for all $i \in \lambda \setminus A$, $\mathbb{P}_{i+1} = \mathbb{P}_i$;*
6. *for all $\beta \leq \lambda$ limit, a set $p \in S(\beta)$ is in $\mathbb{P}_\lambda$ iff for all $i < \beta$, $p \restriction i \in \mathbb{P}_i$.*

Note that if $p \in \mathbb{P}_\lambda$, then $\operatorname{dom}(p) \subseteq A$. If $A = \lambda$, then we say that the sequence is a *special iteration*. The partial ordering $\mathbb{P}_\lambda$ itself is said to be a special A-iteration.

The next two lemmas provide some basic facts about special A-iterations.

Lemma 6.3. *Let $\langle \mathbb{P}_i : i \leq \lambda \rangle$ be a special A-iteration. Let $i < j \leq \lambda$. Then:*

1. *$\mathbb{P}_i \subseteq \mathbb{P}_j$;*
2. *if $p \in \mathbb{P}_j$, then $p \restriction i \in \mathbb{P}_i$;*
3. *if $p \in \mathbb{P}_j$ and $q \leq p \restriction i$ in $\mathbb{P}_i$, then $q \cup p \restriction [i, j)$ is in $\mathbb{P}_j$ and is below p;*
4. *the function $p \mapsto p \restriction i$ is a projection mapping of $\mathbb{P}_j$ onto $\mathbb{P}_i$, and this map satisfies that $p \restriction i = p$ for all $p \in \mathbb{P}_i$, and $q \leq q \restriction i$ for all $q \in \mathbb{P}_j$;*
5. *$\mathbb{P}_i$ is a regular suborder of $\mathbb{P}_j$.*

Proof. (1), (2), and (3) can be easily proven by induction. (4) follows from (3), and (4) implies (5) by Lemma 1.5. □

Lemma 6.4. *Let $\langle \mathbb{P}_i : i \leq \lambda \rangle$ be a special A-iteration, with sequence of names $\langle \dot{T}_i : i \in A \rangle$. Let p and q be conditions in $\mathbb{P}_\lambda$ satisfying that for all $i \in \operatorname{dom}(p) \cap \operatorname{dom}(q)$, $p(i) \subseteq q(i)$. Define $p + q$ as the function with domain equal to $\operatorname{dom}(p) \cup \operatorname{dom}(q)$, such that for all $i \in \operatorname{dom}(p+q)$, if $i \in \operatorname{dom}(q)$ then $(p+q)(i) = q(i)$, and if $i \in \operatorname{dom}(p) \setminus \operatorname{dom}(q)$, then $(p+q)(i) = p(i)$. Then $p + q$ is in $\mathbb{P}_\lambda$, and $p + q \leq p, q$.*

Proof. Let $r := p + q$. It is clear that r is in $S(\lambda)$, and $r \leq p, q$ in $S(\lambda)$. In fact, for all $i \leq \lambda$, $r \restriction i \in S(i)$, and $r \restriction i \leq p \restriction i, q \restriction i$ in $S(i)$. To show that $r \in \mathbb{P}_\lambda$, we will prove by induction that $r \restriction i \in \mathbb{P}_i$ for all $i \leq \lambda$.

For $i = 0$, $r \restriction 0 = \emptyset$ is in $\mathbb{P}_0$ by Definition 6.2(3). If $\beta \leq \lambda$ is a limit ordinal and for all $\gamma < \beta$, $r \restriction \gamma \in \mathbb{P}_\gamma$, then $r \in \mathbb{P}_\beta$ by Definition 6.2(6).

Suppose that $i = i_0 + 1$ and $r \restriction i_0 \in \mathbb{P}_{i_0}$. Then $(r \restriction i) \restriction i_0 = r \restriction i_0 \in \mathbb{P}_{i_0}$. If $i_0 \notin \operatorname{dom}(r)$, then $r \restriction i = r \restriction i_0 \in \mathbb{P}_{i_0} \subseteq \mathbb{P}_i$, and we are done. Suppose that $i_0 \in \operatorname{dom}(r)$. Then $i_0 \in A$, and $r(i_0) = s(i_0)$, where s is either p or q. But $r \restriction i_0 \leq s \restriction i_0$, and $s \restriction i_0 \Vdash_{\mathbb{P}_{i_0}} s(i_0) \in P(\dot{T}_{i_0})$. Hence $r \restriction i_0 \Vdash_{\mathbb{P}_{i_0}} r(i_0) = s(i_0) \in P(\dot{T}_{i_0})$. So $r \in \mathbb{P}_i$. □

The next lemma says that a special A-iteration is forcing equivalent to a finite support iteration of specializing forcings.

Lemma 6.5. *Let $\langle \mathbb{P}_i : i \leq \lambda \rangle$ be a special A-iteration, with sequence of names $\langle \dot{T}_i : i \in A \rangle$. Then there exists a finite support iteration*

$$\langle \mathbb{P}^*_i, \dot{\mathbb{Q}}^*_j : i \leq \lambda, j < \lambda \rangle$$

satisfying the following properties:

(1) *for all $i < \lambda$, the function which send $p \in \mathbb{P}_i$ to $p^* \in \mathbb{P}^*_i$, where* $\mathrm{dom}(p^*) = \mathrm{dom}(p)$ *and for all $\alpha \in \mathrm{dom}(p^*)$, $p^*(\alpha)$ is the canonical $\mathbb{P}^*_\alpha$-name for $p(\alpha)$, is an isomorphism from $\mathbb{P}_i$ into a dense suborder of the separative quotient of $\mathbb{P}^*_i$;*

(2) *for all $i \in A$, $\mathbb{P}^*_i$ forces that $\dot{\mathbb{Q}}^*_i = P(\dot{T}^*_i)$, where $\dot{T}^*_i$ is the canonical translation of the $\mathbb{P}_i$-name $\dot{T}_i$ to a $\mathbb{P}^*_i$-name using the isomorphism described in (1);*

(3) *for all $j \in \lambda \setminus A$, $\mathbb{P}^*_j$ forces that $\dot{\mathbb{Q}}_j = \{\emptyset\}$ is the trivial forcing poset.*

Proof. The proof follows by standard arguments. □

Corollary 6.6. *Let $\mathbb{P}$ be a special A-iteration. Then $\mathbb{P}$ is ω_1-c.c. and has the ω_1-approximation property.*

Proof. By Lemma 6.5, $\mathbb{P}$ is forcing equivalent to a finite support iteration of forcings which specialize trees of height and size ω_1 which have no branches of length ω_1. So $\mathbb{P}$ is forcing equivalent to a finite support iteration of Y-c.c. forcing posets, and hence is Y-c.c. But any Y-c.c. forcing poset is ω_1-c.c. and has the ω_1-approximation property. □

Proposition 6.7. *Let λ be a cardinal of cofinality at least 2^{ω_1}. Then there exists a special iteration $\langle \mathbb{P}_i : i \leq \lambda \rangle$ which forces that every tree with underlying set ω_1 which has no branches of length ω_1 is special.*

Proof. We give a sketch of the proof. The special iteration is constructed by induction. We only need to specify the names $\dot{T}_j$ for $j < \lambda$, since the rest of the definition is determined by conditions 3–6 of Definition 6.2.

For $j < \lambda$, $\mathbb{P}_j$ will be a subset of $S(j)$, and hence will have size at most $\omega_1 \cdot |j|$, which is less than λ. Also $\mathbb{P}_j$ is ω_1-c.c. So it is possible to enumerate all nice $\mathbb{P}_j$-names for trees with underlying set ω_1 with no branches of length ω_1 in order type less than or equal to λ. Using a bookkeeping function, we choose $\dot{T}_j$ to be such a name which was enumerated at some stage less than or equal to j. The bookkeeping function will ensure that any name that is enumerated will eventually be chosen as $\dot{T}_j$ for some $j < \lambda$.

Given a nice $\mathbb{P}_\lambda$-name for a tree with underlying set ω_1 with no branches of length ω_1, since the cofinality of λ is greater than ω_1 and $\mathbb{P}_\lambda$ is ω_1-c.c., it is easy to see that the name is actually a $\mathbb{P}_j$-name for some $j < \lambda$. So at some stage earlier than λ, we forced with $P(\dot{T}_j)$, and specialized the tree $\dot{T}_j$. □

Corollary 6.8. *Let λ be a cardinal with cofinality at least 2^{ω_1}. Then there exists a special iteration $\langle \mathbb{P}_i : i \leq \lambda \rangle$ which forces that every tree with height and size ω_1 which has no branches of length ω_1 is special.*

Proof. If T is a tree with height and size ω_1, then clearly T is isomorphic to a tree with underlying set ω_1. Now apply Proposition 6.7. □

We conclude this section by proving that a tail of a special iteration is forcing equivalent to a special iteration in an intermediate extension. The proof is elementary, but somewhat tedious; the reader should not feel guilty in just accepting the statement of Proposition 6.9 and skipping the proof.

Fix a special A-iteration $\langle \mathbb{P}_i : i \leq \lambda \rangle$, with sequence of names $\langle \dot{T}_i : i \in A \rangle$. Let $i < j \leq \lambda$, and suppose that G_i is a V-generic filter on $\mathbb{P}_i$. By Lemma 1.1, if $G_{i,j}$ is a $V[G_i]$-generic filter on $\mathbb{P}_j / G_i$, then $G_{i,j}$ is a V-generic filter on $\mathbb{P}_j$, $G_{i,j} \cap \mathbb{P}_i = G_i$, and $V[G_i][G_{i,j}] = V[G_{i,j}]$. In particular, if $j \in A$, then in $V[G_i][G_{i,j}] = V[G_{i,j}]$, $\dot{T}_j^{G_{i,j}}$ is a tree with underlying set ω_1 which has no branches of length ω_1. In this situation, we will write $\dot{T}_{i,j}$ for a $\mathbb{P}_j / G_i$-name in $V[G_i]$ for this tree.

Proposition 6.9. *Let $i < \lambda$, and let G_i be a V-generic filter on $\mathbb{P}_i$. Then in $V[G_i]$, there exists a special $(A \setminus i)$-iteration $\langle \mathbb{P}'_j : j \leq \lambda \rangle$, with sequence of names $\langle \dot{T}'_j : j \in A \setminus i \rangle$, satisfying the following properties:*

1. *for each $j \leq \lambda$, the map $p \mapsto p \restriction (A \setminus i)$ is a dense embedding of $\mathbb{P}_j / G_i$ into $\mathbb{P}'_j$;*
2. *for each $j \in A \setminus i$, $\dot{T}'_j$ is the canonical translation of the name $\dot{T}_{i,j}$ to a $\mathbb{P}'_j$-name using the dense embedding from (1).*

Proof. First consider $j \leq i$. Then $\mathbb{P}'_j$ is the trivial poset $\{\emptyset\}$, and $\mathbb{P}_j / G_i$ is equal to $G_i \cap \mathbb{P}_j$. Using the fact that $G_i \cap \mathbb{P}_j$ is a filter to show preservation of incompatibility, easily the map $p \mapsto p \restriction (A \setminus i) = \emptyset$ is a dense embedding of $\mathbb{P}_j / G_i = G_j \cap \mathbb{P}_j$ into $\mathbb{P}'_j = \{\emptyset\}$.

Suppose that $i \leq \beta \leq \lambda$ is limit ordinal, and for all $\gamma < \beta$, the map $p \mapsto p \restriction (A \setminus i)$ is a dense embedding of $\mathbb{P}_\gamma / G_i$ into $\mathbb{P}'_\gamma$. Using the fact that $\mathbb{P}_\beta / G_i = \bigcup \{\mathbb{P}_\gamma / G_i : \gamma < \beta\}$ and $\mathbb{P}'_\beta = \bigcup \{\mathbb{P}'_\gamma : \gamma < \beta\}$, it is easy to check that the same is true of $\mathbb{P}_\beta / G_i$ and $\mathbb{P}'_\beta$.

Finally, assume that $i < j < \lambda$, and the map $p \mapsto p \restriction (A \setminus i)$ is a dense embedding of $\mathbb{P}_j / G_i$ into $\mathbb{P}'_j$. We will prove that the same is true for $\mathbb{P}_{j+1} / G_i$ and $\mathbb{P}'_{j+1}$. First assume that $j \notin A$. Then $\mathbb{P}_{j+1} = \mathbb{P}_j$, so $\mathbb{P}_{j+1} / G_i = \mathbb{P}_j / G_i$. Also $\mathbb{P}'_{j+1} = \mathbb{P}'_j$. So we are done by the inductive hypothesis.

Suppose that $j \in A$. In $V[G_i]$, $\dot{T}_{i,j}$ is a $\mathbb{P}_j / G_i$-name for a tree with underlying set ω_1 which has no branches of length ω_1. Since $p \mapsto p \restriction (A \setminus i)$ is a dense embedding of $\mathbb{P}_j / G_i$ into $\mathbb{P}'_j$, the name $\dot{T}_{i,j}$ translates under this dense embedding to a $\mathbb{P}'_j$-name $\dot{T}'_j$ for the same tree.

Assume that $q \in \mathbb{P}_{j+1} / G_i$, and we will show that $q \restriction (A \setminus i)$ is in $\mathbb{P}'_{j+1}$. This follows from the inductive hypothesis if $j \notin \mathrm{dom}(q)$, so suppose that $j \in \mathrm{dom}(q)$. Since $q \restriction j \in \mathbb{P}_j / G_i$, by the inductive hypothesis $(q \restriction j) \restriction (A \setminus i) = (q \restriction (A \setminus i)) \restriction j$ is in $\mathbb{P}'_j$. As $q \in \mathbb{P}_{j+1}$, $q \restriction j \Vdash^V_{\mathbb{P}_j} q(j) \in P(\dot{T}_j)$. By the choice of $\dot{T}_{i,j}$, $q \restriction j \Vdash^{V[G_i]}_{\mathbb{P}_j / G_i} q(j) \in P(\dot{T}_{i,j})$. Hence $(q \restriction (A \setminus i)) \restriction j \Vdash^{V[G_i]}_{\mathbb{P}'_j} q(j) \in P(\dot{T}'_j)$. So $q \restriction (A \setminus i) \in \mathbb{P}'_{j+1}$.

Let $p \in \mathbb{P}'_{j+1}$. We will find a condition u in $\mathbb{P}_{j+1} / G_i$ such that $u \restriction (A \setminus i) \leq p$. If $j \notin \mathrm{dom}(p)$, then $p \in \mathbb{P}'_j$. By the inductive hypothesis, there is $q \in \mathbb{P}_j / G_i$ such that $q \restriction (A \setminus i) \leq p$. Since $\mathbb{P}_j \subseteq \mathbb{P}_{j+1}$ and $q \restriction i \in G_i$, $q \in \mathbb{P}_{j+1} / G_i$, and we are done.

Suppose that $j \in \mathrm{dom}(p)$. Let $x := p(j)$. By the inductive hypothesis, fix $q \in \mathbb{P}_j / G_i$ such that $q \restriction (A \setminus i) \leq p \restriction i$. The proof would be finished if $q \cup \{(j, x)\}$ was in $\mathbb{P}_{j+1} / G_i$, that is, if $q \cup \{(j, x)\}$ was in $\mathbb{P}_{j+1}$. Unfortunately, we do not know that q forces over V that x is in $P(\dot{T}_j)$, so it could be the case that $q \cup \{(j, x)\}$ is not in $\mathbb{P}_{j+1}$. So we have to work a bit harder.

Since $q \restriction (A \setminus i) \leq p \restriction i$, it follows that

$$q \restriction (A \setminus i) \Vdash^{V[G_i]}_{\mathbb{P}'_j} x \in P(\dot{T}'_j).$$

We claim that

$$q \Vdash^{V[G_i]}_{\mathbb{P}_j/G_i} x \in P(\dot{T}_{i,j}).$$

Let $G_{i,j}$ be a $V[G_i]$-generic filter on $\mathbb{P}_j/G_i$ with $q \in G_{i,j}$. Let $T := \dot{T}^{G_{i,j}}_{i,j}$. We will prove that $x \in P(T)$. The image of $G_{i,j}$ under the dense embedding $s \mapsto s \restriction (A \setminus i)$ generates a $V[G_i]$-generic filter G'_j on $\mathbb{P}'_j$. Since $q \in G_{i,j}$, $q \restriction (A \setminus i) \in G'_j$. And since $q \restriction (A \setminus i) \leq p \restriction j$, $p \restriction j \in G'_j$. As $p \in \mathbb{P}'_{j+1}$, by definition we have that

$$p \restriction j \Vdash^{V[G_i]}_{\mathbb{P}'_j} p(j) = x \in P(\dot{T}'_j).$$

Since $p \in G'_j$, it follows that $x \in P((\dot{T}'_j)^{G'_j})$. But by the choice of the name $\dot{T}'_j$, $(\dot{T}'_j)^{G'_j} = (\dot{T}_{i,j})^{G_{i,j}} = T$. So $x \in P(T)$, proving the claim.

Fix a condition $s \in G_i$ such that

$$s \Vdash^V_{\mathbb{P}_i} q \Vdash^{V[\dot{G}_i]}_{\mathbb{P}_j/\dot{G}_i} x \in P(\dot{T}_{i,j}).$$

Since $q \restriction i \in G_i$, without loss of generality we may assume that $s \leq q \restriction i$. It follows that $t := s \cup (q \restriction [i, \lambda))$ is in $\mathbb{P}_j$, $t \leq q$, and since $t \restriction i = s \in G_i$, $t \in \mathbb{P}_j/G_i$. Now the choice of s implies that

$$t \Vdash^V_{\mathbb{P}_j} p(j) \in \dot{T}_j,$$

as can be easily checked. So $u := t \cup \{(j, x)\}$ is in $\mathbb{P}_{j+1}/G_i$, and $u \restriction (A \setminus i) \leq p$.

It is obvious that the map $p \mapsto p \restriction (A \setminus i)$ preserves order. For the preservation of incompatibility, suppose that p and q are in $\mathbb{P}_{j+1}/G_i$, and $r \leq p \restriction (A \setminus i), q \restriction (A \setminus i)$ in $\mathbb{P}'_{j+1}$. By what we just proved, there is $r_0 \in \mathbb{P}_{j+1}/G_i$ such that $r_0 \restriction (A \setminus i) \leq r$ in $\mathbb{P}'_{j+1}$. Since G_i is a filter, without loss of generality we may assume that $r_0 \restriction i \leq p \restriction i, q \restriction i$. Then easily $r_0 \leq p, q$ in $\mathbb{P}_{j+1}/G_i$. ☐

Corollary 6.10. *Let $\langle \mathbb{P}_i : i \leq \lambda \rangle$ be a special A-iteration. Fix $i < \lambda$, and let G_i be a V-generic filter on $\mathbb{P}_i$. Then in $V[G_i]$, for all $j \leq \lambda$, $\mathbb{P}_j/G_i$ is ω_1-c.c. and has the ω_1-approximation property.*

Proof. By Proposition 6.9, $\mathbb{P}_j/G_i$ is forcing equivalent to a special $(A \setminus i)$-iteration. So we are done by Corollary 6.6. ☐

7. Factoring a special iteration over an elementary substructure

In this section we will prove the following result.

Theorem 7.1. *Let $\vec{P} = \langle \mathbb{P}_i : i \leq \lambda \rangle$ be a special iteration, with sequence of names $\vec{T} = \langle \dot{T}_i : i < \lambda \rangle$. Let $\theta \geq \omega_2$ be a regular cardinal such that $\vec{P}$ and $\vec{T}$ are in $H(\theta)$. Let N be an elementary substructure of $H(\theta)$ such that N has size ω_1, $\omega_1 \subseteq N$, and $\vec{P}$ and $\vec{T}$ are in N.*

Then:

1. *$N \cap \mathbb{P}_\lambda$ is a regular suborder of $\mathbb{P}_\lambda$;*
2. *$N \cap \mathbb{P}_\lambda$ forces that $\mathbb{P}_\lambda/\dot{G}_{N \cap \mathbb{P}_\lambda}$ is forcing equivalent to a special $(\lambda \setminus N)$-iteration;*
3. *$N \cap \mathbb{P}_\lambda$ forces that $\mathbb{P}_\lambda/\dot{G}_{N \cap \mathbb{P}_\lambda}$ is ω_1-c.c. and has the ω_1-approximation property.*

Note that (3) follows from (2) by Corollary 6.6.

The proof of Theorem 7.1(1) is given in Lemma 7.5(2). The rest of the section after that is devoted to proving Theorem 7.1(2). The proof of Theorem 7.1(2) is tedious; we suggest that the reader skip it on a first reading.

Fix for the remainder of the section $\vec{P}$, $\vec{T}$, θ, and N as described in Theorem 7.1.

Lemma 7.2. *Let $i < j \leq \lambda$, where $j \in N$.*

(1) *the map $p \mapsto p \upharpoonright i$ is a projection mapping from $\mathbb{P}_j \cap N$ into $\mathbb{P}_i \cap N$;*
(2) *$p \upharpoonright i = p$ for all $p \in \mathbb{P}_i \cap N$, and $q \leq q \upharpoonright i$ for all $q \in \mathbb{P}_j \cap N$;*
(3) *$\mathbb{P}_i \cap N$ is a regular suborder of $\mathbb{P}_j \cap N$.*

Proof. (2) follows from Lemma 6.3(4), and (3) follows from (1), (2), and Lemma 1.5. It remains to prove (1). Let $i_N := \min((N \cap (\lambda+1)) \setminus i)$. By Lemma 6.3(4), the map $p \mapsto p \upharpoonright i_N$ is a projection mapping of $\mathbb{P}_j$ into $\mathbb{P}_{i_N}$. Using the elementarity of N, it easily follows that the same map restricted to $\mathbb{P}_j \cap N$ is a projection mapping of $\mathbb{P}_j \cap N$ into $\mathbb{P}_{i_N} \cap N$. But if $q \in \mathbb{P}_{i_N}$, then by elementarity, $\mathrm{dom}(q) \subseteq i_N \cap N \subseteq i$, and hence $q \in \mathbb{P}_i$. It follows that $\mathbb{P}_{i_N} \cap N = \mathbb{P}_i \cap N$, and $p \upharpoonright i = p \upharpoonright i_N$ for all $p \in \mathbb{P}_j \cap N$. So this map is a projection mapping of $\mathbb{P}_j \cap N$ into $\mathbb{P}_i \cap N$. $\square$

Definition 7.3. *For each $i \leq \lambda$, let E_i denote the set of $p \in \mathbb{P}_i$ such that $p \upharpoonright N$, the restriction of the function p to $N \cap i$, is in $\mathbb{P}_i$.*

Lemma 7.4. *Let $i < j \leq \lambda$. Then:*

(1) *$E_i \subseteq E_j$;*
(2) *if $p \in E_j$, then $p \upharpoonright i \in E_i$.*

Proof. Let $p \in E_i$. Then $p \in \mathbb{P}_i$ and $p \upharpoonright N \in \mathbb{P}_i$. Since $\mathbb{P}_i \subseteq \mathbb{P}_j$ by Lemma 6.3(1), it follows that $p \in \mathbb{P}_j$ and $p \upharpoonright N \in \mathbb{P}_j$. So $p \in E_j$, which proves (1). For (2), let $p \in E_j$. Then $p \in \mathbb{P}_j$ and $p \upharpoonright N \in \mathbb{P}_j$. By Lemma 6.3(2), $p \upharpoonright i \in \mathbb{P}_i$ and $(p \upharpoonright i) \upharpoonright N = (p \upharpoonright N) \upharpoonright i \in \mathbb{P}_i$. So $p \upharpoonright i \in E_i$. $\square$

We now prove Theorem 7.1(1). We also prove that E_i is dense in $\mathbb{P}_i$, since that fact follows by the same argument.

Lemma 7.5. *Let $i \leq \lambda$. Then:*

(1) *E_i is dense in $\mathbb{P}_i$;*
(2) *$N \cap \mathbb{P}_i$ is a regular suborder of $\mathbb{P}_i$.*

Proof. Suppose that p and q are in $N \cap \mathbb{P}_i$, and are compatible in $\mathbb{P}_i$. We will show that p and q are compatible in $\mathbb{P}_i \cap N$. Let $i_N := \min((N \cap (\lambda + 1)) \setminus i)$. Since $\mathbb{P}_i$ is a regular suborder of $\mathbb{P}_{i_N}$ by Lemma 6.3(5), p and q are compatible in $\mathbb{P}_{i_N}$. By elementarity, there is $r \in N \cap \mathbb{P}_{i_N}$ such that $r \leq p, q$. Then $\mathrm{dom}(r) \subseteq N \cap i_N \subseteq i$, so $r \in \mathbb{P}_i$. Hence $r \in \mathbb{P}_i \cap N$ and $r \leq p, q$.

Let A be a maximal antichain of $N \cap \mathbb{P}_i$, and we will show that A is predense in $\mathbb{P}_i$. Let $p \in \mathbb{P}_i$. We will find $r \in \mathbb{P}_i$ and $s \in A$ such that $r \leq s, p$. Moreover, we will have that $r \upharpoonright N \in \mathbb{P}_i$, and hence that $r \in E_i$. This proves both that A is predense in $\mathbb{P}_i$, and that E_i is dense in $\mathbb{P}_i$.

Since $\omega_1 \subseteq N$, $\mathrm{Fn}(\omega_1, \omega)$ is a subset of N. It follow that $p \upharpoonright N$ is equal to $p \cap N$, which is in N since $p \cap N$ is finite. However, we do not know that $p \cap N$ is in $\mathbb{P}_i$. Since $p \in \mathbb{P}_i \subseteq \mathbb{P}_{i_N}$, by the elementarity of N we can fix p' in $\mathbb{P}_{i_N} \cap N$ such that $p \upharpoonright N \subseteq p'$.

Since A is a maximal antichain in $\mathbb{P}_i \cap N$, A is also a maximal antichain of $\mathbb{P}_{i_N} \cap N$. So we can fix $s \in A$ and $q \in \mathbb{P}_{i_N} \cap N$ such that $q \leq p', s$. Then by elementarity, $\mathrm{dom}(q) \subseteq i_N \cap N \subseteq i$, so $q \in \mathbb{P}_i \cap N$. As $\mathrm{dom}(q) \subseteq N$ and $\mathrm{dom}(p \restriction N) \subseteq \mathrm{dom}(p') \subseteq \mathrm{dom}(q)$, we have that $\mathrm{dom}(p) \cap \mathrm{dom}(q) = \mathrm{dom}(p \restriction N)$. And since $p \restriction N \subseteq p'$ and $q \leq p'$, it follows that for all $i \in \mathrm{dom}(p \restriction N)$, $p(i) \subseteq q(i)$. By Lemma 6.4, $r := p + q$ exists and $r \leq p, q$. Hence $r \leq p, s$. Moreover, by the definition of $p + q$, $r \restriction N = q \in \mathbb{P}_i$, so $r \in E_i$. $\square$

Notation 7.6. *Fix $i \leq \lambda$, and let $G_{i,N}$ be a V-generic filter on $\mathbb{P}_i \cap N$. Let E_i^N denote the set $E_i \cap (\mathbb{P}_i / G_{i,N})$.*

Since E_i is dense in $\mathbb{P}_i$, E_i^N is dense in $\mathbb{P}_i/G_{i,N}$ by Lemma 1.3.

Lemma 7.7. *Fix $i \leq \lambda$, and let $G_{i,N}$ be a V-generic filter on $\mathbb{P}_i \cap N$. Then for all $p \in E_i$, $p \in E_i^N$ iff $p \restriction N \in G_{i,N}$.*

Proof. Using Lemma 6.4, it is easy to check that the map $p \mapsto p \restriction N$ is a projection mapping from E_i into $E_i \cap N = \mathbb{P}_i \cap N$, which satisfies that $p \restriction N = p$ for all $p \in E_i \cap N$, and $q \leq q \restriction N$ for all $q \in E_i$. By Lemma 1.5, it follows that the set of $p \in E_i$ such that p is compatible in E_i with every condition in $G_{i,N}$ is equal to the set of $p \in E_i$ such that $p \restriction N \in G_{i,N}$. But a condition in E_i is compatible in E_i with every condition in $G_{i,N}$ iff it is in $E_i \cap (\mathbb{P}_i/G_{i,N}) = E_i^N$. $\square$

Given a V-generic filter G_N on $\mathbb{P}_\lambda \cap N$, for each $i \leq \lambda$, let $G_{i,N} := G_N \cap \mathbb{P}_i$. Since $\mathbb{P}_i \cap N$ is a regular suborder of $\mathbb{P}_\lambda \cap N$, $G_{i,N}$ is a V-generic filter on $\mathbb{P}_i \cap N$. Also, for all $i < j \leq \lambda$, we have that $G_{i,N} \subseteq G_{j,N}$, and for all $s \in G_{j,N}$, $s \restriction i \in G_{i,N}$.

Lemma 7.8. *Let $i < j \leq \lambda$. Then:*

1. *$E_i^N \subseteq E_j^N$;*
2. *if $p \in E_j^N$, then $p \restriction i \in E_i^N$.*

Proof. (1) Let $p \in E_i^N$. Then $p \in E_i$, and by Lemma 7.7, $p \restriction N \in G_{i,N}$. But $E_i \subseteq E_j$ by Lemma 7.4(1), and $G_{i,N} \subseteq G_{j,N}$. So $p \in E_j$ and $p \restriction N \in G_{j,N}$. By Lemma 7.7, $p \in E_j^N$.

(2) Let $p \in E_j^N$. Then by Lemma 7.7, $p \in E_j$ and $p \restriction N \in G_{j,N}$. By Lemma 7.4(2), $p \restriction i \in E_i$, and by the comments preceding this lemma, $(p \restriction i) \restriction N = (p \restriction N) \restriction i$ is in $G_{i,N}$. By Lemma 7.7, $p \restriction i \in E_i^N$. $\square$

Notation 7.9. *Let $i \leq \lambda$. In $V[G_{i,N}]$, define $\mathbb{P}_i^N$ as the suborder of $S(i)$ consisting of functions of the form $p \restriction (i \setminus N)$, where $p \in E_i^N$.*

So $\mathbb{P}_i^N$ consists of functions in E_i^N, with their fragment belonging to N removed.

Lemma 7.10. *Let $i < j \leq \lambda$. Then:*

1. *$\mathbb{P}_i^N \subseteq \mathbb{P}_j^N$;*
2. *for all $p \in \mathbb{P}_j^N$, $p \restriction i \in \mathbb{P}_i^N$.*

Proof. (1) Let $p \in \mathbb{P}_i^N$. Then there is $p_0 \in E_i^N$ such that $p = p_0 \restriction (i \setminus N)$. Since $E_i^N \subseteq E_j^N$ by Lemma 7.8(1), $p_0 \in E_j^N$. Easily $p = p_0 \restriction (j \setminus N)$, so $p \in \mathbb{P}_j^N$.

(2) Let $p \in \mathbb{P}_j^N$. Then there is $p_0 \in E_j^N$ such that $p = p_0 \restriction (j \setminus N)$. By Lemma 7.8(2), $p_0 \restriction i \in E_i^N$, and $p \restriction i = (p_0 \restriction (j \setminus N)) \restriction i = (p_0 \restriction i) \restriction (i \setminus N)$. So $p \restriction i \in \mathbb{P}_i^N$. $\square$

Definition 7.11. *Define $\pi_i : E_i^N \to \mathbb{P}_i^N$ in $V[G_{i,N}]$ by letting*
$$\pi_i(p) = p \upharpoonright (i \setminus N),$$
for all $p \in E_i^N$.

Lemma 7.12. *The function π_i is a dense embedding of E_i^N onto $\mathbb{P}_i^N$. Hence $\mathbb{P}_i^N$ is forcing equivalent to $\mathbb{P}_i/G_{i,N}$.*

Proof. The second statement follows from the first together with the fact that E_i^N is dense in $\mathbb{P}_i/G_{i,N}$. The function π_i is surjective by the definition of $\mathbb{P}_i^N$, and clearly satisfies that $q \leq p$ in $S(i)$ implies that $\pi_i(q) \leq \pi_i(p)$ in $S(i)$. To see that π_i preserves incompatibility, suppose that p and q are in E_i^N, and $r \leq \pi_i(p), \pi_i(q)$ in $\mathbb{P}_i^N$. We will show that p and q are compatible in E_i^N. Fix $r_0 \in E_i^N$ such that $r = r_0 \upharpoonright (i \setminus N)$.

Since r_0, p, and q are in E_i^N, it follows that $r_0 \upharpoonright N$, $p \upharpoonright N$, and $q \upharpoonright N$ are in $G_{i,N}$ by Lemma 7.7. As $G_{i,N}$ is a filter, we can fix $s \in G_{i,N}$ with $s \leq r_0 \upharpoonright N, p \upharpoonright N, q \upharpoonright N$. Then $\mathrm{dom}(s) \cap \mathrm{dom}(r_0) = \mathrm{dom}(s)$, and for all $\gamma \in \mathrm{dom}(s)$, $r_0(\gamma) \subseteq s(\gamma)$. By Lemma 6.4, $r_0 + s$ is a condition in $\mathbb{P}_i$ which is below r_0 and s. Moreover, $(r_0 + s) \upharpoonright N = s \in G_{i,N}$, so $r_0 + s \in E_i^N$. Now $(r_0 + s) \upharpoonright N = s \leq p \upharpoonright N, q \upharpoonright N$, and $(r_0 + s) \upharpoonright (i \setminus N) = r \leq \pi_i(p) = p \upharpoonright (i \setminus N), \pi_i(q) = q \upharpoonright (i \setminus N)$. So $r_0 + s \leq p, q$. □

Recall that for $i \leq \lambda$, $\mathbb{P}_i \cap N$ is a regular suborder of both $\mathbb{P}_i$ and $\mathbb{P}_\lambda \cap N$, by Lemma 7.2 and Lemma 7.5(2). So if G_i is a V-generic filter on $\mathbb{P}_i$, then $G_i \cap N$ is a V-generic filter on $\mathbb{P}_i \cap N$. Hence we can form the forcing poset $(\mathbb{P}_\lambda \cap N)/(G_i \cap N)$.

Lemma 7.13. *Let $i \in N \cap (\lambda + 1)$. Then:*

1. *$\mathbb{P}_i$ forces that $(\mathbb{P}_\lambda \cap N)/(\dot{G}_i \cap N)$ is equal to $(\mathbb{P}_\lambda/\dot{G}_i) \cap N[\dot{G}_i]$;*
2. *$\mathbb{P}_i$ forces that $(\mathbb{P}_\lambda \cap N)/(\dot{G}_i \cap N)$ is ω_1-c.c. and has the ω_1-approximation property.*

Proof. (1) Let G_i be a V-generic filter on $\mathbb{P}_i$. We will show that
$$(\mathbb{P}_\lambda \cap N)/(G_i \cap N) = (\mathbb{P}_\lambda/G_i) \cap N[G_i].$$
Let $p \in (\mathbb{P}_\lambda \cap N)/(G_i \cap N)$. Then by the definition of the quotient forcing, $p \in \mathbb{P}_\lambda \cap N$ and $p \upharpoonright i \in G_i \cap N$. In particular, $p \upharpoonright i \in G_i$. So $p \in \mathbb{P}_\lambda/G_i$. Also, since $p \in N$ and $N \subseteq N[G_i]$, it follows that $p \in N[G_i]$. Hence $p \in (\mathbb{P}/G_i) \cap N[G_i]$.

Conversely, let $p \in (\mathbb{P}_\lambda/G_i) \cap N[G_i]$. By the definition of the quotient forcing, $p \in \mathbb{P}_\lambda$ and $p \upharpoonright i \in G_i$. Since $\mathbb{P}_i$ is ω_1-c.c., $N[G_i] \cap V = N$ by standard arguments. Therefore $p \in N[G_i] \cap \mathbb{P}_\lambda = N \cap \mathbb{P}_\lambda$. So $p \in \mathbb{P}_\lambda \cap N$. Since p and i are in N, $p \upharpoonright i \in N$. So $p \upharpoonright i \in G_i \cap N$. Hence $p \in (\mathbb{P}_\lambda \cap N)/(G_i \cap N)$.

(2) By Proposition 6.9, there exists in $V[G_i]$ a special $(\lambda \setminus i)$-iteration $\mathbb{P}'_\lambda$ such that the map $f(p) = p \upharpoonright (\lambda \setminus i)$ is a dense embedding of $\mathbb{P}_\lambda/G_i$ into $\mathbb{P}'_\lambda$. By the elementarity of $N[G_i]$, we can choose $\mathbb{P}'_\lambda$ to be in $N[G_i]$. Again by elementarity, it is easy to check that $f \upharpoonright N[G_i]$ is a dense embedding of $(\mathbb{P}_\lambda/G_i) \cap N[G_i]$ into $\mathbb{P}'_\lambda \cap N[G_i]$. Hence $(\mathbb{P}_\lambda/G_i) \cap N[G_i]$ is forcing equivalent to $\mathbb{P}'_\lambda \cap N[G_i]$. By (1), it follows that $(\mathbb{P}_\lambda \cap N)/(G_i \cap N)$ is forcing equivalent to $\mathbb{P}'_\lambda \cap N[G_i]$.

By Lemma 7.5(2) applied in $V[G_i]$ to $\mathbb{P}'_\lambda$ and $N[G_i]$, $\mathbb{P}'_\lambda \cap N[G_i]$ is a regular suborder of $\mathbb{P}'_\lambda$. But $\mathbb{P}'_\lambda$ is a special $(\lambda \setminus i)$-iteration, and hence is ω_1-c.c. and has the ω_1-approximation property. Since $\mathbb{P}'_\lambda \cap N[G_i]$ is a regular suborder of $\mathbb{P}'_\lambda$, it is also ω_1-c.c., and by Lemma 1.8, it has the ω_1-approximation property. Hence $(\mathbb{P}_\lambda \cap N)/(G_i \cap N)$ is ω_1-c.c. and has the ω_1-approximation property. □

Lemma 7.14. *Let $i \leq \lambda$. Then $\mathbb{P}_i$ forces that $(\mathbb{P}_\lambda \cap N)/(\dot{G}_i \cap N)$ has the ω_1-approximation property.*

Proof. Let G_i be a V-generic filter on $\mathbb{P}_i$, and let H be a $V[G_i]$-generic filter on $(\mathbb{P}_\lambda \cap N)/(G_i \cap N)$. We will show that the pair $(V[G_i], V[G_i][H])$ has the ω_1-approximation property.

Let $j := \min((N \cap (\lambda + 1)) \setminus i)$. Let $G_{i,j}$ be a $V[G_i][H]$-generic filter on $\mathbb{P}_j/G_i$. By the product lemma and Lemma 1.1,

$$V[G_i][H][G_{i,j}] = V[G_i][G_{i,j}][H] = V[G_{i,j}][H],$$

and $G_{i,j}$ is a V-generic filter on $\mathbb{P}_j$ such that $G_{i,j} \cap \mathbb{P}_i = G_i$.

We claim that $G_i \cap N = G_{i,j} \cap N$. Since $G_i \subseteq G_{i,j}$, the forward inclusion is immediate. Conversely, let $p \in G_{i,j} \cap N$, and we will show that $p \in G_i$. Then $p \in \mathbb{P}_j \cap N$. By elementarity, $\mathrm{dom}(p) \subseteq N \cap j \subseteq i$, so $p \in G_{i,j} \cap \mathbb{P}_i = G_i$. It follows from the claim that $(\mathbb{P}_\lambda \cap N)/(G_i \cap N) = (\mathbb{P}_\lambda \cap N)/(G_{i,j} \cap N)$.

Since $j \in N$, by Lemma 7.13 it follows that $(\mathbb{P}_\lambda \cap N)/(G_{i,j} \cap N)$ has the ω_1-c.c. and the ω_1-approximation property in $V[G_{i,j}]$. Therefore the pair

$$(V[G_{i,j}], V[G_{i,j}][H])$$

has the ω_1-covering property and the ω_1-approximation property. That is, the pair

$$(V[G_i][G_{i,j}], V[G_i][G_{i,j}][H])$$

has the ω_1-covering property and the ω_1-approximation property. By Corollary 6.10, the forcing poset $\mathbb{P}_j/G_i$ is ω_1-c.c. and has the ω_1-approximation property in $V[G_i]$. So the pair

$$(V[G_i], V[G_i][G_{i,j}])$$

has the ω_1-covering property and the ω_1-approximation property. By Lemma 1.7, it follows that the pair

$$(V[G_i], V[G_i][G_{i,j}][H])$$

has the ω_1-approximation property. Since $V[G_i] \subseteq V[G_i][H] \subseteq V[G_i][G_{i,j}][H]$, by Lemma 1.8 the pair

$$(V[G_i], V[G_i][H])$$

has the ω_1-approximation property. $\square$

For the remainder of the section, fix a V-generic filter G_N on $\mathbb{P}_\lambda \cap N$. Our goal is to prove that in $V[G_N]$, $\mathbb{P}_\lambda/G_N$ is forcing equivalent to a special $(\lambda \setminus N)$-iteration. For $i \leq \lambda$, let $G_{i,N} := G_N \cap \mathbb{P}_i$. Recall that E_i^N is dense in $\mathbb{P}_i/G_{i,N}$, and $\pi_i : E_i^N \to \mathbb{P}_i^N$ is a dense embedding, where $\pi_i(p) := p \restriction (i \setminus N)$ for all $p \in E_i^N$. So it suffices to show that in $V[G_N]$, $\mathbb{P}_i^N$ is forcing equivalent to a special $(i \setminus N)$-iteration, for all $i \leq \lambda$.

Notation 7.15. *In $V[G_{i,N}]$, let $\dot{G}_i$ be a $\mathbb{P}_i^N$-name for the $V[G_{i,N}]$-generic filter on $\mathbb{P}_i/G_{i,N}$ generated by $\pi_i^{-1}(\dot{G}_{\mathbb{P}_i^N})$. Also in $V[G_{i,N}]$, let $\dot{T}_i^N$ be a $\mathbb{P}_i^N$-name for $(\dot{T}_i)^{\dot{G}_i}$.*

Lemma 7.16. *The forcing poset $\mathbb{P}_i^N$ forces over $V[G_N]$ that $\dot{T}_i^N$ is a tree with underlying set ω_1 which has no branches of length ω_1.*

Proof. Let G_i^N be a $V[G_N]$-generic filter on $\mathbb{P}_i^N$. We will show that in $V[G_N][G_i^N]$, $(\dot{T}_i^N)^{G_i^N}$ is a tree with underlying set ω_1 which has no branches of length ω_1. Let G_i

denote the filter on $\mathbb{P}_i/G_{i,N}$ generated by $\pi_i^{-1}(G_i^N)$. Since π_i is a dense embedding and E_i^N is dense in $\mathbb{P}_i/G_{i,N}$, G_i is a $V[G_N]$-generic filter on $\mathbb{P}_i/G_{i,N}$. Moreover,

$$V[G_N][G_i^N] = V[G_N][G_i].$$

We claim that $G_i \cap N = G_{i,N}$ and $V[G_{i,N}][G_i] = V[G_i]$. Since G_i is a $V[G_N]$-generic filter on $\mathbb{P}_i/G_{i,N}$, it is also a $V[G_{i,N}]$-generic filter on $\mathbb{P}_i/G_{i,N}$. By Lemma 1.1, G_i is a V-generic filter on $\mathbb{P}_i$, $G_i \cap (\mathbb{P}_i \cap N) = G_{i,N}$, and $V[G_{i,N}][G_i] = V[G_i]$. But $G_i \cap (\mathbb{P}_i \cap N) = G_i \cap N$, proving the claim.

Note that both of the forcing posets $\mathbb{P}_i/G_{i,N}$ and $(\mathbb{P}_\lambda \cap N)/G_{i,N}$ are in $V[G_{i,N}]$. By Lemma 1.1,

$$V[G_N] = V[G_{i,N}][G_N],$$

and G_N is a $V[G_{i,N}]$-generic filter on $(\mathbb{P}_\lambda \cap N)/G_{i,N}$. So G_i is a $V[G_{i,N}][G_N]$-generic filter on $\mathbb{P}_i/G_{i,N}$. By the product lemma,

$$V[G_N][G_i] = V[G_{i,N}][G_N][G_i] = V[G_{i,N}][G_i][G_N],$$

and G_N is a $V[G_{i,N}][G_i]$-generic filter on $(\mathbb{P}_\lambda \cap N)/G_{i,N}$.

By the above claim, $G_{i,N} = G_i \cap N$ and $V[G_{i,N}][G_i] = V[G_i]$. So

$$V[G_N][G_i] = V[G_{i,N}][G_i][G_N] = V[G_i][G_N],$$

and G_N is a $V[G_i]$-generic filter on $(\mathbb{P}_\lambda \cap N)/(G_i \cap N)$. By Lemma 7.14, it follows that in $V[G_i]$, $(\mathbb{P}_\lambda \cap N)/(G_i \cap N)$ has the ω_1-approximation property. Hence the pair $(V[G_i], V[G_i][G_N])$ has the ω_1-approximation property.

Recall that $\dot{T}_i$ is a $\mathbb{P}_i$-name for a tree with underlying set ω_1 which has no branches of length ω_1. Let $T_i := \dot{T}_i^{G_i}$. Then in $V[G_i]$, T_i is a tree with underlying set ω_1 which has no branches of length ω_1. By upwards absoluteness, in $V[G_N][G_i^N] = V[G_N][G_i] = V[G_i][G_N]$, T_i is a tree with underlying set ω_1. Since the pair $(V[G_i], V[G_i][G_N])$ has the ω_1-approximation property, T_i has no branches of length ω_1 in $V[G_i][G_N]$. By Notation 7.15, $(\dot{T}_i^N)^{G_i^N}$ is equal to $\dot{T}_i^{G_i} = T_i$, and this is a tree with underlying set ω_1 with no branches of length ω_1 in $V[G_i][G_N] = V[G_N][G_i] = V[G_N][G_i^N]$. $\square$

Let us return to proving that in $V[G_N]$, $\mathbb{P}_i^N$ is forcing equivalent to a special $(i \setminus N)$-iteration, for all $i \leq \lambda$. In the model $V[G_N]$, consider the sequence

$$\langle \mathbb{P}_i' : i \leq \lambda \rangle$$

which is the special $(\lambda \setminus N)$-iteration defined from the sequence $\langle \dot{T}_i^N : i \in \lambda \setminus N \rangle$. In other words, the sequence is defined inductively using (3)–(6) of Definition 6.2. Of course, this only makes sense if for each $i \in \lambda \setminus N$, the name $\dot{T}_i^N$ from Notation 7.15 is a $\mathbb{P}_i'$-name for a tree with underlying set ω_1 which has no branches of length ω_1.

We will prove inductively that for each $i \leq \lambda$, $\mathbb{P}_i^N$ is a dense subset of $\mathbb{P}_i'$. Then for all $i \in \lambda \setminus N$, $\dot{T}_i^N$ literally is a $\mathbb{P}_i'$-name. And since $\mathbb{P}_i^N$ forces over $V[G_N]$ that $\dot{T}_i^N$ is a tree with underlying set ω_1 which has no branches of length ω_1 by Lemma 7.16, so does $\mathbb{P}_i'$. So in the end, we have that $\mathbb{P}_\lambda^N$ is forcing equivalent to $\mathbb{P}_\lambda'$, which is a special $(\lambda \setminus N)$-iteration, completing the proof of Theorem 7.1.

It remains to prove the following lemma.

Lemma 7.17. *For all $i \leq \lambda$, $\mathbb{P}_i^N$ is a dense subset of $\mathbb{P}_i'$.*

Proof. This is trivial for $i = 0$ by Definition 6.2(3), and it follows easily from the inductive hypothesis when i is a limit ordinal by Definition 6.2(6).

Let $i < \lambda$, and suppose that $\mathbb{P}_i^N$ is a dense subset of $\mathbb{P}'_i$. We will prove that $\mathbb{P}_{i+1}^N$ is a dense subset of $\mathbb{P}'_{i+1}$. First, assume the easier case that $i \in N$. Then $\mathbb{P}'_{i+1} = \mathbb{P}'_i$ by Definition 6.2(5). So it suffices to show that $\mathbb{P}_{i+1}^N = \mathbb{P}_i^N$. But $\mathbb{P}_i^N \subseteq \mathbb{P}_{i+1}^N$ by Lemma 7.10(1). Conversely, let $p \in \mathbb{P}_{i+1}^N$, and we will show that $p \in \mathbb{P}_i^N$. Then for some $p_0 \in E_{i+1}^N$, $p = p_0 \restriction ((i+1) \setminus N)$. But $i \in N$, so $(i+1) \setminus N = i \setminus N$. Hence $p = p_0 \restriction ((i+1)\setminus N) = p_0 \restriction (i \setminus N) = (p_0 \restriction i) \restriction (i \setminus N)$. But $p_0 \restriction i \in E_i^N$ by Lemma 7.8(2), so $p \in \mathbb{P}_i^N$.

Secondly, assume that $i \notin N$. We will show that $\mathbb{P}_{i+1}^N$ is dense in $\mathbb{P}'_{i+1}$. Let $p \in \mathbb{P}'_{i+1}$, and we will find a condition in $\mathbb{P}_{i+1}^N$ which is below p. If $i \notin \mathrm{dom}(p)$, then $p = p \restriction i \in \mathbb{P}'_i$. By the inductive hypothesis, there is $q \leq p$ in $\mathbb{P}_i^N$. Then $q \in \mathbb{P}_{i+1}^N$ by Lemma 7.10(1), and $q \leq p$, so we are done.

Suppose that $i \in \mathrm{dom}(p)$. Then by the definition of $\mathbb{P}'_{i+1}$, $p \restriction i \in \mathbb{P}'_i$, and

$$p \restriction i \Vdash^{V[G_N]}_{\mathbb{P}'_i} p(i) \in P(\dot{T}_i^N).$$

Let $x = p(i)$. By the inductive hypothesis, we can fix $q \leq p \restriction i$ in $\mathbb{P}_i^N$. Then

$$q \Vdash^{V[G_N]}_{\mathbb{P}'_i} x \in P(\dot{T}_i^N).$$

Since $\mathbb{P}'_i$ is a dense suborder of $\mathbb{P}_i^N$ and $\dot{T}_i^N$ is a $\mathbb{P}_i^N$-name, we have that

$$q \Vdash^{V[G_N]}_{\mathbb{P}_i^N} x \in P(\dot{T}_i^N).$$

Claim 7.18. $q \Vdash^{V[G_{i,N}]}_{\mathbb{P}_i^N} x \in P(\dot{T}_i^N)$.

Proof. If not, then there is $r \leq q$ in $\mathbb{P}_i^N$ such that

$$r \Vdash^{V[G_{i,N}]}_{\mathbb{P}_i^N} x \notin P(\dot{T}_i^N).$$

Let K be a $V[G_N]$-generic filter on $\mathbb{P}'_i$ with $r \in K$, and let $T := (\dot{T}_i^N)^K$. Since $r \leq p \restriction i$, $p \restriction i \in K$. As $p \restriction i \in K$, in $V[G_N][K]$ we have that $p(i) = x \in P(T)$. On the other hand, $K \cap \mathbb{P}_i^N$ is a $V[G_{i,N}]$-generic filter on $\mathbb{P}_i^N$, and $T = (\dot{T}_i^N)^{K \cap \mathbb{P}_i^N}$. Since $r \in K \cap \mathbb{P}_i^N$, in $V[G_{i,N}][K \cap \mathbb{P}_i^N]$ we have that $x \notin P(T)$. But $P(T)$ is the same in $V[G_{i,N}][K \cap \mathbb{P}_i^N]$ and $V[G_N][K]$, which is a contradiction. $\square$

Since $V[G_{i,N}]$ models that q forces in $\mathbb{P}_i^N$ that $x \in P(\dot{T}_i^N)$, we can fix $s \in G_{i,N}$ such that

$$s \Vdash^{V}_{\mathbb{P}_i \cap N} q \Vdash^{V[\dot{G}_{i,N}]}_{\mathbb{P}_i^N} x \in P(\dot{T}_i^N).$$

As $q \in \mathbb{P}_i^N$, fix $q_0 \in E_i^N$ such that $q = q_0 \restriction (i \setminus N)$. Then $q_0 \in \mathbb{P}_i/G_{i,N}$. By Lemma 1.4, q_0 and s are compatible in $\mathbb{P}_i/G_{i,N}$, so fix $r_0 \leq q_0, s$ in $\mathbb{P}_i/G_{i,N}$. By extending r_0 if necessary, we may assume that $r_0 \in E_i^N$. By Lemma 7.7, $r_0 \restriction N \in G_{i,N}$, and since $s \in N$ and $r_0 \leq s$, $r_0 \restriction N \leq s$.

Claim 7.19. $r_0 \Vdash^V_{\mathbb{P}_i} x \in P(\dot{T}_i)$.

Proof. To prove this, let H_i be a V-generic filter on $\mathbb{P}_i$ with $r_0 \in H_i$. We will show that $x \in P((\dot{T}_i)^{H_i})$. Let $H_{i,N} := H_i \cap N$, which is a V-generic filter on $\mathbb{P}_i \cap N$ by Lemma 7.5(2). Since $r_0 \leq s$ and $s \in \mathbb{P}_i \cap N$, $s \in H_i \cap N = H_{i,N}$. By Lemma 1.1, $V[H_i] = V[H_{i,N}][H_i]$, and H_i is a $V[H_{i,N}]$-generic filter on $\mathbb{P}_i/H_{i,N}$. Therefore $\pi_i[H_i \cap E_i^N]$ generates a $V[H_{i,N}]$-generic filter H_i^N on $\mathbb{P}_i^N$, where E_i^N and

$\mathbb{P}_i^N$ are defined in $V[H_{i,N}]$ from $H_{i,N}$, instead of from $G_{i,N}$ as above. Moreover, $V[H_i] = V[H_{i,N}][H_i] = V[H_{i,N}][H_i^N]$. Since $r_0 \in H_i$, $r_0 \restriction (i \setminus N) \in H_i^N$. Now $r_0 \leq q_0$, so $r_0 \restriction (i \setminus N) \leq q_0 \restriction (i \setminus N) = q$. Hence $q \in H_i^N$. Let $T := (\dot{T}_i^N)^{H_i^N}$.

Recall that

$$s \Vdash^V_{\mathbb{P}_i \cap N} q \Vdash^{V[\dot{G}_{i,N}]}_{\mathbb{P}_i^N} x \in P(\dot{T}_i^N).$$

Since $s \in H_{i,N}$ and $q \in H_i^N$, it follows that $x \in P(T)$. By Notation 7.15, T is equal to $\dot{T}_i^L$, where L is the filter on $\mathbb{P}_i/H_{i,N}$ generated by $\pi_i^{-1}(H_i^N)$. By the definition of H_i^N, we have that $\pi_i[H_i \cap E_i^N] \subseteq H_i^N$, and therefore $H_i \cap E_i^N \subseteq \pi_i^{-1}(H_i^N) \subseteq L$. Since clearly $H_i \cap E_i^N$ generates the filter H_i, $H_i \subseteq L$. As H_i and L are both $V[H_{i,N}]$-generic filters on $\mathbb{P}_i/H_{i,N}$, it follows that $H_i = L$. So $T = (\dot{T}_i)^{H_i}$. It follows that in $V[H_i]$, x is in $P((\dot{T}_i)^{H_i})$, which proves the claim. $\square$

Since $r_0 \in \mathbb{P}_i$ and $r_0 \Vdash^V_{\mathbb{P}_i} x \in P(\dot{T}_i)$, by the definition of $\mathbb{P}_{i+1}$ it follows that $r_1 := r_0 \cup \{(i,x)\}$ is in $\mathbb{P}_{i+1}$. Moreover, since $i \notin N$ and $r_0 \in E_i^N$,

$$r_1 \restriction N = r_0 \restriction N \in G_{i,N} \subseteq G_{i+1,N}.$$

So $r_1 \in E_{i+1}^N$. Hence $r := r_1 \restriction ((i+1) \setminus N)$ is in $\mathbb{P}_{i+1}^N$. Since $r_0 \leq q_0$,

$$r \restriction i = r_0 \restriction (i \setminus N) \leq q_0 \restriction (i \setminus N) = q,$$

and so $r \restriction i \leq q \leq p \restriction i$. Since $p(i) = x$, $r = (r \restriction i) \cup \{(i,x)\} \leq p$. $\square$

8. IGMP and the Continuum

We now construct a model in which IGMP holds and $2^\omega > \omega_2$. We begin with a ground model V which satisfies GCH, in which κ is a supercompact cardinal, and $\lambda \geq \kappa$ is a cardinal with cofinality at least ω_2. We will define a forcing poset of the form

$$(\mathbb{P} \times \mathrm{Add}(\omega, \lambda)) * \dot{\mathbb{Q}}$$

which forces that $\kappa = \omega_2$, $2^\omega = \lambda$, and IGMP.

Let $\mathbb{P}$ be the strongly proper collapse of κ to become ω_2 which was discussed in Section 5. Then $\mathbb{P}$ is strongly proper, κ-c.c., has size κ, and collapses κ to become ω_2. Since GCH holds in the ground model, standard arguments show that $\mathbb{P}$ forces that 2^ω and 2^{ω_1} are bounded by κ (in fact, they are both equal to κ).

Consider the product forcing $\mathbb{P} \times \mathrm{Add}(\omega, \lambda)$. Note that the forcing poset $\mathrm{Add}(\omega, \lambda)$ is the same in V and $V^{\mathbb{P}}$. Since $\mathbb{P}$ forces that $\mathrm{Add}(\omega, \lambda)$ is ω_1-c.c., it follows that $\mathbb{P} \times \mathrm{Add}(\omega, \lambda)$ is κ-c.c., and it has size λ. So by standard arguments, $\mathbb{P} \times \mathrm{Add}(\omega, \lambda)$ forces that $2^\omega = 2^{\omega_1} = \lambda$.

Applying Corollary 6.8, let $\dot{\mathbb{Q}}$ be a $\mathbb{P} \times \mathrm{Add}(\omega, \lambda)$-name for a special iteration of length λ which forces that every tree of height and size ω_1 which has no branches of length ω_1 is special. Then $\mathbb{P} \times \mathrm{Add}(\omega, \lambda)$ forces that $\dot{\mathbb{Q}}$ is ω_1-c.c., and has size λ. So the forcing poset

$$(\mathbb{P} \times \mathrm{Add}(\omega, \lambda)) * \dot{\mathbb{Q}}$$

is κ-c.c., and forces that $2^\omega = 2^{\omega_1} = \lambda$.

By Corollary 4.5, in order to prove that $(\mathbb{P} \times \mathrm{Add}(\omega, \lambda)) * \dot{\mathbb{Q}}$ forces IGMP, it suffices to show that $(\mathbb{P} \times \mathrm{Add}(\omega, \lambda)) * \dot{\mathbb{Q}}$ forces that for all sufficiently large regular cardinals $\theta \geq \omega_2$, there are stationarily many sets N in $P_{\omega_2}(H(\theta))$ such that N is internally unbounded and ω_1-guessing.

Fix a regular cardinal $\theta \geq \kappa$ such that the forcing poset $(\mathbb{P} \times \mathrm{Add}(\omega, \lambda)) * \dot{\mathbb{Q}}$ is a member of $H(\theta)$. Let $\dot{F}$ be a $(\mathbb{P} \times \mathrm{Add}(\omega, \lambda)) * \dot{\mathbb{Q}}$-name for a function $\dot{F} : (H(\theta))^{<\omega} \to H(\theta)$. We will prove that $(\mathbb{P} \times \mathrm{Add}(\omega, \lambda)) * \dot{\mathbb{Q}}$ forces that there exists a set N satisfying:

(1) N is in $P_\kappa(H(\theta))$;
(2) $N \prec H(\theta)$;
(3) N is closed under $\dot{F}$;
(4) N is internally unbounded and ω_1-guessing.

This will complete the proof.[4]

Let us abbreviate $\mathrm{Add}(\omega, \lambda)$ by Add.

Notation 8.1. *Fix a V-generic filter $G \times H$ on $\mathbb{P} \times \mathrm{Add}$, and fix a $V[G \times H]$-generic filter I on $\mathbb{Q} := (\dot{\mathbb{Q}})^{G \times H}$. Let $F := (\dot{F})^{(G \times H) * I}$.*

We will prove that in $V[(G \times H) * I]$, there exists a set N in $P_\kappa(H(\theta))$ such that $N \prec H(\theta)$, N is closed under F, and N is internally unbounded and ω_1-guessing.

Notation 8.2. *By the supercompactness of κ, fix in V an elementary embedding $j : V \to M$ with critical point κ such that $j(\kappa) > |H(\theta)|$ and $M^{|H(\theta)|} \subseteq M$.*

Note that by the closure of M, we have that $H(\theta)^V = H(\theta)^M$ and $j \restriction H(\theta)^V \in M$. Since $H(\theta)^V = H(\theta)^M$ and $\mathbb{P} \times \mathrm{Add} \in H(\theta)^V$, it follows that

$$H(\theta)^{V[G \times H]} = H(\theta)^V[G \times H] = H(\theta)^M[G \times H] = H(\theta)^{M[G \times H]},$$

and also

$$H(\theta)^{V[(G \times H) * I]} = H(\theta)^V[(G \times H) * I] = H(\theta)^M[(G \times H) * I] = H(\theta)^{M[(G \times H) * I]}.$$

Since the critical point of j is κ and $\mathbb{P} \times \mathrm{Add}$ is κ-c.c., it follows that $j \restriction (\mathbb{P} \times \mathrm{Add})$ is a regular embedding of $\mathbb{P} \times \mathrm{Add}$ into $j(\mathbb{P} \times \mathrm{Add})$, as proven in Section 1. Hence $j[\mathbb{P} \times \mathrm{Add}]$ is a regular suborder of $j(\mathbb{P} \times \mathrm{Add})$, $j[G \times H]$ is a V-generic filter on $j[\mathbb{P} \times \mathrm{Add}]$, and $V[G \times H] = V[j[G \times H]]$. So in $V[G \times H]$, we can form the forcing poset $j(\mathbb{P} \times \mathrm{Add})/j[G \times H]$.

Notation 8.3. *Let J be a $V[(G \times H) * I]$-generic filter on the forcing poset $j(\mathbb{P} \times \mathrm{Add})/j[G \times H]$.*

Then in particular, J is a $V[G \times H]$-generic filter on $j(\mathbb{P} \times \mathrm{Add})/j[G \times H]$. So by Lemma 1.1, J is a V-generic filter on $j(\mathbb{P} \times \mathrm{Add})$, $J \cap j[\mathbb{P} \times \mathrm{Add}] = j[G \times H]$, and $V[G \times H][J] = V[J]$. It follows that J is an M-generic filter on $j(\mathbb{P} \times \mathrm{Add})$, and $j[G \times H] \subseteq J$. So in $V[J]$, standard arguments show that we can extend j to

$$j : V[G \times H] \to M[J]$$

by letting

$$j(\dot{a}^{G \times H}) := j(\dot{a})^J$$

for any $\mathbb{P} \times \mathrm{Add}$-name $\dot{a}$ in V.

[4]The proof actually shows the existence of stationarily many N satisfying (1)–(4) which are internally stationary. The reason is that for all regular $\omega_2 \leq \theta < \kappa$, $\mathbb{P}$ forces that $H(\theta)$ is internally stationary. In fact, we can strengthen this to internally club, by modifying the adequate set forcing $\mathbb{P}$ by simultaneously adding κ many club subsets of ω_1 to get that for all regular $\omega_2 \leq \theta < \kappa$, $\mathbb{P}$ forces that $H(\theta)$ is internally club. This modification of adequate set forcing is beyond the scope of the paper. Alternatively, this stronger result could also be obtained by replacing $\mathbb{P}$ with the decorated sequence forcing of Neeman [6].

As noted above, $j \restriction H(\theta)^V \in M$. Since $H(\theta)^{V[G\times H]} = H(\theta)^V[G \times H]$, the definition of j above shows that $j \restriction H(\theta)^{V[G\times H]}$ is definable in $M[J]$ from $H(\theta)^V$, $j \restriction H(\theta)^V$, $G \times H$, and J. Hence $j \restriction H(\theta)^{V[G\times H]}$ is in $M[J]$.

Since J is a $V[G \times H][I]$-generic filter on $j(\mathbb{P} \times \mathrm{Add})/j[G \times H]$, by the product lemma, I is a $V[G \times H][J]$-generic filter on $\mathbb{Q}$. But $V[G \times H][J] = V[J]$, so I is a $V[J]$-generic filter on $\mathbb{Q}$. Hence I is an $M[J]$-generic filter on $\mathbb{Q}$. Also by the product lemma,

$$V[J][I] = V[G \times H][J][I] = V[G \times H][I][J].$$

Since $j \restriction \mathbb{Q}$ is an isomorphism in $M[J]$ from $\mathbb{Q}$ to $j[\mathbb{Q}]$, $j[I]$ is an $M[J]$-generic filter on $j[\mathbb{Q}]$.

As J is a $V[G\times H]$-generic filter on $j(\mathbb{P}\times \mathrm{Add})/j[G\times H]$, it is also an $M[G\times H]$-generic filter on $j(\mathbb{P} \times \mathrm{Add})/j[G \times H]$. So by Lemma 1.1, $M[G \times H][J] = M[J]$. By the product lemma,

$$M[J][I] = M[G \times H][J][I] = M[G \times H][I][J].$$

Let

$$N_0 := j[H(\theta)^{V[G\times H]}]$$

Note that since $j \restriction H(\theta)^{V[G\times H]}$ is in $M[J]$, N_0 is in $M[J]$. We would like to apply Theorem 7.1 in $M[J]$ to the special iteration $j(\mathbb{Q})$ and the model N_0. Now N_0 has the same cardinality as $H(\theta)^{V[G\times H]}$, which in turn has the same cardinality as $H(\theta)^V$. But $|H(\theta)|^V < j(\kappa)$, and in $M[J]$, $j(\kappa)$ is equal to ω_2 and ω_1^M is preserved. It follows that in $M[J]$, N_0 has size ω_1. Also clearly $\omega_1 \subseteq N_0$.

We claim that N_0 is an elementary substructure of $H(j(\theta))^{M[J]}$. Let F^* be a Skolem function for the structure $H(\theta)^{V[G\times H]}$ in $V[G \times H]$. Since $H(\theta)^{V[G\times H]}$ is closed under F^*, it easily follows that N_0 is closed under $j(F^*)$. By the elementarity of j, $j(F^*)$ is a Skolem function for $H(j(\theta))^{M[J]}$ in $M[J]$, which proves the claim.

So all of the assumptions of Theorem 7.1 for $j(\mathbb{Q})$ and N_0 hold in $M[J]$. By Theorem 7.1, it follows that in $M[J]$, $N_0 \cap j(\mathbb{Q})$ is a regular suborder of $j(\mathbb{Q})$, and $N_0 \cap j(\mathbb{Q})$ forces that $j(\mathbb{Q})/\dot{G}_{N_0\cap j(\mathbb{Q})}$ is ω_1-c.c. and has the ω_1-approximation property.

Lemma 8.4. *In the model $M[J]$, $j[\mathbb{Q}]$ is a regular suborder of $j(\mathbb{Q})$, and $j[\mathbb{Q}]$ forces that $j(\mathbb{Q})/\dot{G}_{j[\mathbb{Q}]}$ is ω_1-c.c. and has the ω_1-approximation property.*

Proof. By the comments preceding this lemma, it suffices to show that $j[\mathbb{Q}] = j(\mathbb{Q}) \cap N_0$. Let $q \in \mathbb{Q}$, and we will show that $j(q) \in j(\mathbb{Q}) \cap N_0$. Since $\dot{\mathbb{Q}} \in H(\theta)^V$, $\mathbb{Q} \in H(\theta)^{V[G\times H]}$. So $q \in H(\theta)^{V[G\times H]}$. Hence $j(q) \in N_0$. Also $j(q) \in j(\mathbb{Q})$ by the elementarity of j. Conversely, let $q^* \in j(\mathbb{Q}) \cap N_0$, and we will show that $q^* \in j[\mathbb{Q}]$. Then by the definition of N_0, there is $q \in H(\theta)^{V[G\times H]}$ such that $j(q) = q^*$. Since $j(q) = q^* \in j(\mathbb{Q})$, it follows that $q \in \mathbb{Q}$ by the elementarity of j. Hence $q^* \in j[\mathbb{Q}]$. □

Since $j[I]$ is an $M[J]$-generic filter on $j[\mathbb{Q}]$, we can form the forcing poset $j(\mathbb{Q})/j[I]$ in $M[J][I]$. In particular, this forcing poset is in $V[G \times H][I][J]$.

Notation 8.5. *Let K be a $V[G \times H][I][J]$-generic filter on $j(\mathbb{Q})/j[I]$.*

Since $V[G\times H][I][J] = V[J][I]$, K is a $V[J][I]$-generic filter on $j(\mathbb{Q})/j[I]$. Hence K is an $M[J][I]$-generic filter on $j(\mathbb{Q})/j[I]$. By Lemma 1.1, it follows that K is an $M[J]$-generic filter on $j(\mathbb{Q})$, $K \cap j[\mathbb{Q}] = j[I]$, and $M[J][I][K] = M[J][K]$. In particular, $j[I] \subseteq K$. By standard arguments, in $V[J][I][K]$ we can extend the elementary embedding $j : V[G \times H] \to M[J]$ to

$$j : V[G \times H][I] \to M[J][K]$$

by letting

$$j(\dot{a}^I) := j(\dot{a})^K$$

for any $\dot{a}$ which is a $\mathbb{Q}$-name in $V[G \times H]$. Note that since $j \restriction H(\theta)^{V[G\times H]}$ is in $M[J]$, and $H(\theta)^{V[G\times H][I]} = H(\theta)^{V[G\times H]}[I]$, it follows that $j \restriction H(\theta)^{V[G\times H][I]}$ is definable in $M[J][K]$ from the parameters $H(\theta)^{V[G\times H]}$, $j \restriction H(\theta)^{V[G\times H]}$, I, and K. Therefore $j \restriction H(\theta)^{V[G\times H][I]}$ is in $M[J][K]$.

Lemma 8.6. *The pair*

$$(M[G \times H][I], M[J][K])$$

has the ω_1-covering property and the ω_1-approximation property.

Proof. We have that

$$M[J][K] = M[J][I][K] = M[G \times H][J][I][K] = M[G \times H][I][J][K].$$

So it suffices to show that the pair

$$(M[G \times H][I], M[G \times H][I][J][K])$$

has the ω_1-covering property and the ω_1-approximation property.

By Proposition 5.6, for any regular suborder $\mathbb{P}_0$ of $\mathbb{P} \times \mathrm{Add}$, $\mathbb{P}_0$ forces that $(\mathbb{P} \times \mathrm{Add})/\dot{G}_{\mathbb{P}_0}$ is strongly proper on a stationary set. By the elementarity of j, the same is true of $j(\mathbb{P} \times \mathrm{Add})$ in M. In particular, this is true in M of the regular suborder $j[\mathbb{P} \times \mathrm{Add}]$ of $j(\mathbb{P} \times \mathrm{Add})$. It follows that in $M[G \times H]$, the forcing poset $j(\mathbb{P} \times \mathrm{Add})/j[G \times H]$ is strongly proper on a stationary set.

In $M[G\times H]$, $\mathbb{Q}$ is a special iteration, and hence is ω_1-c.c., and therefore proper. By Theorem 5.5, since I is an $M[G \times H]$-generic filter on $\mathbb{Q}$, in $M[G \times H][I]$ the forcing poset $j(\mathbb{P} \times \mathrm{Add})/j[G \times H]$ is strongly proper on a stationary set. Therefore in $M[G \times H][I]$, $j(\mathbb{P} \times \mathrm{Add})/j[G \times H]$ has the ω_1-covering property and the ω_1-approximation property. Since J is an $M[G \times H][I]$-generic filter on $j(\mathbb{P} \times \mathrm{Add})/j[G \times H]$, it follows that the pair

$$(M[G \times H][I], M[G \times H][I][J])$$

has the ω_1-covering property and the ω_1-approximation property.

By Lemma 8.4, in the model $M[J] = M[G \times H][J]$, $j[\mathbb{Q}]$ is a regular suborder of $j(\mathbb{Q})$, and $j[\mathbb{Q}]$ forces that $j(\mathbb{Q})/\dot{G}_{j[\mathbb{Q}]}$ is ω_1-c.c. and has the ω_1-approximation property. Also $j[I]$ is an $M[J]$-generic filter on $j[\mathbb{Q}]$. So in the model $M[J][I]$, $j(\mathbb{Q})/j[I]$ is ω_1-c.c. and has the ω_1-approximation property. Since K is an $M[J][I]$-generic filter on $j(\mathbb{Q})/j[I]$, the pair

$$(M[J][I], M[J][I][K])$$

has the ω_1-covering property and the ω_1-approximation property. But $M[J] = M[G \times H][J]$, so the pair

$$(M[G \times H][J][I], M[G \times H][J][I][K])$$

has the ω_1-covering property and the ω_1-approximation property. Since $M[G \times H][J][I] = M[G \times H][I][J]$, the pair

$$(M[G \times H][I][J], M[G \times H][I][J][K])$$

has the ω_1-covering property and the ω_1-approximation property.

To summarize, we have shown that the pairs

$$(M[G \times H][I], M[G \times H][I][J])$$

and

$$(M[G \times H][I][J], M[G \times H][I][J][K])$$

both have the ω_1-covering property and the ω_1-approximation property. By Lemma 1.7, it follows that the pair

$$(M[G \times H][I], M[G \times H][I][J][K])$$

has the ω_1-covering property and the ω_1-approximation property. $\square$

Recall that we are trying to prove that in $V[G \times H][I]$, there is a set N in $P_\kappa(H(\theta))$ such that $N \prec H(\theta)$, N is closed under F, and N is internally unbounded and ω_1-guessing. In the model $V[J][I][K]$, we have an elementary embedding $j : V[G \times H][I] \to M[J][K]$. So by the elementarity of j, it suffices to prove that in $M[J][K]$, there exists a set N satisfying:

(a) N is in $P_{j(\kappa)}(H(j(\theta))^{M[J][K]})$;
(b) $N \prec H(j(\theta))^{M[J][K]}$;
(c) N is closed under $j(F)$;
(d) N is internally unbounded and ω_1-guessing.

Let

$$N := j[H(\theta)^{V[G\times H][I]}].$$

We will show that N is in $M[J][K]$, and $M[J][K]$ models that N satisfies properties (a)–(d) above.

Since $j \restriction H(\theta)^{V[G\times H][I]} \in M[J][K]$, as observed above, $N \in M[J][K]$. Also by the elementarity of j, $N \subseteq j(H(\theta)^{V[G\times H][I]}) = H(j(\theta))^{M[J][K]}$. Now N has the same cardinality as $H(\theta)^{V[G\times H][I]}$, which in turn has the same cardinality as $H(\theta)^V$. But $|H(\theta)|^V < j(\kappa)$, and in $M[J][K]$, $j(\kappa)$ is equal to ω_2 and ω_1^M is preserved. It follows that in $M[J][K]$, N has size ω_1. Hence N is in $P_{j(\kappa)}(H(j(\theta))^{M[J][K]})$.

We claim that N is an elementary substructure of $H(j(\theta))^{M[J][K]}$. Let F^* be a Skolem function for the structure $H(\theta)^{V[G\times H][I]}$ in $V[G\times H][I]$. Since $H(\theta)^{V[G\times H][I]}$ is closed under F^*, it easily follows that N is closed under $j(F^*)$. By the elementarity of j, $j(F^*)$ is a Skolem function for $H(j(\theta))^{M[J][I]}$ in $M[J][I]$. So N is an elementary substructure of $H(j(\theta))^{M[J][I]}$. The same argument shows that N is closed under $j(F)$.

It remains to show that N is internally unbounded and ω_1-guessing in $M[J][K]$. To show that N is internally unbounded, let a be a countable subset of N in $M[J][K]$. Then $b := j^{-1}[a]$ is a countable subset of $H(\theta)^{V[G\times H][I]}$ in $M[J][K]$. Since the pair $(M[G\times H][I], M[J][K])$ has the ω_1-covering property by Lemma 8.6, we can fix a countable set $c \subseteq H(\theta)^{V[G\times H][I]} = H(\theta)^{M[G\times H][I]}$ in $M[G\times H][I]$ such that $b \subseteq c$. But θ is regular and uncountable in $M[G\times H][I]$, so $c \in H(\theta)^{M[G\times H][I]}$. Hence $j(c) = j[c]$ is in N, $j(c)$ is countable, and $a = j[b] \subseteq j[c] = j(c)$.

To show that N is ω_1-guessing in $M[J][K]$, by Lemma 2.2 it suffices to show that the pair

$$(\overline{N}, M[J][K])$$

satisfies the ω_1-approximation property, where $\overline{N}$ is the transitive collapse of N. Since N is isomorphic to $H(\theta)^{V[G\times H][I]}$, which is transitive, we have that $\overline{N} = H(\theta)^{V[G\times H][I]} = H(\theta)^{M[G\times H][I]}$. Hence it suffices to show that the pair

$$(H(\theta)^{M[G\times H][I]}, M[J][K])$$

has the ω_1-approximation property.

Let d be a bounded subset of $H(\theta)^{M[G\times H][I]} \cap On = \theta$ in $M[J][K]$ which is countably approximated by $H(\theta)^{M[G\times H][I]}$. We will show that d is in $H(\theta)^{M[G\times H][I]}$. We claim that d is countably approximated by $M[G\times H][I]$. Consider a countable set a in $M[G\times H][I]$. Then $a\cap\theta$ is a countable subset of θ in $M[G\times H][I]$, and hence is in $H(\theta)^{M[G\times H][I]}$. Since $d \subseteq \theta$, $a\cap d = (a\cap\theta)\cap d$. Since d is countably approximated by $H(\theta)^{M[G\times H][I]}$, $a\cap d = (a\cap\theta)\cap d$ is in $H(\theta)^{M[G\times H][I]}$, and hence is in $M[G\times H][I]$. Thus d is countably approximated by $M[G\times H][I]$. Since $(M[G\times H][I], M[J][K])$ has the ω_1-approximation property by Lemma 8.6, $d \in M[G\times H][I]$. But d is a bounded subset of θ and θ is regular in $M[G\times H][I]$, so $d \in H(\theta)^{M[G\times H][I]}$.

We conclude the paper with two questions.

(1) In Corollary 3.5, we proved that IGMP implies SCH. Does GMP imply SCH?
(2) At the end of Section 2, we noted that GMP is consistent with $\mathfrak{p} = \omega_1$. Does IGMP imply $\mathfrak{p} > \omega_1$?

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