

THE APPROACHABILITY IDEAL WITHOUT A MAXIMAL SET

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ABSTRACT. We develop a forcing poset with finite conditions which adds a partial square sequence on a given stationary set, with adequate sets of models as side conditions. We then develop a kind of side condition product forcing for simultaneously adding partial square sequences on multiple stationary sets. We show that certain quotients of such forcings have the ω_1 -approximation property. We apply these ideas to prove, assuming the consistency of a greatly Mahlo cardinal, that it is consistent that the approachability ideal $I[\omega_2]$ does not have a maximal set modulo clubs.

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Introduction

Let κ be an uncountable cardinal. For a given sequence $\vec{a} = \langle a_i : i < \kappa^+ \rangle$ of subsets of κ^+ with size less than κ , define $S_{\vec{a}}$ to be the set of limit ordinals $\beta < \kappa^+$ for which there exists a set $c \subseteq \beta$, which is cofinal in β , and has order type equal to $\text{cf}(\beta)$, which is approximated by the sequence $\vec{a} \restriction \beta$ in the sense that for all $\gamma < \beta$, $c \cap \gamma \in \{a_i : i < \beta\}$.

Define the approachability ideal $I[\kappa^+]$ as the collection of sets $S \subseteq \kappa^+$ such that for some sequence $\vec{a}$ as above, and for some club $C \subseteq \kappa^+$, $S \cap C \subseteq S_{\vec{a}}$. In other words, $I[\kappa^+]$ is the ideal on κ^+ which is generated over the nonstationary ideal on κ^+ by sets of the form $S_{\vec{a}}$. The collection $I[\kappa^+]$ is a normal ideal on κ^+ .

The approachability ideal was introduced by Shelah in the 1970's ([10]), and since then it has played a role as an important tool in combinatorial set theory and forcing. A major result on the approachability ideal is that if κ is a regular uncountable cardinal, then the set $\kappa^+ \cap \text{cof}(<\kappa)$ is a member of $I[\kappa^+]$ ([11, Section 4]). Hence, when κ is a regular uncountable cardinal, the structure of $I[\kappa^+]$ is completely determined by which stationary subsets of $\kappa^+ \cap \text{cof}(\kappa)$ are in $I[\kappa^+]$.

Shelah [11] raised the question whether it is consistent that there are no stationary subsets of $\kappa^+ \cap \text{cof}(\kappa)$ in $I[\kappa^+]$. This problem was solved by Mitchell [8], who proved that it is consistent, relative to the consistency of a greatly Mahlo cardinal, that there is no stationary subset of $\omega_2 \cap \text{cof}(\omega_1)$ in $I[\omega_2]$. In the process of solving this problem, Mitchell introduced a number of powerful new ideas in forcing, including strongly generic conditions, strong properness, and a method for using side conditions to add by forcing a club subset of ω_2 with finite conditions (see Friedman [4] for a similar method which was introduced independently).

Assuming that $(\kappa^+)^{<\kappa} = \kappa^+$, we can enumerate all subsets of κ^+ of size less than κ in a single sequence $\vec{b} = \langle b_i : i < \kappa^+ \rangle$. It is not hard to show that if $\vec{a} = \langle a_i : i < \kappa^+ \rangle$ is any sequence of subsets of κ^+ of size less than κ , then there exists a club $C \subseteq \kappa^+$ such that $S_{\vec{a}} \cap C \subseteq S_{\vec{b}}$. It follows that $I[\kappa^+]$ is generated over the nonstationary ideal on κ^+ by the single set $S_{\vec{b}}$. Another way of describing this conclusion is that $I[\kappa^+]$ has a maximum set modulo clubs, which is easily seen to be equivalent to it having a *maximal* set modulo clubs.

A natural question is whether the approachability ideal $I[\kappa^+]$ must *always* have a maximal set modulo clubs, regardless of any cardinal arithmetic assumptions. That is, is it consistent that $I[\kappa^+]$ does not have a single generator over the nonstationary ideal. By the normality of $I[\kappa^+]$, this possibility is equivalent to having not fewer than κ^{++} many generators. This question was first raised by Shelah in [11], in the same place where he mentions the possibility of $I[\kappa^+]$ not containing any stationary subset of $\kappa^+ \cap \text{cof}(\kappa)$. The problem also appears at the end of [8], where Mitchell suggests that the methods introduced in his paper are likely to be useful for answering the question. In this paper we will solve this problem.

Theorem 1. *Assuming the consistency of a greatly Mahlo cardinal, it is consistent that $I[\omega_2]$ does not have a maximal set modulo clubs.*

Let us give an overview of the structure of the paper and the ideas which will be used in the proof of the theorem. In Part 1, we present the material on side conditions which will be the foundation for everything else in the paper. We will use the adequate set framework of side conditions. This framework was first introduced in [7]. In this paper, we will follow the presentation which was given later in [5].

In Part II, we develop a forcing poset with finite conditions for adding a partial square sequence to a given stationary set $S \subseteq \omega_2 \cap \text{cof}(\omega_1)$. Recall that a *partial square sequence on S* is a sequence $\langle c_\alpha : \alpha \in S \rangle$ satisfying that for each $\alpha \in S$, c_α is a club subset of α with order type equal to ω_1 , and whenever γ is a common limit point of c_α and c_β , then $c_\alpha \cap \gamma = c_\beta \cap \gamma$.

If there exists a partial square sequence on S , then S is in $I[\omega_2]$. Namely, define a sequence $\vec{a} = \langle a_\gamma : \gamma < \omega_2 \rangle$ as follows. For a given ordinal γ , if there exists some $\alpha \in S$ strictly greater than γ such that γ is a limit point of c_α , then let $a_\gamma := c_\alpha \cap \gamma$. Define a_ξ for all other ordinals ξ in such a way as to include any initial segment of any set of the form a_γ , where γ is an ordinal of the first type. One can easily check that for some club $C \subseteq \omega_2$, $S \cap C \subseteq S_{\vec{a}}$. Therefore, $S \in I[\omega_2]$.

Forcing a square sequence with finite conditions was first achieved by Dolinar and Dzamonja [3], using Mitchell's style of models as side conditions [8]. Later, Krueger [6] developed a forcing poset for adding a square sequence with finite conditions using the framework of coherent adequate sets. And Neeman [9] defined a forcing poset for adding a square sequence using his framework of two-type side conditions.

The forcing poset we present in Part II for adding a partial square sequence is similar to the forcings of [6] and [9] for adding a square sequence. However, we will need to develop the properties of our forcing poset in much greater detail than was done in those papers, so that we can use it to prove the consistency result. In particular, in Sections 8 and 9 we will derive some very technical information in order to prove that certain quotients of the forcing poset satisfy the ω_1 -approximation property.

In Part III, we develop a forcing poset $\mathbb{Q}$ which simultaneously adds a partial square sequence on multiple sets. This forcing poset is similar to a product forcing, since the different posets which are incorporated in the forcing are independent of each other, except for the presence of a shared side condition. We believe that it is likely that this kind of side condition product will have other applications in the future.

A crucial property of the product forcing $\mathbb{Q}$ presented in Section 10 for proving our consistency result on $I[\omega_2]$ is that certain quotients of it satisfy the ω_1 -approximation property. More specifically, in Section 12 we will show that for certain uncountable models P , $P \cap \mathbb{Q}$ is a regular suborder of $\mathbb{Q}$, and the quotient forcing $\mathbb{Q}/\dot{G}_{P \cap \mathbb{Q}}$ has the ω_1 -approximation property in $V^{P \cap \mathbb{Q}}$.

A similar result about certain quotients having the ω_1 -approximation property was used by Mitchell [8] in his proof of the consistency that $I[\omega_2]$ does not contain a stationary subset of $\omega_2 \cap \text{cof}(\omega_1)$. This result followed from the equation

$$(p \wedge q) \restriction P = (p \restriction P) \wedge (q \restriction P),$$

where $\wedge$ denotes greatest lower bound, which holds below a strongly P -generic condition which is tidy (see [8, Definition 2.20, Lemma 2.22]). Unfortunately, our forcing poset $\mathbb{Q}$ does not satisfy this equation. First, our forcing poset $\mathbb{Q}$ does not even have greatest lower bounds. Secondly, even if the definition of $\mathbb{Q}$ is adjusted so that $\mathbb{Q}$ has greatest lower bounds, which is possible, the above equation still fails, even on any dense set.

Nonetheless, we are able to make use of some of the ideas in Mitchell's original argument for the ω_1 -approximation property [8, Lemma 2.22], by replacing the

above equation with something weaker, and more complicated, namely,

$$(q \oplus^N p) \restriction P = (q \restriction P) \oplus^{N \cap P} (p \restriction P).$$

In this equation, $a \oplus^M b$ denotes the amalgam of a condition b with a condition a which is in the model M and is below the projection $b \restriction M$ (see Proposition 12.6). We believe that this equation will be useful in future applications for verifying the approximation property, in cases where Mitchell's original tidy property fails.

Finally, in Section 13 we complete the proof of the consistency that $I[\omega_2]$ does not have a maximal set modulo clubs. Assuming that κ is a greatly Mahlo cardinal, we get a sequence $\langle B_i : i < \kappa^+ \rangle$ of Mahlo sets. We use the forcing poset $\mathbb{Q}$ from Part 3 to simultaneously add partial square sequences on $B_i \setminus B_{i+1}$, for each $i < \kappa^+$, while collapsing κ to become ω_2 . This will place each such set in the approachability ideal $I[\omega_2]$. We make use of the approximation property of certain quotients of $\mathbb{Q}$ to show that $I[\omega_2]$ does not have a maximal set.

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Part 1. Background

§1. Preliminaries

The prerequisites for reading this paper are a background of one year of graduate level study in set theory, a working knowledge of forcing, and some basic familiarity with proper forcing and generalized stationarity.

For a regular uncountable cardinal λ and a set X with $\lambda \subseteq X$, we let $P_\lambda(X)$ denote the set $\{a \subseteq X : |a| < \lambda\}$. A set $S \subseteq P_\lambda(X)$ is stationary iff for any function $F : X^{<\omega} \rightarrow X$, there exists $b \in S$ such that $b \cap \lambda \in \lambda$ and b is closed under F .

In this paper, a *forcing poset* is a pair $(\mathbb{P}, \leq_{\mathbb{P}})$, where $\mathbb{P}$ is a nonempty set and $\leq_{\mathbb{P}}$ is a reflexive and transitive relation on $\mathbb{P}$. To simplify notation, we usually refer to $\mathbb{P}$ itself as a forcing poset, with the relation $\leq_{\mathbb{P}}$ being implicit. If $\mathbb{Q}$ is a forcing poset, we will write $\dot{G}_{\mathbb{Q}}$ for the canonical $\mathbb{Q}$ -name for a generic filter on $\mathbb{Q}$.

Let $\mathbb{P}$ and $\mathbb{Q}$ be forcing posets. Then $\mathbb{P}$ is a *suborder* of $\mathbb{Q}$ if $\mathbb{P} \subseteq \mathbb{Q}$ and $\leq_{\mathbb{P}} = \leq_{\mathbb{Q}} \restriction (\mathbb{P} \times \mathbb{P})$. Let $\mathbb{P}$ be a suborder of $\mathbb{Q}$. We say that $\mathbb{P}$ is a *regular suborder* of $\mathbb{Q}$ if:

- (1) whenever p and q are in $\mathbb{P}$ and are incompatible in $\mathbb{P}$, then p and q are incompatible in $\mathbb{Q}$;
- (2) if A is a maximal antichain of $\mathbb{P}$, then A is predense in $\mathbb{Q}$.

Lemma 1.1. *Suppose that $\mathbb{P}_0$ is a suborder of $\mathbb{P}_1$, and $\mathbb{P}_1$ is a suborder of $\mathbb{Q}$. Assume, moreover, that $\mathbb{P}_0$ and $\mathbb{P}_1$ are both regular suborders of $\mathbb{Q}$. Then $\mathbb{P}_0$ is a regular suborder of $\mathbb{P}_1$.*

Proof. Straightforward. □

Let $\mathbb{P}$ be a regular suborder of $\mathbb{Q}$, and assume that G is a generic filter on $\mathbb{P}$. In $V[G]$, define the forcing poset $\mathbb{Q}/G$ to consist of conditions $q \in \mathbb{Q}$ such that for all $s \in G$, q and s are compatible in $\mathbb{Q}$, with the same ordering as $\mathbb{Q}$. Then $\mathbb{Q}$ is forcing equivalent to the two-step iteration $\mathbb{P} * (\mathbb{Q}/\dot{G}_{\mathbb{P}})$. Moreover:

Lemma 1.2. *Let $\mathbb{P}$ be a regular suborder of $\mathbb{Q}$.*

- (1) *Suppose that H is a V -generic filter on $\mathbb{Q}$. Then $H \cap \mathbb{P}$ is a V -generic filter on $\mathbb{P}$, and H is a $V[H \cap \mathbb{P}]$ -generic filter on $\mathbb{Q}/(H \cap \mathbb{P})$.*
- (2) *Suppose that G is a V -generic filter on $\mathbb{P}$ and H is a $V[G]$ -generic filter on $\mathbb{Q}/G$. Then H is a V -generic filter on $\mathbb{Q}$, $G = H \cap \mathbb{P}$, and $V[G][H] = V[H]$.*

Proof. See [2, Lemma 1.6]. $\square$

Let $\mathbb{P}$ and $\mathbb{Q}$ be forcing posets with maximum conditions. A function $\pi : \mathbb{Q} \rightarrow \mathbb{P}$ is said to be a *projection mapping* if:

- (1) π maps the maximum condition in $\mathbb{Q}$ to the maximum condition in $\mathbb{P}$;
- (2) if $q \leq p$ in $\mathbb{Q}$, then $\pi(q) \leq \pi(p)$ in $\mathbb{P}$;
- (3) if $v \leq \pi(q)$ in $\mathbb{P}$, then there is $r \leq q$ in $\mathbb{Q}$ such that $\pi(r) \leq v$ in $\mathbb{P}$.

If $\pi : \mathbb{Q} \rightarrow \mathbb{P}$ is a projection mapping, and G is a generic filter on $\mathbb{Q}$, then the set $\{s \in \mathbb{P} : \exists p \in G \ \pi(p) \leq s\}$ is a generic filter on $\mathbb{P}$.

Let $\mathbb{Q}$ be a forcing poset. For a set N and a condition $q \in \mathbb{Q}$, we say that q is *strongly N -generic* if whenever D is a dense subset of $N \cap \mathbb{Q}$, then D is predense below q in $\mathbb{Q}$.

We say that $\mathbb{Q}$ is *strongly proper on a stationary set* if for all sufficiently large cardinals λ with $\mathbb{Q} \subseteq H(\lambda)$, there are stationarily many N in $P_{\omega_1}(H(\lambda))$ such that for all $p \in N \cap \mathbb{Q}$, there is $q \leq p$ which is strongly N -generic. If $\mathbb{Q}$ is strongly proper on a stationary set, then $\mathbb{Q}$ preserves ω_1 , because being a strongly N -generic condition implies being an N -generic condition in the sense of proper forcing.

Let $\lambda_{\mathbb{Q}}$ denote the smallest cardinal such that $\mathbb{Q} \subseteq H(\lambda_{\mathbb{Q}})$. Note that a condition q is strongly N -generic iff q is strongly $(N \cap H(\lambda_{\mathbb{Q}}))$ -generic. Using this fact, standard arguments show that $\mathbb{Q}$ is strongly proper on a stationary set iff there are stationarily many N in $P_{\omega_1}(H(\lambda_{\mathbb{Q}}))$ such that for all $p \in N \cap \mathbb{Q}$, there is $q \leq p$ which is strongly N -generic.

Let $V \subseteq W$ be transitive class models of ZFC. A set $X \subseteq V$ is said to be *countably approximated by V* if for any set $a \in V$ which is countable in V , $a \cap X \in V$. We say that the pair (V, W) has the ω_1 -*approximation property* if whenever X is a subset of V in W which is countably approximated by V , then $X \in V$. A forcing poset $\mathbb{Q}$ is said to have the ω_1 -*approximation property* if $\mathbb{Q}$ forces that the pair $(V, V^{\mathbb{Q}})$ has the ω_1 -approximation property.

Note that the ω_1 -approximation property is equivalent to the definition in the previous paragraph, except replacing the assumption that X is a subset of V with the assumption that X is a set of ordinals. Namely, if $X \subseteq V$, then for some α , $X \subseteq V_\alpha$. And in V we can fix a bijection $g : V_\alpha \rightarrow \mu$ for some ordinal μ . Then X is countably approximated by V iff $g[X]$ is countably approximated by V , and $X \in V$ iff $g[X] \in V$.

The next lemma shows that the ω_1 -approximation property defined above is equivalent to the version of the property used in [8].

Lemma 1.3. *A pair (V, W) has the ω_1 -approximation property iff whenever μ is an ordinal, $k : \mu \rightarrow On$ is in W , and for any set a in V which is countable in V , $k \restriction a \in V$, then $k \in V$.*

Proof. Assume that the pair (V, W) has the ω_1 -approximation property, and let $k : \mu \rightarrow On$ be a function in W satisfying that for any countable set a in V ,

$k \restriction a \in V$. We will show that $k \in V$. It suffices to show that whenever $x \in V$ is countable in V , then $k \cap x \in V$.

Suppose that x is a countable set in V , and we will show that $k \cap x \in V$. Define $x_0 := \{\xi \in \mu : \exists z (\xi, z) \in x\}$. Then x_0 is a countable subset of μ in V , so $k \restriction x_0 \in V$. It is easy to check that $k \cap x = (k \restriction x_0) \cap x$, and hence $k \cap x \in V$.

Conversely, suppose that whenever μ is an ordinal, $k : \mu \rightarrow On$ is in W , and for any set a in V which is countable in V , $k \restriction a \in V$, then $k \in V$. We will prove that (V, W) has the ω_1 -approximation property. Let X be a subset of V in W which is countably approximated by V , and we will show that $X \in V$. By the comments preceding the lemma, we may assume that $X \subseteq \mu$ for some ordinal μ .

Let $k : \mu \rightarrow 2$ be the characteristic function of X , so that $k(\alpha) = 1$ iff $\alpha \in X$. Then $k \in V$ iff $X \in V$, so it suffices to show that $k \in V$. To show that $k \in V$, it suffices to show that whenever a is a countable set in V , then $k \restriction a \in V$. So let a be a countable set in V . Then $X \cap a \in V$ by assumption. But $k \restriction a$ is equal to the function with domain $a \cap \mu$ such that for all $\alpha \in a \cap \mu$, α is mapped to 1 iff $\alpha \in X \cap a$. Since $a \cap \mu$ and $X \cap a$ are in V , so is $k \restriction a$. $\square$

Lemma 1.4. *Let $\mathbb{P}$ be a regular suborder of $\mathbb{Q}$, and suppose that $\mathbb{Q}$ forces that the pair $(V[\dot{G}_{\mathbb{Q}} \cap \mathbb{P}], V[\dot{G}_{\mathbb{Q}}])$ has the ω_1 -approximation property. Then $\mathbb{P}$ forces that $\mathbb{Q}/\dot{G}_{\mathbb{P}}$ has the ω_1 -approximation property.*

Proof. Let G be a V -generic filter on $\mathbb{P}$. Then by Lemma 1.2(2), whenever H is a $V[G]$ -generic filter on $\mathbb{Q}/G$, then H is a V -generic filter on $\mathbb{Q}$, $H \cap \mathbb{P} = G$, and $V[H] = V[G][H]$. By assumption, the pair $(V[H \cap \mathbb{P}], V[H])$ has the ω_1 -approximation property. But $(V[H \cap \mathbb{P}], V[H]) = (V[G], V[G][H])$. Thus, for any $V[G]$ -generic filter H on $\mathbb{Q}/G$, the pair $(V[G], V[G][H])$ has the ω_1 -approximation property. This means that $\mathbb{Q}/G$ has the ω_1 -approximation property in $V[G]$. $\square$

§2. Side conditions

In this section, we lay out the basic framework of side conditions which will serve as the foundation for almost everything in the paper. Our goal is to make this material as self-contained as possible. However, we do not want to prove all of the results from scratch, since that has already been done in other papers, and some of the proofs are tedious. So several of the assumptions and results in this section will be stated without proof; we will provide specific references here and in Section 13 so that an interested reader can easily find the complete details.

The basic objects we introduce are the cardinals κ and λ , the set Λ , the structure $\mathcal{A}$ on $H(\lambda)$, the classes of countable models $\mathcal{X}$ and uncountable models $\mathcal{Y}$, and the comparison point $\beta_{M,N}$, for all M and N in $\mathcal{X}$.

Notation 2.1. *For the remainder of the paper, κ is a regular cardinal satisfying that $\omega_2 \leq \kappa$, and λ is a regular cardinal such that $\kappa \leq \lambda$.*

In this paper, our interest will be in the cases where either $\kappa = \omega_2$, or κ is an inaccessible cardinal which is intended to become ω_2 in some generic extension. In the context of adding a single object by forcing, it is natural to let $\lambda = \kappa$. If multiple objects are being added by forcing, then λ will be at least κ^+ .

Notation 2.2. Fix a set Λ such that for some club $C^* \subseteq \kappa$,

$$\Lambda = C^* \cap \text{cof}(> \omega).$$

Notation 2.3. Fix a structure $\mathcal{A}$, whose underlying set is $H(\lambda)$, which has a well-ordering of $H(\lambda)$ as a predicate, and for which the sets κ and Λ are definable predicates.

Since $\mathcal{A}$ has a well-ordering as a predicate, it has definable Skolem functions. For any set $a \subseteq H(\lambda)$, let $Sk(a)$ denote the Skolem hull of a in $\mathcal{A}$ under some (any) complete set of definable Skolem functions.

Notation 2.4. Fix sets $\mathcal{X}$ and $\mathcal{Y}$ satisfying:

- (1) for all $M \in \mathcal{X}$, M is a countable elementary substructure of $\mathcal{A}$;
- (2) for all $P \in \mathcal{Y}$, P is an elementary substructure of $\mathcal{A}$, $|P| < \kappa$, $P \cap \kappa \in \kappa$, and $\text{cf}(P \cap \kappa) > \omega$.

The next assumption describes some closure properties of $\mathcal{X}$ and $\mathcal{Y}$.

Assumption 2.5. (1) If P and Q are in $\mathcal{Y}$, then $P \cap Q \in \mathcal{Y}$.
 (2) If $M \in \mathcal{X}$ and $P \in \mathcal{Y}$, then $M \cap P \in \mathcal{X}$.

Following Friedman [4], we say that a stationary set $T^* \subseteq P_{\omega_1}(\kappa)$ is *thin* if for all $\beta < \kappa$,

$$|\{a \cap \beta : a \in T^*\}| < \kappa.$$

Implicit in our listed assumptions is the existence of a thin stationary set T^* , which is used in the actual definitions of $\mathcal{A}$ and $\mathcal{X}$. For example, it will be the case that for all $M \in \mathcal{X}$, $M \cap \kappa \in T^*$. In particular, there are a limited number of sets of the form $M \cap \kappa$, where $M \in \mathcal{X}$. The next assumption follows as a consequence; see [7, Proposition 1.11] for the details.

Assumption 2.6. If $M \in \mathcal{X}$, $\alpha \in \Lambda \cup \{\kappa\}$, and $Sk(\alpha) \cap \kappa = \alpha$ if $\alpha < \kappa$, then $M \cap \alpha \in Sk(\alpha)$.

Note that there are club many $\alpha < \kappa$ such that $Sk(\alpha) \cap \kappa = \alpha$. Also, $Sk(\kappa) \cap \kappa = \kappa$ is immediate.

We have enough information now to derive some useful properties.

Lemma 2.7. Let $M \in \mathcal{X}$ and $\alpha \in \Lambda \cup \{\kappa\}$. If $\alpha < \kappa$, assume that $Sk(\alpha) \cap \kappa = \alpha$. Then:

- (1) $Sk(M \cap \alpha) = M \cap Sk(\alpha)$;
- (2) $Sk(M \cap \alpha) \cap \kappa = M \cap \alpha$.

Proof. (1) The forward inclusion is immediate. For the reverse inclusion, suppose that $x \in M \cap Sk(\alpha)$. Then since $x \in Sk(\alpha)$, there are ordinals $\beta_0, \dots, \beta_{n-1}$ in α and a definable Skolem function f of $\mathcal{A}$ such that $x = f(\beta_0, \dots, \beta_{n-1})$. Since κ and f are definable in $\mathcal{A}$, the lexicographically least tuple of ordinals $\gamma_0, \dots, \gamma_{n-1}$ in κ such that $x = f(\gamma_0, \dots, \gamma_{n-1})$ is definable in $\mathcal{A}$ from x . Since $x \in M \cap Sk(\alpha)$, it follows that $\gamma_0, \dots, \gamma_{n-1}$ are in $M \cap Sk(\alpha)$. But $Sk(\alpha) \cap \kappa = \alpha$, so $\gamma_0, \dots, \gamma_{n-1} \in M \cap \alpha$. Therefore, $x = f(\gamma_0, \dots, \gamma_{n-1}) \in Sk(M \cap \alpha)$.

(2) Using (1) and our assumption about α , we have

$$Sk(M \cap \alpha) \cap \kappa = (M \cap Sk(\alpha)) \cap \kappa = M \cap (Sk(\alpha) \cap \kappa) = M \cap \alpha.$$

□

Lemma 2.8. *Let M and N be in $\mathcal{X}$ and $\alpha \in \Lambda \cup \{\kappa\}$. If $\alpha < \kappa$, assume that $Sk(\alpha) \cap \kappa = \alpha$. Then:*

- (1) *If $M \cap \alpha \in N$, then $M \cap \alpha \in Sk(N \cap \alpha)$;*
- (2) *If $M \cap \kappa \subseteq \alpha$ and $M \cap \kappa \in N$, then $M \cap \kappa \in Sk(N \cap \alpha)$.*

Proof. (1) By Assumption 2.6, $M \cap \alpha \in Sk(\alpha)$. By Lemma 2.7(1), $M \cap \alpha \in N \cap Sk(\alpha) = Sk(N \cap \alpha)$. (2) If $M \cap \kappa \subseteq \alpha$, then $M \cap \kappa = M \cap \alpha$. So (2) follows immediately from (1). $\square$

We now introduce the *comparison point* $\beta_{M,N}$, for all M and N in $\mathcal{X}$. The actual definition of $\beta_{M,N}$ is not important for us in this paper. The only properties of $\beta_{M,N}$ which we will need are stated in Lemma 2.10 and Proposition 2.11 below.

Notation 2.9. *For M and N in $\mathcal{X}$, $\beta_{M,N}$ will denote the comparison point of M and N , as defined in [5, Definition 1.14].*

Lemma 2.10. *Let M and N be in $\mathcal{X}$. Then:*

- (1) *$\beta_{M,N} \in \Lambda$ and $\beta_{M,N} = \beta_{N,M}$;*
- (2) *if $\beta < \beta_{M,N}$ and $\beta \in \Lambda$, then $M \cap [\beta, \beta_{M,N}) \neq \emptyset$;*
- (3) *if $K \in \mathcal{X}$ and $K \subseteq M$, then $\beta_{K,N} \leq \beta_{M,N}$.*

Proof. See [5, Definition 1.14, Lemma 1.16(1,3)]. $\square$

For a set of ordinals a , let $\text{cl}(a)$ denote the union of a together with the set of limit points of a .

Proposition 2.11. *Let M and N be in $\mathcal{X}$. Then*

$$\text{cl}(M \cap \kappa) \cap \text{cl}(N \cap \kappa) \subseteq \beta_{M,N}.$$

Proof. See [5, Lemma 1.15]. $\square$

Since the property described in Proposition 2.11 is extremely important for what follows, let us review it for emphasis. The property says that if γ is an ordinal which is either in $M \cap \kappa$ or is a limit point of $M \cap \kappa$, and at the same time, is either in $N \cap \kappa$ or is a limit point of $N \cap \kappa$, then $\gamma < \beta_{M,N}$.

Lemma 2.12. *Let $P \in \mathcal{Y}$ and $M \in P \cap \mathcal{X}$. Then for all $K \in \mathcal{X}$, $\beta_{K,M} < P \cap \kappa$.*

Proof. Roughly speaking, the reason why this is true is because, given M , there are only countably many possibilities for the value of $\beta_{K,M}$, and hence they are all in P by elementarity. See [5, Lemma 1.34(1)] for the proof. $\square$

We now use the comparison point $\beta_{M,N}$ to introduce a way of comparing models in $\mathcal{X}$.

Definition 2.13. *Let M and N be in $\mathcal{X}$.*

- (1) *Let $M < N$ if $M \cap \beta_{M,N} \in N$.*
- (2) *Let $M \sim N$ if $M \cap \beta_{M,N} = N \cap \beta_{M,N}$.*
- (3) *Let $M \leq N$ if either $M < N$ or $M \sim N$.*

If $M < N$, then by elementarity, the set $\text{cl}(M \cap \beta_{M,N})$ is a member of N . Since $\text{cl}(M \cap \beta_{M,N})$ is countable, it follows that $\text{cl}(M \cap \beta_{M,N}) \subseteq N$. Also, every initial segment of the set of ordinals $M \cap \beta_{M,N}$ is in N , since there are only countably many initial segments.

Definition 2.14. A finite set $A \subseteq \mathcal{X}$ is said to be adequate if for all M and N in A , either $M < N$, $M \sim N$, or $N < M$.

Note that A is adequate iff for all M and N in A , $\{M, N\}$ is adequate. If A is adequate and $B \subseteq A$, then B is adequate. If M and N are in an adequate set A , then either $M \leq N$ or $N \leq M$.

Lemma 2.15. Suppose that $M \leq N$. Then

$$M \cap \beta_{M,N} = M \cap N \cap \kappa = M \cap N \cap \beta_{M,N}.$$

Proof. Since $M \leq N$, either $M \cap \beta_{M,N} \in N$ or $M \cap \beta_{M,N} = N \cap \beta_{M,N}$. In either case, $M \cap \beta_{M,N} \subseteq N$. So $M \cap \beta_{M,N} \subseteq M \cap N \cap \kappa$. Conversely, by Proposition 2.11, $M \cap N \cap \kappa \subseteq \beta_{M,N}$, so $M \cap N \cap \kappa \subseteq M \cap \beta_{M,N}$. This proves the first equality. For the second equality, the reverse inclusion is trivial, and the forward inclusion follows from Proposition 2.11. $\square$

Lemma 2.16. If A is adequate, $N \in \mathcal{X}$, and $A \in N$, then $A \cup \{N\}$ is adequate.

Proof. Since A is finite, $A \subseteq N$. So it suffices to show that if $M \in N \cap \mathcal{X}$, then $M < N$. But as M is countable, by elementarity every initial segment of $M \cap \kappa$ is in N , and in particular, $M \cap \beta_{M,N}$ is in N (in fact, Proposition 2.11 implies that $M \cap \kappa = M \cap \beta_{M,N}$). So $M < N$. $\square$

It turns out that if $\{M, N\}$ is adequate, then which relation holds between M and N is determined by comparing the ordinals $M \cap \omega_1$ and $N \cap \omega_1$.

Lemma 2.17. Let $\{M, N\}$ be adequate. Then:

- (1) $M < N$ iff $M \cap \omega_1 < N \cap \omega_1$;
- (2) $M \sim N$ iff $M \cap \omega_1 = N \cap \omega_1$;
- (3) $M \leq N$ iff $M \cap \omega_1 \leq N \cap \omega_1$.

Proof. Suppose that $M < N$, and we will show that $M \cap \omega_1 < N \cap \omega_1$. Since $\beta_{M,N}$ has uncountable cofinality, $\omega_1 \leq \beta_{M,N}$. Therefore, $M \cap \omega_1$ is an initial segment of $M \cap \beta_{M,N}$, and hence is in N . So $M \cap \omega_1 < N \cap \omega_1$.

Suppose that $M \sim N$, and we will show that $M \cap \omega_1 = N \cap \omega_1$. Then $M \cap \beta_{M,N} = N \cap \beta_{M,N}$. Since $\omega_1 \leq \beta_{M,N}$, $M \cap \omega_1 = N \cap \omega_1$.

Conversely, if $M \cap \omega_1 < N \cap \omega_1$, then the implications which we just proved rule out the possibilities that $N < M$ and $N \sim M$. Therefore, $M < N$. This completes the proof of (1) and (2), and (3) follows immediately. $\square$

Lemma 2.18. Let A be an adequate set. Then the relation $<$ is irreflexive and transitive on A , $\sim$ is an equivalence relation on A , $\leq$ is transitive on A , and the relations $<$ and $\leq$ respect $\sim$.

Proof. Immediate from Lemma 2.17. $\square$

We state a closure property of $\mathcal{X}$ as an assumption.

Assumption 2.19. Suppose that M and N are in $\mathcal{X}$ and $\{M, N\}$ is adequate. Then $M \cap N \in \mathcal{X}$.

The proof of this assumption depends on the actual definition of $\mathcal{X}$, which we are not giving here. The actual definitions of the sets of models $\mathcal{X}$ and $\mathcal{Y}$ which we will use in the main consistency result, as well as the proofs that they satisfy the assumptions listed in this section, will not appear until Section 13. We note that

the last assumption is fairly easy to verify in the case that $\kappa = \lambda$. However, when $\lambda > \kappa$, the assumption is highly nontrivial, and requires a very careful definition of $\mathcal{X}$. Similar comments apply to the existence of stationarily many simple models, which we describe next.

Definition 2.20. *A set $N \in \mathcal{X}$ is said to be simple if for all $M \in \mathcal{X}$, if $M < N$ then $M \cap N \in N$.*

Definition 2.21. *A set $P \in \mathcal{Y}$ is said to be simple if for all $M \in \mathcal{X}$, $M \cap P \in P$.*

Assumption 2.22. *The set of $N \in P_{\omega_1}(H(\lambda))$ such that $N \in \mathcal{X}$ and N is simple is stationary.*

Assumption 2.23. *The set of $P \in P_\kappa(H(\lambda))$ such that $P \in \mathcal{Y}$ and P is simple is stationary.*

We now state the main theorems of the basic theory of adequate sets. The proofs of these results depend on technical, and sometimes tedious, facts about comparison points, so we will omit them.

First, we handle countable models.

Proposition 2.24. *Let A be adequate and $N \in A$. Then the set*

$$B := A \cup \{M \cap N : M \in A, M < N\}$$

is adequate. Moreover, for all $M \in B$, if $M < N$ then $M \cap N \in B$.

Proof. The first statement is proven in [5, Proposition 1.25]. The second statement is easy to prove. $\square$

Proposition 2.25 (Amalgamation over countable models). *Let A be adequate, $N \in A$, and suppose that for all $M \in A$, if $M < N$ then $M \cap N \in A$. Assume that N is simple. Suppose that B is adequate and*

$$A \cap N \subseteq B \subseteq N.$$

Then $A \cup B$ is adequate.

Proof. See [5, Proposition 1.29]. $\square$

Let us derive an easy consequence of Proposition 2.24.

Lemma 2.26. *Suppose that $M < N$. Then $M \sim M \cap N$.*

Note that by Assumption 2.19, $M \cap N \in \mathcal{X}$.

Proof. Applying Proposition 2.24 to the adequate set $\{M, N\}$, we get that the set $\{M, N, M \cap N\}$ is adequate. In particular, $\{M, M \cap N\}$ is adequate. By Lemma 2.15, $M < N$ implies that $M \cap \beta_{M,N} = M \cap N \cap \beta_{M,N}$. Since $\omega_1 \leq \beta_{M,N}$, it follows that $M \cap \omega_1 = M \cap N \cap \omega_1$. As $\{M, M \cap N\}$ is adequate, Lemma 2.17(2) implies that $M \sim M \cap N$. $\square$

Next, we handle uncountable models.

Proposition 2.27. *Let A be adequate and $P \in \mathcal{Y}$. Then the set*

$$B := A \cup \{M \cap P : M \in A\}$$

is adequate. Moreover, for all $M \in B$, $M \cap P \in B$.

Proof. The first statement is proven in [5, Proposition 1.33]. The second statement is easy to prove. $\square$

Proposition 2.28 (Amalgamation over uncountable models). *Let A be adequate, $P \in \mathcal{Y}$, and suppose that for all $M \in A$, $M \cap P \in A$. Assume that P is simple. Suppose that B is adequate and*

$$A \cap P \subseteq B \subseteq P.$$

Then $A \cup B$ is adequate.

Proof. See [5, Proposition 1.35]. $\square$

Lemma 2.29. *Suppose that $M \in \mathcal{X}$ and $P \in \mathcal{Y}$. Then $M \sim M \cap P$.*

Note that by Assumption 2.5(2), $M \cap P \in \mathcal{X}$.

Proof. Applying Proposition 2.27 to the adequate set $\{M\}$, we get that $\{M, M \cap P\}$ is adequate. Since $\omega_1 \leq P \cap \kappa$, we have that $M \cap \omega_1 = M \cap P \cap \omega_1$. Hence, by Lemma 2.17(2), $M \sim M \cap P$. $\square$

We will need one more result about simple models.

Lemma 2.30. *Suppose that $N \in \mathcal{X}$ is simple and $P \in \mathcal{Y} \cap N$ is simple. Then $N \cap P$ is simple.*

Note that by Assumption 2.5(2), $N \cap P \in \mathcal{X}$.

Proof. Let $M \in \mathcal{X}$ be such that $M < N \cap P$, and we will show that $M \cap (N \cap P) \in N \cap P$. It suffices to show that $M \cap N \cap P < N$. For then, since N is simple,

$$M \cap (N \cap P) = (M \cap N \cap P) \cap N \in N,$$

and since P is simple,

$$M \cap (N \cap P) = (M \cap N \cap P) \cap P \in P.$$

So $M \cap (N \cap P) \in N \cap P$.

Since $M < N \cap P$, we have that $M \cap \beta_{M, N \cap P} \in N \cap P$. In particular, $M \cap \beta_{M, N \cap P} \in N$. We claim that $\beta_{M \cap N \cap P, N} \leq \beta_{M, N \cap P}$. If not, then by Lemma 2.10(2), we can fix $\gamma \in (M \cap N \cap P) \cap [\beta_{M, N \cap P}, \beta_{M \cap N \cap P, N})$. Then by Proposition 2.11, $\gamma < \beta_{M, N \cap P}$, which is a contradiction.

Since $M \cap \beta_{M, N \cap P} \in N$ and $\beta_{M \cap N \cap P, N} \leq \beta_{M, N \cap P}$, it follows that

$$M \cap \beta_{M \cap N \cap P, N} \in N.$$

But $M < N \cap P$ implies that

$$M \cap \beta_{M \cap N \cap P, N} \subseteq M \cap \beta_{M, N \cap P} \subseteq N \cap P.$$

Thus,

$$M \cap \beta_{M \cap N \cap P, N} = (M \cap N \cap P) \cap \beta_{M \cap N \cap P, N},$$

and so this set is in N . Hence, $M \cap N \cap P < N$. $\square$

We end this section by deriving some specialized consequences of adequacy, which will play a role in the arguments concerning the forcing poset developed in Part II.

Definition 2.31. *A set $a \subseteq P_{\omega_1}(\kappa)$ is said to be an $\in$ -chain if for all x and y in a , either $x = y$, $x \in Sk(y)$, or $y \in Sk(x)$.*

Lemma 2.32. *Suppose that $a \subseteq P_{\omega_1}(\kappa)$ is an $\in$ -chain. Assume that for all $x \in a$, $Sk(x) \cap \kappa = x$. Then for all x and y in a , $x \in Sk(y)$ iff $\sup(x) < \sup(y)$, and $x = y$ iff $\sup(x) = \sup(y)$.*

Proof. Straightforward. $\square$

Lemma 2.33. *Let A be an adequate set, $\alpha \in \Lambda$, and assume that $Sk(\alpha) \cap \kappa = \alpha$. Then the set*

$$C := \{M \cap \alpha : M \in A, \alpha \in M\}$$

is a finite $\in$ -chain and a subset of $Sk(\alpha)$.

Proof. By Assumption 2.6, C is a subset of $Sk(\alpha)$. Let M and N be in A such that $\alpha \in M \cap N$. We will show that either $M \cap \alpha = N \cap \alpha$, $M \cap \alpha \in Sk(N \cap \alpha)$, or $N \cap \alpha \in Sk(M \cap \alpha)$. Without loss of generality, assume that $M \leq N$. Then either $M \cap \beta_{M,N} = N \cap \beta_{M,N}$, or $M \cap \beta_{M,N} \in N$.

Since $\alpha \in M \cap N \cap \kappa$, it follows that $\alpha < \beta_{M,N}$ by Proposition 2.11. So if $M \cap \beta_{M,N} = N \cap \beta_{M,N}$, then intersecting both sides of this equation with α we get that $M \cap \alpha = N \cap \alpha$. Assume that $M \cap \beta_{M,N} \in N$. Then since $M \cap \alpha$ is an initial segment of $M \cap \beta_{M,N}$, $M \cap \alpha \in N$. By Lemma 2.8(1), $M \cap \alpha \in Sk(N \cap \alpha)$. $\square$

§3. Remainders

In the standard development of the basic ideas of adequate sets, the next topic which comes up is the idea of a remainder point (see [5, Section 2]). In this paper, however, only a particular kind of remainder point will be relevant, namely, those which are in the set r^* defined next.

Definition 3.1. *Let A be an adequate set. Define $r^*(A)$ as the set of ordinals γ satisfying that for some K and M in A with $K \sim M$,*

$$\gamma = \min((M \cap \kappa) \setminus \beta_{K,M}).$$

Note that $r^*(A)$ is finite. Also, $A \subseteq B$ implies that $r^*(A) \subseteq r^*(B)$.

Before analyzing the set $r^*(A)$, we first prove a very useful lemma.

Lemma 3.2. *Suppose that M and N are in $\mathcal{X}$ and $\{M, N\}$ is adequate. Assume that α and γ are uncountable ordinals satisfying:*

- (1) $\alpha \in M \cap \kappa$;
- (2) $\gamma \in (N \cap \kappa) \cup \{\kappa\}$;
- (3) $\alpha \neq \gamma$;
- (4) $M \cap \alpha = N \cap \gamma$.

Then $M \sim N$ and $\alpha = \min((M \cap \kappa) \setminus \beta_{M,N})$.

Proof. Since α and γ are uncountable, $M \cap \alpha = N \cap \gamma$ implies that $M \cap \omega_1 = N \cap \omega_1$. As $\{M, N\}$ is adequate, Lemma 2.17(2) implies that $M \sim N$. In particular, $M \cap \beta_{M,N} = N \cap \beta_{M,N}$.

Since $\alpha \neq \gamma$, either $\gamma < \alpha$ or $\alpha < \gamma$. First, assume that $\gamma < \alpha$. Then $\gamma \in N \cap \kappa$. Since $\gamma < \alpha$ and $M \cap \alpha = N \cap \gamma$, it follows that $\gamma \notin M$. But $\gamma \in N \cap \kappa$. As $M \cap \beta_{M,N} = N \cap \beta_{M,N}$ and $\gamma \notin M$, $\beta_{M,N} \leq \gamma$. So $\beta_{M,N} \leq \gamma < \alpha$.

We claim that $\alpha = \min((M \cap \kappa) \setminus \beta_{M,N})$. If not, then there is $\alpha_0 \in M \cap \kappa$ such that $\beta_{M,N} \leq \alpha_0 < \alpha$. Then $\alpha_0 \in M \cap \alpha = N \cap \gamma$, so $\alpha_0 \in N$. By Proposition 2.11, $\alpha_0 \in M \cap N \cap \kappa \subseteq \beta_{M,N}$. So $\alpha_0 < \beta_{M,N}$, which contradicts the choice of α_0 .

Secondly, assume that $\alpha < \gamma$. Since $M \cap \alpha = N \cap \gamma$, we have that $\alpha \notin N$. As $M \cap \beta_{M,N} = N \cap \beta_{M,N}$ and $\alpha \in M \cap \kappa$, it follows that $\beta_{M,N} \leq \alpha$.

We claim that $\alpha = \min((M \cap \kappa) \setminus \beta_{M,N})$. If not, then there is $\alpha_0 \in M \cap \kappa$ such that $\beta_{M,N} \leq \alpha_0 < \alpha$. But then $\alpha_0 \in M \cap \alpha = N \cap \gamma$. By Proposition 2.11, it follows that $\alpha_0 \in M \cap N \cap \kappa \subseteq \beta_{M,N}$. So $\alpha_0 < \beta_{M,N}$, which contradicts the choice of α_0 . $\square$

The main goal of this section is to prove Proposition 3.5, which handles models in $\mathcal{X}$, and Proposition 3.8, which handles models in $\mathcal{Y}$.

First, we analyze r^* in the context of models in $\mathcal{X}$.

Lemma 3.3. *Suppose that $M < N$ and $(M \cap \kappa) \setminus \beta_{M,N} \neq \emptyset$. Then*

$$\min((M \cap \kappa) \setminus \beta_{M,N}) = \min((M \cap \kappa) \setminus \beta_{M, M \cap N}).$$

Note that by Assumption 2.19, $M < N$ implies that $M \cap N \in \mathcal{X}$.

Proof. By Lemma 2.26, $M \sim M \cap N$. Let $\alpha := \min((M \cap \kappa) \setminus \beta_{M,N})$. Then by Lemma 2.15 and the minimality of α ,

$$M \cap N \cap \kappa = M \cap \beta_{M,N} = M \cap \alpha.$$

So $M \cap \alpha = M \cap N \cap \kappa$. Applying Lemma 3.2 to M , $M \cap N$, α , and κ , we get that $\alpha = \min((M \cap \kappa) \setminus \beta_{M, M \cap N})$. $\square$

Lemma 3.4. *Let $\{K, M, N\}$ be adequate, and assume that $M \in N \cap \mathcal{X}$ and $K \sim M$. Then:*

- (1) $K < N$ and $M \sim K \cap N$;
- (2) if $\alpha = \min((M \cap \kappa) \setminus \beta_{K,M})$, then $\alpha = \min((M \cap \kappa) \setminus \beta_{M, K \cap N})$;
- (3) if $\alpha = \min((K \cap \kappa) \setminus \beta_{K,M})$, then either $\alpha = \min((K \cap N \cap \kappa) \setminus \beta_{K \cap N, M})$, or $\alpha = \min((K \cap \kappa) \setminus \beta_{K, K \cap N})$.

Proof. (1) Since $K \sim M$, it follows that $K \cap \omega_1 = M \cap \omega_1$ by Lemma 2.17(2). As $M \in N$, $M \cap \omega_1 < N \cap \omega_1$. Therefore, $K \cap \omega_1 < N \cap \omega_1$. By Lemma 2.17(1), $K < N$. By Proposition 2.24, $\{K, M, N, K \cap N\}$ is adequate. By Lemma 2.26, $K \sim K \cap N$. So $M \sim K \sim K \cap N$, which by Lemma 2.18 implies that $M \sim K \cap N$.

(2) We apply Lemma 3.2 to the objects M , $K \cap N$, α , and γ , where

$$\gamma := \min(((K \cap N \cap \kappa) \cup \{\kappa\}) \setminus \beta_{K,M}).$$

Provided that the assumptions of Lemma 3.2 are true for these objects, we get that $\alpha = \min((M \cap \kappa) \setminus \beta_{M, K \cap N})$, which finishes the proof of (2).

Assumptions (1) and (2) of Lemma 3.2 are immediate, and (3) follows from Proposition 2.11 since α and γ are greater than or equal to $\beta_{K,M}$. For assumption (4), we need to show that $M \cap \alpha = K \cap N \cap \gamma$. By the minimality of α and γ , we have that $M \cap \alpha = M \cap \beta_{K,M}$ and $K \cap N \cap \gamma = K \cap N \cap \beta_{K,M}$. So it suffices to show that

$$M \cap \beta_{K,M} = K \cap N \cap \beta_{K,M}.$$

Since $K \sim M$, $M \cap \beta_{K,M} = K \cap \beta_{K,M}$. This last equation implies the reverse inclusion of the displayed equation, and noting that $M \in N$ implies that $M \subseteq N$, it implies the forward inclusion as well.

(3) Let

$$\gamma := \min((M \cap \kappa) \cup \{\kappa\}) \setminus \beta_{K,M}.$$

Then by the minimality of α and γ and the fact that $K \sim M$, we have that

$$K \cap \alpha = K \cap \beta_{K,M} = M \cap \beta_{K,M} = M \cap \gamma.$$

First, assume that $\alpha \in N$. We apply Lemma 3.2 to the objects $K \cap N$, M , α , and γ . Provided that the assumptions of Lemma 3.2 hold for these objects, we can conclude that

$$\alpha = \min((K \cap N \cap \kappa) \setminus \beta_{K \cap N, M}),$$

which completes the proof of (3). Assumptions (1) and (2) of Lemma 3.2 are immediate, and (3) follows from Proposition 2.11 since α and γ are greater than or equal to $\beta_{K,M}$.

For (4), we need to show that $K \cap N \cap \alpha = M \cap \gamma$. From the above, we already know that $K \cap \alpha = M \cap \gamma$, so it suffices to show that

$$K \cap \alpha = K \cap N \cap \alpha.$$

Since $K < N$, $K \cap \beta_{K,N} = K \cap N \cap \beta_{K,N}$ by Lemma 2.15. As $\alpha \in K \cap N \cap \kappa$, we have that $\alpha < \beta_{K,N}$ by Proposition 2.11. Hence,

$$K \cap \alpha = K \cap \beta_{K,N} \cap \alpha = K \cap N \cap \beta_{K,N} \cap \alpha = K \cap N \cap \alpha.$$

Secondly, assume that $\alpha \notin N$. Since $K < N$ by (1), we have that $K \cap \beta_{K,N} \subseteq N$, and therefore $\beta_{K,N} \leq \alpha$. To show that $\alpha = \min((K \cap \kappa) \setminus \beta_{K, K \cap N})$, by Lemma 3.3 it suffices to show that $\alpha = \min((K \cap \kappa) \setminus \beta_{K,N})$.

Suppose for a contradiction that α is not equal to $\min((K \cap \kappa) \setminus \beta_{K,N})$. Then there is $\alpha_0 \in K$ such that $\beta_{K,N} \leq \alpha_0 < \alpha$. But then $\alpha_0 \in K \cap \alpha$. Since $\alpha = \min((K \cap \kappa) \setminus \beta_{K,M})$, it follows that $\alpha_0 < \beta_{K,M}$. Now $K \sim M$ means that $K \cap \beta_{K,M} = M \cap \beta_{K,M}$. As $\alpha_0 \in K \cap \beta_{K,M}$, we have that $\alpha_0 \in M$. But $M \in N$, so $M \subseteq N$, and hence $\alpha_0 \in N$. Therefore, $\alpha_0 \in K \cap N \cap \kappa$, which implies that $\alpha_0 < \beta_{K,N}$ by Proposition 2.11, contradicting the choice of α_0 . $\square$

Proposition 3.5. *Assume that A is adequate, $N \in A$ is simple, and for all $M \in A$, if $M < N$ then $M \cap N \in A$. Let B be adequate, and suppose that*

$$A \cap N \subseteq B \subseteq N.$$

Then

$$r^*(A \cup B) = r^*(A) \cup r^*(B).$$

Proof. By Proposition 2.25, $A \cup B$ is adequate. The reverse inclusion is immediate. For the forward inclusion, it suffices to show that if $K \in A$, $M \in B$, and $K \sim M$, then $r^*({K, M}) \subseteq r^*(A) \cup r^*(B)$.

Since $M \in B \subseteq N$, we have that $M \in N$. Since K and N are in A and $K \sim M$, Lemma 3.4(1) implies that $K < N$ and $M \sim K \cap N$. So by our assumptions about A , $K \cap N \in A$. And since N is simple, $K \cap N \in N$. Therefore, $K \cap N \in A \cap N \subseteq B$. So $K \cap N$ and M are both in B . Also, by Lemma 2.26, $K \sim K \cap N$.

Suppose that $\alpha \in r^*({K, M})$, and we will show that $\alpha \in r^*(A) \cup r^*(B)$. First, assume that $\alpha = \min((M \cap \kappa) \setminus \beta_{K,M})$. Then by Lemma 3.4(2), $\alpha = \min((M \cap \kappa) \setminus \beta_{M, K \cap N})$. Since $K \cap N$ and M are in B and $M \sim K \cap N$, $\alpha \in r^*(B)$.

Secondly, assume that $\alpha = \min((K \cap \kappa) \setminus \beta_{K,M})$. Then by Lemma 3.4(3), either $\alpha = \min((K \cap N \cap \kappa) \setminus \beta_{K \cap N, M})$, or $\alpha = \min((K \cap \kappa) \setminus \beta_{K, K \cap N})$. In the first case, $\alpha \in r^*(B)$, since $K \cap N$ and M are in B and $M \sim K \cap N$. In the second case, $\alpha \in r^*(A)$, since K and $K \cap N$ are in A and $K \sim K \cap N$. $\square$

Next, we analyze the set r^* in the context of models in $\mathcal{Y}$.

Lemma 3.6. *Suppose that $M \in \mathcal{X}$, $P \in \mathcal{Y}$, and $(M \cap \kappa) \setminus (P \cap \kappa) \neq \emptyset$. Then*

$$\min((M \cap \kappa) \setminus (P \cap \kappa)) = \min((M \cap \kappa) \setminus \beta_{M \cap P, M}).$$

Proof. By Lemma 2.29, $M \sim M \cap P$. Let $\alpha := \min((M \cap \kappa) \setminus (P \cap \kappa))$. Then by the minimality of α ,

$$M \cap \alpha = M \cap P \cap \kappa.$$

Applying Lemma 3.2 to M , $M \cap P$, α , and κ , we get that $\alpha = \min((M \cap \kappa) \setminus \beta_{M, M \cap P})$. $\square$

Lemma 3.7. *Let $K \in \mathcal{X}$, $P \in \mathcal{Y}$, and assume that $M \in P \cap \mathcal{X}$ and $K \sim M$. Then:*

- (1) $K \cap P \sim M$;
- (2) if $\alpha = \min((M \cap \kappa) \setminus \beta_{K, M})$, then $\alpha = \min((M \cap \kappa) \setminus \beta_{M, K \cap P})$;
- (3) if $\alpha = \min((K \cap \kappa) \setminus \beta_{K, M})$, then either $\alpha = \min((K \cap P \cap \kappa) \setminus \beta_{K \cap P, M})$, or $\alpha = \min((K \cap \kappa) \setminus \beta_{K \cap P, K})$.

Proof. (1) Since $K \sim M$, $K \cap \beta_{K, M} = M \cap \beta_{K, M}$. By Lemma 2.12, since $M \in P$, $\beta_{K, M} < P \cap \kappa$. Thus,

$$K \cap P \cap \beta_{K, M} = M \cap \beta_{K, M}.$$

By Lemma 2.10(3), $\beta_{K \cap P, M} \leq \beta_{K, M}$. Therefore,

$$K \cap P \cap \beta_{K \cap P, M} = M \cap \beta_{K \cap P, M}.$$

So $K \cap P \sim M$.

(2) We apply Lemma 3.2 to the objects M , $K \cap P$, α , and γ , where

$$\gamma := \min(((K \cap P \cap \kappa) \cup \{\kappa\}) \setminus \beta_{K, M}).$$

Provided that the assumptions of Lemma 3.2 are true for these objects, it follows that $\alpha = \min((M \cap \kappa) \setminus \beta_{M, K \cap P})$, and we are done.

Assumptions (1) and (2) are immediate, and (3) follows from Proposition 2.11 since α and γ are greater than or equal to $\beta_{K, M}$. It remains to show that $M \cap \alpha = K \cap P \cap \gamma$. By the minimality of α and γ , we have that $M \cap \alpha = M \cap \beta_{K, M}$ and $K \cap P \cap \gamma = K \cap P \cap \beta_{K, M}$. So it suffices to show that

$$M \cap \beta_{K, M} = K \cap P \cap \beta_{K, M}.$$

Since $K \sim M$, $M \cap \beta_{K, M} = K \cap \beta_{K, M}$. Hence, it is enough to show that

$$K \cap \beta_{K, M} = K \cap P \cap \beta_{K, M}.$$

But this follows immediately from the fact that $\beta_{K, M} < P \cap \kappa$, which is true by Lemma 2.12.

(3) Let

$$\gamma := \min(((M \cap \kappa) \cup \{\kappa\}) \setminus \beta_{K, M}).$$

By Proposition 2.11, $\alpha \neq \gamma$, since α and γ are greater than or equal to $\beta_{K, M}$. By the minimality of α and γ and the fact that $K \sim M$, we have that

$$K \cap \alpha = K \cap \beta_{K, M} = M \cap \beta_{K, M} = M \cap \gamma.$$

First, assume that $\alpha < P \cap \kappa$. Then $K \cap \alpha = K \cap P \cap \alpha$. Therefore, by the last paragraph,

$$K \cap P \cap \alpha = M \cap \gamma.$$

Applying Lemma 3.2 to the objects $K \cap P$, M , α , and γ , it follows that

$$\alpha = \min((K \cap P \cap \kappa) \setminus \beta_{K \cap P, M}).$$

Secondly, assume that $P \cap \kappa \leq \alpha$. To show that $\alpha = \min((K \cap \kappa) \setminus \beta_{K \cap P, K})$, by Lemma 3.6 it suffices to show that $\alpha = \min((K \cap \kappa) \setminus (P \cap \kappa))$.

Suppose for a contradiction that α is not equal to $\min((K \cap \kappa) \setminus (P \cap \kappa))$. Then there is $\alpha_0 \in K \cap \kappa$ such that $P \cap \kappa \leq \alpha_0 < \alpha$. Then $\alpha_0 \in K \cap \alpha$. Since $\alpha = \min((K \cap \kappa) \setminus \beta_{K, M})$, it follows that $\alpha_0 < \beta_{K, M}$. Hence, $\alpha_0 \in K \cap \beta_{K, M}$. As $K \sim M$, we have that $K \cap \beta_{K, M} = M \cap \beta_{K, M}$. So $\alpha_0 \in M$. But $M \in P$, so $M \subseteq P$. Therefore, $\alpha_0 \in P \cap \kappa$, which contradicts the choice of α_0 . $\square$

Proposition 3.8. *Suppose that A is adequate, $P \in \mathcal{Y}$ is simple, and for all $M \in A$, $M \cap P \in A$. Let B be adequate, and suppose that*

$$A \cap P \subseteq B \subseteq P.$$

Then

$$r^*(A \cup B) = r^*(A) \cup r^*(B).$$

Proof. By Proposition 2.28, $A \cup B$ is adequate. The reverse inclusion is immediate. For the forward inclusion, it suffices to show that if $K \in A$, $M \in B$, and $K \sim M$, then $r^*(\{K, M\}) \subseteq r^*(A) \cup r^*(B)$.

Since $M \in B \subseteq P$, we have that $M \in P$. Thus, the assumptions of Lemma 3.7 are satisfied. By Lemma 3.7(1), $M \sim K \cap P$. By the assumptions on A , $K \cap P \in A$. As P is simple, $K \cap P \in P$. So $K \cap P \in A \cap P \subseteq B$. Hence, $K \cap P$ is in B . By Lemma 2.29, $K \sim K \cap P$.

Suppose that $\alpha \in r^*(\{K, M\})$, and we will show that $\alpha \in r^*(A) \cup r^*(B)$. First, assume that $\alpha = \min((M \cap \kappa) \setminus \beta_{K, M})$. Then by Lemma 3.7(2), $\alpha = \min((M \cap \kappa) \setminus \beta_{M, K \cap P})$. Since $K \cap P$ and M are in B and $M \sim K \cap P$, it follows that $\alpha \in r^*(B)$.

Secondly, assume that $\alpha = \min((K \cap \kappa) \setminus \beta_{K, M})$. Then by Lemma 3.7(3), either $\alpha = \min((K \cap P \cap \kappa) \setminus \beta_{K \cap P, M})$, or $\alpha = \min((K \cap \kappa) \setminus \beta_{K \cap P, K})$. In the first case, since $K \cap P$ and M are in B and $K \cap P \sim M$, it follows that $\alpha \in r^*(B)$. In the second case, since K and $K \cap P$ are in A and $K \sim K \cap P$, it follows that $\alpha \in r^*(A)$. $\square$

Part 2. The Single Forcing

§4. The forcing poset

We introduce a forcing poset for adding a partial square sequence to a stationary set $S \subseteq \kappa \cap \text{cof}(> \omega)$. This forcing poset will preserve ω_1 , is κ -c.c., and if $\kappa > \omega_2$, then it will collapse κ to become ω_2 .

Notation 4.1. *Fix, for the remainder of Part 2, a set $S \subseteq \Lambda$ which is stationary in κ and satisfies that for all $\alpha \in S$, $Sk(\alpha) \cap \kappa = \alpha$.*

Definition 4.2. *Let $\mathbb{P}$ be the forcing poset whose conditions are triples of the form $p = (f_p, g_p, A_p)$ satisfying the following requirements:¹*

- (1) A_p is an adequate set;

¹We will sometimes refer to f_p , g_p , and A_p as the f , g , and A components of p .

- (2) f_p is a function with a finite domain, and for all $x \in \text{dom}(f_p)$, either $x \in S$, or there is $M \in A_p$ and

$$\alpha \in (M \cap \text{dom}(f_p) \cap S) \cup \{\kappa\}$$

satisfying that $x = M \cap \alpha$; moreover, for all $x \in \text{dom}(f_p)$, $f_p(x)$ is a finite $\in$ -chain and a subset of $Sk(x)$;

- (3) if $x \in \text{dom}(f_p)$, then $f_p(x) \subseteq \text{dom}(f_p)$, and for all $K \in f_p(x)$,

$$f_p(K) = f_p(x) \cap Sk(K);$$

- (4) g_p is a function whose domain is the set of all pairs (K, x) such that $K \in f_p(x)$, and for all $(K, x) \in \text{dom}(g_p)$, $g_p(K, x)$ is a finite subset of $x \setminus \text{sup}(K)$;

- (5) if $K \in f_p(L)$ and $L \in f_p(x)$, then $g_p(K, x) \subseteq g_p(K, L)$;²

- (6) if $\alpha \in \text{dom}(f_p) \cap S$, $M \in A_p$, and $\alpha \in M$, then $M \cap \alpha \in f_p(\alpha)$;

- (7) $r^*(A_p) \cap S \subseteq \text{dom}(f_p)$.

For conditions p and q in $\mathbb{P}$, we let $q \leq p$ if:

- (a) $A_p \subseteq A_q$;
- (b) $\text{dom}(f_p) \subseteq \text{dom}(f_q)$, and for all $x \in \text{dom}(f_p)$, $f_p(x) \subseteq f_q(x)$;
- (c) for all $(K, x) \in \text{dom}(g_p)$, $g_p(K, x) \subseteq g_q(K, x)$;
- (d) if K and x are in $\text{dom}(f_p)$ and $K \in f_q(x)$, then $K \in f_p(x)$.

Let us summarize some of the main properties which we will prove about $\mathbb{P}$. The forcing poset $\mathbb{P}$ will be shown to be strongly proper on a stationary set, and thus preserve ω_1 , and to be κ -c.c. In particular, $\mathbb{P}$ preserves the stationarity of S . If $\kappa > \omega_2$, then $\mathbb{P}$ collapses all cardinals μ such that $\omega_1 < \mu < \kappa$ to have size ω_1 , and hence forces that $\kappa = \omega_2$. Finally, $\mathbb{P}$ forces that there exists a partial square sequence on S , and in particular, forces that S is in the approachability ideal $I[\omega_2]$.

The properties of $\mathbb{P}$ just listed will be proved in Sections 4–7. In Sections 8 and 9, we will derive some additional information about the forcing poset $\mathbb{P}$, and use this information to show that certain quotients of $\mathbb{P}$ have the ω_1 -approximation property.

In the remainder of the current section, we will prove some basic facts about $\mathbb{P}$ which we will need.

Lemma 4.3. *Let $p \in \mathbb{P}$ and $x \in \text{dom}(f_p)$. Then:*

- (1) $Sk(x) \cap \kappa = x$;
- (2) for all $K \in f_p(x)$, $K \subseteq x$;
- (3) if $N \in \mathcal{X}$, $\alpha \in S$, $x \in N \setminus S$, and $x \subseteq \alpha$, then $x \in Sk(N \cap \alpha)$;
- (4) if $P \in \mathcal{Y}$ and $\text{sup}(x) < P \cap \kappa$, then $x \in P$.

Proof. (1) By Definition 4.2(2), either $x \in S$, or there is $M \in A_p$ and $\alpha \in (M \cap \text{dom}(f_p) \cap S) \cup \{\kappa\}$ such that $x = M \cap \alpha$. Then $Sk(x) \cap \kappa = x$ holds by Notation 4.1 in the first case, and by Lemma 2.7(2) in the second case.

(2) Suppose that $K \in f_p(x)$, and we will show that $K \subseteq x$. By Definition 4.2(2,3), K is a countable subset of κ in $Sk(x)$. By the elementarity of $Sk(x)$, $K \subseteq Sk(x)$. As $K \subseteq \kappa$, it follows by (1) that $K \subseteq Sk(x) \cap \kappa = x$.

(3) Fix $M \in A_p$ and $\beta \in (M \cap \text{dom}(f_p) \cap S) \cup \{\kappa\}$ such that $x = M \cap \beta$. Since $M \cap \beta = x \in N$, we have that $x = M \cap \beta \in Sk(N \cap \beta)$ by Lemma 2.8(1). If $\beta \leq \alpha$, then $x \in Sk(N \cap \beta) \subseteq Sk(N \cap \alpha)$, and we are done. If $\alpha < \beta$, then since

²Note that if $K \in f_p(L)$ and $L \in f_p(x)$, then $K \in f_p(x)$ by requirement (3).

$x = M \cap \beta \subseteq \alpha$, $M \cap \beta = M \cap \alpha$. So $M \cap \alpha = x \in N$. Hence, $x \in Sk(N \cap \alpha)$ by Lemma 2.8(1).

(4) If $x \in S$, then $x = \sup(x) \in P$. Otherwise by Definition 4.2(2) there is $M \in A_p$ and $\beta \in (M \cap \text{dom}(f_p) \cap S) \cup \{\kappa\}$ such that $x = M \cap \beta$. Let $\alpha := P \cap \kappa$. Then by the elementarity of P , $Sk(\alpha) \cap \kappa = \alpha$. By Assumption 2.6, $M \cap \alpha \in Sk(\alpha) \subseteq P$. So $M \cap \alpha \in P$. If $\beta \leq \alpha$, then $x = M \cap \beta$ is an initial segment of $M \cap \alpha$, and hence is in P . If $\alpha < \beta$, then since $\sup(x) = \sup(M \cap \beta) < \alpha$, $M \cap \alpha = M \cap \beta = x$, which is in P . $\square$

Lemma 4.4. *Let $p \in \mathbb{P}$ and $z \in \text{dom}(f_p)$. Then for all x and y in $f_p(z)$, $x \in Sk(y)$ iff $\sup(x) < \sup(y)$, and $x = y$ iff $\sup(x) = \sup(y)$.*

Proof. By Definition 4.2(3), $f_p(z) \subseteq \text{dom}(f_p)$. So by Lemma 4.3(1), if $x \in f_p(z)$ then $Sk(x) \cap \kappa = x$. The lemma now follows from Lemma 2.32, letting $a = f_p(z)$. $\square$

Lemma 4.5. *Let $p \in \mathbb{P}$ and $\alpha \in S \setminus \text{dom}(f_p)$. If $M \in A_p$ and $\alpha \in M$, then $M \cap \alpha$ is not in $\text{dom}(f_p)$.*

Proof. Suppose for a contradiction that $M \cap \alpha \in \text{dom}(f_p)$. Then by Definition 4.2(2), there is $M_1 \in A_p$ and $\beta \in (M_1 \cap \text{dom}(f_p) \cap S) \cup \{\kappa\}$ such that $M \cap \alpha = M_1 \cap \beta$. Since $\beta \in \text{dom}(f_p)$ and $\alpha \notin \text{dom}(f_p)$, $\alpha \neq \beta$. By Lemma 3.2 applied to M , M_1 , α , and β , we have that $M \sim M_1$ and

$$\alpha = \min((M \cap \kappa) \setminus \beta_{M, M_1}).$$

In particular, $\alpha \in r^*(A_p) \cap S$. So by Definition 4.2(7), $\alpha \in \text{dom}(f_p)$, which contradicts our assumptions. $\square$

We show next that for any condition p and any ordinal α in S , there is $q \leq p$ with $\alpha \in \text{dom}(f_q)$. Among other things, this fact will allow us to prove that $\mathbb{P}$ adds a partial square sequence whose domain is all of S .

Lemma 4.6. *Let $p \in \mathbb{P}$, and let α and β be distinct ordinals in $S \setminus \text{dom}(f_p)$. Then the sets*

$$\text{dom}(f_p), \{M \cap \alpha : M \in A_p, \alpha \in M\}, \{N \cap \beta : N \in A_p, \beta \in N\}$$

are pairwise disjoint.

Proof. By Lemma 4.5, the first and second sets are disjoint, and the first and third sets are disjoint. If the second and third sets are not disjoint, then $M \cap \alpha = N \cap \beta$, for some M and N in A_p with $\alpha \in M$ and $\beta \in N$. Applying Lemma 3.2 to M , N , α , and β , we get that $M \sim N$ and $\alpha = \min((M \cap \kappa) \setminus \beta_{M, N})$. In particular, $\alpha \in r^*(A_p) \cap S$. So by Definition 4.2(7), $\alpha \in \text{dom}(f_p)$, which contradicts our assumptions. $\square$

Definition 4.7. *Let $p \in \mathbb{P}$, and let x be a finite subset of $S \setminus \text{dom}(f_p)$. Define $p+x$ as the triple (f, g, A) satisfying:*

- (1) $A := A_p$;
- (2) $\text{dom}(f) := \text{dom}(f_p) \cup x \cup \{M \cap \alpha : M \in A_p, \alpha \in M \cap x\}$;
- (3) for each $z \in \text{dom}(f_p)$, $f(z) := f_p(z)$;
- (4) for each $\alpha \in x$, $f(\alpha) := \{M \cap \alpha : M \in A_p, \alpha \in M\}$;
- (5) for each $\alpha \in x$ and $M \in A_p$ with $\alpha \in M$, $f(M \cap \alpha) := f(\alpha) \cap Sk(M \cap \alpha)$, where $f(\alpha)$ was defined in (4);
- (6) the domain of g is the set of pairs (K, z) such that $K \in f(z)$;

- (7) $g(K, z) = g_p(K, z)$ if $(K, z) \in \text{dom}(g_p)$, and $g(K, z) = \emptyset$ if $(K, z) \in \text{dom}(g) \setminus \text{dom}(g_p)$.

Note that by Lemma 4.6, $f(M \cap \alpha)$ in (5) is well-defined, since the set $M \cap \alpha$ is not in $\text{dom}(f_p)$ and uniquely determines α .

Lemma 4.8. *Let $p \in \mathbb{P}$, and let x be a finite subset of $S \setminus \text{dom}(f_p)$. Then $p + x$ is a condition in $\mathbb{P}$, and $p + x \leq p$.*

Proof. Let $q := p + x$. Assuming that q is a condition, it is easy to check that $q \leq p$ from the definition of q and the fact that for all M and z in $\text{dom}(f_p)$, $M \in f_p(z)$ iff $M \in f_q(z)$.

To show that q is a condition, we verify requirements (1)–(7) of Definition 4.2. (1), (4), (6), and (7) are immediate from the definition of q , and (3) is easy to check using the definition of f_q . It remains to prove (2) and (5).

(2) Clearly f_q is a function with a finite domain, and every $z \in \text{dom}(f_q)$ has the right form. Let $z \in \text{dom}(f_q)$, and we will show that $f_q(z)$ is a finite $\in$ -chain and $f_q(z) \subseteq \text{Sk}(z) \setminus S$. This is immediate if $z \in \text{dom}(f_p)$.

If $z = M \cap \alpha$, for some $M \in A_p$ with $\alpha \in M \cap x$, then $f_q(M \cap \alpha) = f_q(\alpha) \cap \text{Sk}(M \cap \alpha)$ is obviously a subset of $\text{Sk}(M \cap \alpha)$, and will be a finite $\in$ -chain disjoint from S provided that $f_q(\alpha)$ is. So it suffices to show that $f_q(\alpha) = \{M \cap \alpha : M \in A_p, \alpha \in M\}$ is a finite $\in$ -chain and is a subset of $\text{Sk}(\alpha) \setminus S$. This set is obviously disjoint from S , and it is a finite $\in$ -chain and a subset of $\text{Sk}(\alpha)$ by Lemma 2.33.

(5) Suppose that $K \in f_q(L)$ and $L \in f_q(z)$. We will show that $g_q(K, z) \subseteq g_q(K, L)$. If $z \in \text{dom}(f_p)$, then $f_q(z) = f_p(z)$; this implies that K and L are in $\text{dom}(f_p)$ as well, so $K \in f_p(L)$ and $L \in f_p(z)$. Hence, $g_q(K, z) = g_p(K, z) \subseteq g_p(K, L) = g_q(K, L)$. On the other hand, if z is not in $\text{dom}(f_p)$, then $(K, z) \notin \text{dom}(g_p)$. Therefore, $g_q(K, z) = \emptyset \subseteq g_q(K, L)$. $\square$

The next lemma will be needed in the amalgamation arguments of Section 7.

Lemma 4.9. *Let p be a condition. Then there is $q \leq p$ satisfying that whenever $K \in f_q(x)$ and $x \in f_q(y)$, then*

$$g_q(K, x) \subseteq g_q(K, y).^3$$

Moreover, $f_q = f_p$ and $A_q = A_p$.

Proof. Define q as follows. Let $f_q := f_p$ and $A_q := A_p$. For any K and y such that $K \in f_p(y)$, define

$$g_q(K, y) := \bigcup \{g_p(K, x) : x = y, \text{ or } (K \in f_p(x) \text{ and } x \in f_p(y))\}.$$

It is trivial to check that if q is a condition, then $q \leq p$. To show that q is a condition, we verify requirements (1)–(7) of Definition 4.2. (1), (2), (3), (6), and (7) are immediate. It remains to prove (4) and (5).

(4) The domain of g_q is equal to the set of pairs (K, y) , where $K \in f_q(y)$. Let $K \in f_q(y)$, and we will show that $g_q(K, y)$ is a finite subset of $y \setminus \text{sup}(K)$. By the definition of $g_q(K, y)$ and the fact that p is a condition, it is clear that $g_q(K, y)$ is finite and every ordinal in $g_q(K, y)$ is greater than or equal to $\text{sup}(K)$. It remains to show that $g_q(K, y) \subseteq y$.

³Note that by Definition 4.2(5), any such condition satisfies that whenever $K \in f_q(x)$ and $x \in f_q(y)$, then $g_q(K, x) = g_q(K, y)$.

Let $\xi \in g_q(K, y)$, and we will show that $\xi \in y$. By the definition of g_q , either $\xi \in g_p(K, y)$, or else there is some x satisfying that $K \in f_p(x)$, $x \in f_p(y)$, and $\xi \in g_p(K, x)$. In the first case, $\xi \in y$, since p is a condition. In the second case, $\xi \in g_p(K, x) \subseteq x$. But $x \in f_p(y)$, which implies that $x \subseteq y$ by Lemma 4.3(2). So $\xi \in y$.

(5) Suppose that $K \in f_q(L)$ and $L \in f_q(y)$. We will show that $g_q(K, y) \subseteq g_q(K, L)$. Since $f_q = f_p$, we have that $K \in f_p(L)$ and $L \in f_p(y)$. Let $\xi \in g_q(K, y)$, and we will show that $\xi \in g_q(K, L)$. By the definition of $g_q(K, y)$, either

(a) $\xi \in g_p(K, y)$, or

(b) there is x such that $K \in f_p(x)$, $x \in f_p(y)$, and $\xi \in g_p(K, x)$.

In case a, since p is a condition, $g_p(K, y) \subseteq g_p(K, L)$. Also, $g_p(K, L) \subseteq g_q(K, L)$ by the definition of g_q . So $\xi \in g_p(K, y) \subseteq g_p(K, L) \subseteq g_q(K, L)$.

Consider case b. Since x and L are both in $f_p(y)$, either $L \in f_p(x)$, $x = L$, or $x \in f_p(L)$. Assume that $L \in f_p(x)$. Then, since p is a condition,

$$\xi \in g_p(K, x) \subseteq g_p(K, L) \subseteq g_q(K, L).$$

Assume that $x = L$. Then

$$\xi \in g_p(K, x) = g_p(K, L) \subseteq g_q(K, L).$$

Finally, assume that $x \in f_p(L)$. Then by the definition of $g_q(K, L)$, since $K \in f_p(x)$ and $x \in f_p(L)$, $g_p(K, x) \subseteq g_q(K, L)$. So $\xi \in g_q(K, L)$.

This completes the proof that q is a condition. To show that q is as required, suppose that $K \in f_q(x)$ and $x \in f_q(y)$, and we will show that $g_q(K, x) \subseteq g_q(K, y)$. Then $K \in f_p(x)$ and $x \in f_p(y)$.

Let $\xi \in g_q(K, x)$, and we will show that $\xi \in g_q(K, y)$. By the definition of $g_q(K, x)$, either $\xi \in g_p(K, x)$, or $\xi \in g_p(K, x_0)$ for some x_0 with $K \in f_p(x_0)$ and $x_0 \in f_p(x)$. In the second case, $x_0 \in f_p(x)$ and $x \in f_p(y)$ imply that $x_0 \in f_p(y)$.

Let x' be equal to x or x_0 depending on the first or the second case. Then in either case, $K \in f_p(x')$ and $x' \in f_p(y)$, and also $\xi \in g_p(K, x')$. By the definition of g_q , $g_p(K, x') \subseteq g_q(K, y)$. Thus, $\xi \in g_q(K, y)$, as required. $\square$

§5. A partial square sequence

In Sections 6 and 7, we will prove that $\mathbb{P}$ preserves ω_1 , is κ -c.c., and collapses κ to become ω_2 . The proofs of these facts are quite involved. So it makes sense, from the expositional point of view, to assume for the time being that they are true, and show that the forcing poset $\mathbb{P}$ does what it is intended to do, namely, to add a partial square sequence on S .

To be precise, in this section we will assume exactly that $\mathbb{P}$ preserves ω_1 , forces that κ is equal to ω_2 , and that Lemma 7.1 from Section 7 below holds.

Let $\dot{f}$ be a $\mathbb{P}$ -name for a function such that $\mathbb{P}$ forces that for all x ,

$$\dot{f}(x) = \bigcup \{f_p(x) : p \in \dot{G}_{\mathbb{P}}, x \in \text{dom}(f_p)\}.$$

It is easy to check that $\mathbb{P}$ forces that $\dot{f}(x)$ is an $\in$ -chain of countable subsets of κ and is a subset of $Sk(x) \setminus S$. Note that by Lemma 4.4, $\mathbb{P}$ forces that if J and K are in $\dot{f}(x)$, then $\sup(J) < \sup(K)$ iff $J \in Sk(K)$.

For each $\alpha \in S$, let $\dot{c}_\alpha$ be a $\mathbb{P}$ -name such that $\mathbb{P}$ forces that

$$\dot{c}_\alpha = \{\sup(M) : M \in \dot{f}(\alpha)\}.$$

We will prove that $\mathbb{P}$ forces that the sequence

$$\langle \dot{c}_\alpha : \alpha \in S \rangle$$

is a partial square sequence on S .

Lemma 5.1. *Let $\alpha \in S$. Then $\mathbb{P}$ forces that for all $K \in \dot{f}(\alpha)$,*

$$\dot{f}(K) = \dot{f}(\alpha) \cap Sk(K)$$

and

$$\dot{c}_\alpha \cap \sup(K) = \{\sup(J) : J \in \dot{f}(K)\}.$$

Proof. Straightforward. $\square$

Lemma 5.2. *Let $\alpha \in S$. Then $\mathbb{P}$ forces that $\dot{c}_\alpha$ is a cofinal subset of α with order type ω_1 .*

Proof. We first show that $\dot{c}_\alpha$ is forced to be a cofinal subset of α . So let $\gamma < \alpha$ and $p \in \mathbb{P}$. Using Lemma 4.8, we can fix $q \leq p$ with $\alpha \in \text{dom}(f_q)$. By Lemma 7.1, fix $r \leq q$ such that for some $N \in \mathcal{X}$ with γ and α in N , $N \in A_r$. By Definition 4.2(6), $N \cap \alpha \in f_r(\alpha)$. So r forces that $\gamma < \sup(N \cap \alpha) \in \dot{c}_\alpha$.

Now we show that $\dot{c}_\alpha$ is forced to have order type equal to ω_1 . Since α has uncountable cofinality and $\dot{c}_\alpha$ is forced to be cofinal in α , clearly $\dot{c}_\alpha$ is forced to have an order type of uncountable cofinality. If it is not forced to have order type equal to ω_1 , then some condition forces that it has a proper initial segment of order type ω_1 . Hence, for some $p \in \mathbb{P}$ and $K \in f_p(\alpha)$, p forces that $\dot{c}_\alpha \cap \sup(K)$ has order type equal to ω_1 .

Let G be a generic filter on $\mathbb{P}$ which contains p , and let $c_\alpha := \dot{c}_\alpha^G$ and $f := \dot{f}^G$. By Lemma 5.1, $c_\alpha \cap \sup(K) = \{\sup(J) : J \in f(K)\}$. Since $c_\alpha \cap \sup(K)$ is uncountable, it follows that $f(K)$ is uncountable. But since $K \in f(\alpha)$, by Lemma 5.1 we have that $f(K) = f(\alpha) \cap Sk(K) \subseteq Sk(K)$. As $Sk(K)$ is countable, so is $f(K)$, and we have a contradiction. $\square$

Proposition 5.3. *Let $\alpha \in S$. Suppose that $p \in \mathbb{P}$ and p forces that $\xi < \alpha$ is a limit point of $\dot{c}_\alpha$. Then there is $q \leq p$ such that for some $M \in f_q(\alpha)$, $\sup(M) = \xi$. In particular, $\mathbb{P}$ forces that $\dot{c}_\alpha$ is closed.*

Proof. Note that by Lemma 5.2, ξ must have cofinality ω . Extend p to q so that for some $M \in f_q(\alpha)$, q forces that M is the membership least element of $\dot{f}(\alpha)$ with $\xi \leq \sup(M)$. We will prove that $\sup(M) = \xi$, which finishes the proof.

Claim 1: If $K \in f_q(\alpha)$ and $\sup(K \cap \xi) < \xi$, then $\sup(K) < \xi$.

Suppose for a contradiction that $K \in f_q(\alpha)$, $\sup(K \cap \xi) < \xi$, but $\xi \leq \sup(K)$. Since ξ is forced by q to be a limit point of $\dot{c}_\alpha$, there exist $t \leq q$ and N such that $N \in f_t(\alpha)$ and

$$\sup(K \cap \xi) < \sup(N) < \xi.$$

As K and N are in $f_t(\alpha)$ and $\sup(N) < \xi \leq \sup(K)$, it follows that $N \in Sk(K)$ by Lemma 4.4. By elementarity, $\sup(N) \in Sk(K)$. By Lemma 4.3(1), $Sk(K) \cap \kappa = K$. So $\sup(N) \in K$. Thus, $\sup(N) \in K \cap \xi$, contradicting that $\sup(K \cap \xi) < \sup(N)$.

It easily follows from Claim 1 that $f_q(\alpha)$ is the union of the sets A_1 and A_2 defined by

$$A_1 := \{K \in f_q(\alpha) : \sup(K) < \xi\},$$

$$A_2 := \{K \in f_q(\alpha) : \sup(K \cap \xi) = \xi\}.$$

Namely, if $K \in f_q(\alpha)$, then either $\sup(K \cap \xi) < \xi$, in which case $\sup(K) < \xi$ by Claim 1, or else $\sup(K \cap \xi) = \xi$. Note that since $f_q(\alpha)$ is an $\in$ -chain, if $K \in A_1$ and $L \in A_2$, then $\sup(K) < \sup(L)$, and therefore $K \in Sk(L)$ by Lemma 4.4.

Observe that since $M \in f_q(\alpha)$ and $\xi \leq \sup(M)$, we have that $M \in A_2$. So for all $K \in A_1$, $K \in Sk(M)$. Also, since M is the membership least element of $f_q(\alpha)$ with $\xi \leq \sup(M)$, we have that for all $N \in A_2$, either $M = N$ or $M \in Sk(N)$. In particular, for all $N \in A_2$, $M \subseteq N$.

Now we prove the proposition. Assume for a contradiction that $\sup(M) \neq \xi$. Then $M \cap [\xi, \alpha) \neq \emptyset$. Fix $\gamma < \xi$ large enough so that if $J \in A_1$, then $\sup(J) < \gamma$. This is possible since A_1 is finite.

As q forces that ξ is a limit point of $\dot{c}_\alpha$, we can fix $s \leq q$ and K such that:

- (1) K is the membership largest element of $f_s(\alpha)$ with $\sup(K) < \xi$;
- (2) $K \notin \text{dom}(f_q)$;
- (3) $\gamma < \sup(K)$;
- (4) $K \cap \omega_1$ is different from $L \cap \omega_1$, for all $L \in A_q$.

By Definition 4.2(2), fix $K_1 \in A_s$ and $\beta \in (K_1 \cap \text{dom}(f_s) \cap S) \cup \{\kappa\}$ such that $K = K_1 \cap \beta$.

We use s , K , K_1 , and β to define an extension r of q . Let

$$A_r := A_q \cup \{K_1\}.$$

If $\beta = \kappa$, then let

$$\text{dom}(f_r) := \text{dom}(f_q) \cup \{K\} \cup \{K_1 \cap \delta : \delta \in K_1 \cap \text{dom}(f_q) \cap S\},$$

and if $\beta < \kappa$, then let

$$\text{dom}(f_r) := \text{dom}(f_q) \cup \{K\} \cup \{K_1 \cap \delta : \delta \in K_1 \cap \text{dom}(f_q) \cap S\} \cup \{\beta\} \cup \{L \cap \beta : L \in A_q, \beta \in L\}.$$

Note that the domain of f_r is a subset of the domain of f_s , since $s \leq q$, $K \in f_s(\alpha)$, $K_1 \in A_s$, and $\beta \in (K_1 \cap \text{dom}(f_s) \cap S) \cup \{\kappa\}$. Thus, it makes sense to define, for each $x \in \text{dom}(f_r)$,

$$f_r(x) := f_s(x) \cap \text{dom}(f_r).$$

Observe that since $K \in f_s(\alpha)$ and K and α are in $\text{dom}(f_r)$, we have that $K \in f_r(\alpha)$.

Let $J \in f_r(x)$, and we will define $g_r(J, x)$. We let $g_r(J, x) := g_s(J, x)$, unless $J = K$ and $x \in f_r(\alpha) \cup \{\alpha\}$, in which case we let

$$g_r(J, x) := g_s(J, x) \cup \{\zeta\},$$

where

$$\zeta := \min(M \setminus \xi).$$

The ordinal ζ exists because we are assuming for a contradiction that $\sup(M) > \xi$. Note that in either case, we have that $g_s(J, x) \subseteq g_r(J, x)$.

We will prove that r is a condition and $r \leq q$. Let us see that this gives us a contradiction. If r is a condition and $r \leq q$, then r forces that ξ is a limit

point of $\dot{c}_\alpha$. But $\sup(K) < \xi$, so we can find $u \leq r$ and $L \in f_u(\alpha)$ such that $\sup(K) < \sup(L) < \xi$. Then K and L are in $f_u(\alpha)$, and since $\sup(K) < \sup(L)$, $K \in Sk(L)$ by Lemma 4.4. So $K \in f_u(\alpha) \cap Sk(L) = f_u(L)$. Therefore, by Definition 4.2(4,5),

$$g_u(K, \alpha) \subseteq g_u(K, L) \subseteq L.$$

By the definition of g_r , $\zeta \in g_r(K, \alpha)$. Hence,

$$\zeta \in g_r(K, \alpha) \subseteq g_u(K, \alpha) \subseteq L.$$

So $\zeta \in L$. This is a contradiction, since $\sup(L) < \xi \leq \zeta$.

Suppose for a moment that r is a condition, and let us prove that $r \leq q$. We verify properties (a)–(d) of Definition 4.2. (a,b) By the definition of r , $A_q \subseteq A_r$ and $\text{dom}(f_q) \subseteq \text{dom}(f_r)$. If $x \in \text{dom}(f_q)$, then since $s \leq q$,

$$f_q(x) \subseteq f_s(x) \cap \text{dom}(f_q) \subseteq f_s(x) \cap \text{dom}(f_r) = f_r(x).$$

(c) Suppose that $(J, x) \in \text{dom}(g_q)$. Then $J \neq K$, since $K \notin \text{dom}(f_q)$. So by the definition of g_r , $g_r(J, x) = g_s(J, x)$. Since $s \leq q$, we have that $g_q(J, x) \subseteq g_s(J, x) = g_r(J, x)$. (d) Assume that J and x are in $\text{dom}(f_q)$ and $J \in f_r(x)$. Then by the definition of f_r , $J \in f_s(x)$. Since $s \leq q$, it follows that $J \in f_q(x)$.

It remains to prove that r is a condition. We verify requirements (1)–(7) of Definition 4.2.

(1) We have that $A_r = A_q \cup \{K_1\} \subseteq A_s$. Since A_s is adequate, so is A_r .

(2) It is obvious that f_r is a function with a finite domain. Let $x \in \text{dom}(f_r)$, and we will show that either $x \in S$, or there is $L \in A_r$ and $\delta \in (L \cap \text{dom}(f_r) \cap S) \cup \{\kappa\}$ such that $x = L \cap \delta$. If $x \in \text{dom}(f_q)$, then this statement follows from the fact that q is a condition. If $x = K$, then $x = K_1 \cap \beta$, where $K_1 \in A_r$ and $\beta \in (K_1 \cap \text{dom}(f_r) \cap S) \cup \{\kappa\}$. If $x = K_1 \cap \delta$, where $\delta \in K_1 \cap \text{dom}(f_q) \cap S$, then we are done since $K_1 \in A_r$ and $\text{dom}(f_q) \subseteq \text{dom}(f_r)$.

If $\beta = \kappa$, then we have already handled all possibilities for x . Suppose that $\beta < \kappa$. Then we also have the possibility that $x = \beta$, in which case $x \in S$, or $x = L \cap \beta$, where $\beta \in L$ and $L \in A_q$. In the second case, $L \in A_r$ and $\beta \in L \cap \text{dom}(f_r) \cap S$, so we are done.

Let $x \in \text{dom}(f_r)$, and we will show that $f_r(x)$ is a finite $\in$ -chain and a subset of $Sk(x) \setminus S$. But by the definition of f_r , $f_r(x) \subseteq f_s(x)$. Since s is a condition, $f_s(x)$ is a finite $\in$ -chain and a subset of $Sk(x) \setminus S$. Hence, $f_r(x)$ is as well.

(3) Let $x \in \text{dom}(f_r)$. Then $f_r(x) = f_s(x) \cap \text{dom}(f_r)$, and therefore $f_r(x) \subseteq \text{dom}(f_r)$. Let $K \in f_r(x)$, and we will show that $f_r(K) = f_r(x) \cap Sk(K)$. But $f_r(K) = f_s(K) \cap \text{dom}(f_r)$, and since s is a condition, $f_s(K) = f_s(x) \cap Sk(K)$. Thus,

$$\begin{aligned} f_r(K) &= f_s(K) \cap \text{dom}(f_r) = (f_s(x) \cap Sk(K)) \cap \text{dom}(f_r) = \\ &= (f_s(x) \cap \text{dom}(f_r)) \cap Sk(K) = f_r(x) \cap Sk(K). \end{aligned}$$

(4) Consider $J \in f_r(x)$, and we will show that $g_r(J, x)$ is a finite subset of $x \setminus \sup(J)$. If $g_r(J, x) = g_s(J, x)$, then since s is a condition, $g_s(J, x)$ is a finite subset of $x \setminus \sup(J)$, and we are done.

Suppose that $g_r(J, x) \neq g_s(J, x)$. Then by the definition of g_r , $J = K$, $x \in f_r(\alpha) \cup \{\alpha\}$, and $g_r(J, x) = g_s(K, x) \cup \{\zeta\}$, where $\zeta = \min(M \setminus \xi)$. Since s is a condition, $g_s(J, x)$ is a finite subset of $x \setminus \sup(J)$. So it suffices to show that $\zeta \in x \setminus \sup(K)$. We already know that $\sup(K) < \xi \leq \zeta$, so $\zeta \notin \sup(K)$.

It remains to show that $\zeta \in x$. If $x = \alpha$, then certainly $\zeta < \alpha$, since $\zeta \in M$ and $M \subseteq \alpha$. Assume that $x \in f_r(\alpha)$. We consider the different possibilities for why x is in $f_r(\alpha)$.

First, assume that $x \in \text{dom}(f_q)$. Then, since $x \in f_r(\alpha) \subseteq f_s(\alpha)$, we have that $x \in f_s(\alpha)$. But x and α are in $\text{dom}(f_q)$. Since $s \leq q$, it follows that $x \in f_q(\alpha)$. So either $x \in A_1$ or $x \in A_2$.

By the choice of γ and K , for all $J \in A_1$, $\sup(J) < \gamma < \sup(K)$. Since $K \in f_r(x)$, $K \in Sk(x)$, and so $\sup(K) < \sup(x)$. Therefore, $x \notin A_1$. So $x \in A_2$. As noted above, the minimality of M implies that for all $J \in A_2$, $M \subseteq J$. So $M \subseteq x$. Since $\zeta \in M$, $\zeta \in x$, and we are done.

Secondly, assume that $x \notin \text{dom}(f_q)$. Since $K \in f_r(x)$, $K \in Sk(x)$. So

$$K_1 \cap \omega_1 = K \cap \omega_1 < x \cap \omega_1.$$

It follows that x is not equal to K , and x is not equal to $K_1 \cap \delta$ for any $\delta \in K_1 \cap \text{dom}(f_q) \cap S$.

The remaining possibility is that $\beta < \kappa$ and $x = L \cap \beta$, where $L \in A_q$ and $\beta \in L$. Since $x \in f_r(\alpha) \subseteq f_s(\alpha)$, we have that $x \in f_s(\alpha)$. As x and M are both in $f_s(\alpha)$, they are membership comparable.

Since $K \in f_s(x) = f_s(L \cap \beta)$ and K is the membership largest member of $f_s(\alpha)$ with $\sup(K) < \xi$, we have that $\xi \leq \sup(L \cap \beta)$. But recall that q forces that M is the membership least element of $\dot{f}(\alpha)$ with $\xi \leq \sup(M)$, and $s \leq q$. Hence, it is not the case that $L \cap \beta \in Sk(M)$. So either $L \cap \beta = M$ or $M \in Sk(L \cap \beta)$. In either case, $\zeta \in M \subseteq L \cap \beta$, so $\zeta \in L \cap \beta = x$.

(5) Suppose that $J \in f_r(L)$ and $L \in f_r(x)$. Then $J \in f_s(L)$ and $L \in f_s(x)$. We will show that $g_r(J, x) \subseteq g_r(J, L)$. If $g_r(J, x) = g_s(J, x)$, then since s is a condition and $g_s(J, L) \subseteq g_r(J, L)$, we have that

$$g_r(J, x) = g_s(J, x) \subseteq g_s(J, L) \subseteq g_r(J, L).$$

Assume that $g_r(J, x) \neq g_s(J, x)$. Then by the definition of g_r , we have that $J = K$, $x \in f_r(\alpha) \cup \{\alpha\}$, and $g_r(K, x) = g_s(K, x) \cup \{\zeta\}$, where $\zeta = \min(M \setminus \xi)$. Again, $g_s(K, x) \subseteq g_s(K, L) \subseteq g_r(K, L)$. So it suffices to show that $\zeta \in g_r(K, L)$.

By the definition of g_r , in order to show that $\zeta \in g_r(K, L)$, it is enough to show that $L \in f_r(\alpha) \cup \{\alpha\}$, for then $g_r(K, L)$ is defined as $g_s(K, L) \cup \{\zeta\}$. But $L \in f_r(x)$ and $x \in f_r(\alpha) \cup \{\alpha\}$. So if $x = \alpha$, then $L \in f_r(\alpha)$, and if $x \in f_r(\alpha)$, then $L \in f_r(x) = f_r(\alpha) \cap Sk(x)$ by requirement (3), so $L \in f_r(\alpha)$.

(6) Suppose that $\delta \in \text{dom}(f_r) \cap S$, $L \in A_r$, and $\delta \in L$. We will show that $L \cap \delta \in f_r(\delta)$. Since $\text{dom}(f_r) \subseteq \text{dom}(f_s)$, we have that $\delta \in \text{dom}(f_s) \cap S$. And $A_r \subseteq A_s$. As s is a condition, it follows that $L \cap \delta \in f_s(\delta)$. Since $f_r(\delta) = f_s(\delta) \cap \text{dom}(f_r)$, it suffices to show that $L \cap \delta \in \text{dom}(f_r)$.

By the definition of $\text{dom}(f_r)$, δ being in $\text{dom}(f_r) \cap S$ implies that either $\delta \in \text{dom}(f_q)$ or $\delta = \beta$. Also, $A_r = A_q \cup \{K_1\}$, so L being in A_r means that either $L \in A_q$ or $L = K_1$.

If $\delta \in \text{dom}(f_q)$ and $L \in A_q$, then $L \cap \delta \in f_q(\delta) \subseteq \text{dom}(f_q) \subseteq \text{dom}(f_r)$, since q is a condition. If $\delta \in \text{dom}(f_q)$ and $L = K_1$, then $L \cap \delta = K_1 \cap \delta$ is in $\text{dom}(f_r)$ by the definition of $\text{dom}(f_r)$. If $\delta = \beta$ and $L \in A_q$, then $L \cap \delta = L \cap \beta$ is in $\text{dom}(f_r)$ by the definition of $\text{dom}(f_r)$. And if $\delta = \beta$ and $L = K_1$, then $L \cap \delta = K_1 \cap \beta = K$ is in $\text{dom}(f_r)$ by the definition of $\text{dom}(f_r)$.

(7) Suppose that $\tau \in r^*(A_r) \cap S$, and we will show that $\tau \in \text{dom}(f_r)$. Fix J and L in A_r such that $J \sim L$ and $\tau = \min((J \cap \kappa) \setminus \beta_{J,L})$. Then obviously J and L are different, and since $J \sim L$, $J \cap \omega_1 = L \cap \omega_1$ by Lemma 2.17(2). Moreover, $A_r = A_q \cup \{K_1\}$, and by the choice of K , $K_1 \cap \omega_1 = K \cap \omega_1$ is different from $N \cap \omega_1$ for all $N \in A_q$. Therefore, J and L must both be in A_q . Thus, $\tau \in r^*(A_q) \cap S \subseteq \text{dom}(f_q) \subseteq \text{dom}(f_r)$. $\square$

Proposition 5.4. *The forcing poset $\mathbb{P}$ forces that $\langle \dot{c}_\alpha : \alpha \in S \rangle$ is a partial square sequence.*

Proof. By Lemma 5.2 and Proposition 5.3, for each $\alpha \in S$, $\mathbb{P}$ forces that $\dot{c}_\alpha$ is a club subset of α with order type equal to ω_1 . Let G be a generic filter on $\mathbb{P}$. Consider α and β in S , and let $c_\alpha := \dot{c}_\alpha^G$ and $c_\beta := \dot{c}_\beta^G$. Assume that ξ is a common limit point of c_α and c_β . We will show that $c_\alpha \cap \xi = c_\beta \cap \xi$.

Since c_α and c_β are closed, it follows that $\xi \in c_\alpha \cap c_\beta$. Thus, there are $K \in f(\alpha)$ and $L \in f(\beta)$ such that $\sup(K) = \xi = \sup(L)$. By Lemma 5.1,

$$c_\alpha \cap \xi = c_\alpha \cap \sup(K) = \{\sup(J) : J \in f(K)\},$$

and

$$c_\beta \cap \xi = c_\beta \cap \sup(L) = \{\sup(J) : J \in f(L)\}.$$

Thus, to show that $c_\alpha \cap \xi = c_\beta \cap \xi$, it suffices to show that $f(K) = f(L)$. We will prove, in fact, that $K = L$.

Since $K \in f(\alpha)$ and $L \in f(\beta)$, we can fix $p \in G$ such that $K \in f_p(\alpha)$ and $L \in f_p(\beta)$. Then K and L are in $\text{dom}(f_p)$, by Definition 4.2(3). By Definition 4.2(2), fix K_1 in A_p and $\theta \in (K_1 \cap \text{dom}(f_p) \cap S) \cup \{\kappa\}$ such that $K = K_1 \cap \theta$, and L_1 in A_p and $\tau \in (L_1 \cap \text{dom}(f_p) \cap S) \cup \{\kappa\}$ such that $L = L_1 \cap \tau$.

Since $\sup(K_1 \cap \theta) = \sup(K) = \xi$, it follows that $\xi \leq \theta$, and similarly, $\xi \leq \tau$. As θ and τ are in $S \cup \{\kappa\}$, they have uncountable cofinality. On the other hand, since ξ is the supremum of the countable set K , ξ has countable cofinality. Therefore, $\xi < \theta$ and $\xi < \tau$. Since the sets $K = K_1 \cap \theta$ and $L = L_1 \cap \tau$ are closed under successor ordinals by elementarity, ξ is not a member of K_1 nor L_1 . Therefore,

$$K = K_1 \cap \theta = K_1 \cap \xi,$$

and

$$L = L_1 \cap \tau = L_1 \cap \xi.$$

The ordinal ξ , which is the supremum of K and L , is a common limit point of $K_1 \cap \kappa$ and $L_1 \cap \kappa$. So by Proposition 2.11, $\xi < \beta_{K_1, L_1}$.

We claim that $K_1 \sim L_1$. Suppose not, and without loss of generality, assume that $K_1 < L_1$. Then $K_1 \cap \beta_{K_1, L_1}$ is in L_1 . Since $\xi < \beta_{K_1, L_1}$, and ξ is a limit point of $K = K_1 \cap \theta$, we have that ξ is a limit point of $K_1 \cap \beta_{K_1, L_1}$. But $K_1 \cap \beta_{K_1, L_1}$ is in L_1 , and therefore ξ is in L_1 by elementarity. So $\xi \in L_1 \cap \tau = L$, which is a contradiction.

So indeed, $K_1 \sim L_1$. Hence $K_1 \cap \beta_{K_1, L_1} = L_1 \cap \beta_{K_1, L_1}$. Since $\xi < \beta_{K_1, L_1}$, it follows that $K_1 \cap \xi = L_1 \cap \xi$. Thus, $K = K_1 \cap \xi = L_1 \cap \xi = L$. $\square$

We point out that if S is chosen so that $S = D \cap \text{cof}(> \omega)$, for some club set $D \subseteq \kappa$, then adding a partial square sequence on S will imply that $\square_{\omega_1}$ holds in the generic extension (see the end of [6]). So as a special case, our forcing poset provides another way to force $\square_{\omega_1}$ with finite conditions.

§6. Amalgamation over uncountable models

We now turn to proving that $\mathbb{P}$ is strongly proper on a stationary set, and hence preserves ω_1 , and is κ -c.c. Strong properness is proven using amalgamation of conditions over countable models, and the κ -c.c. is proven using amalgamation of conditions over uncountable models. The uncountable case is similar to, but not as complicated as, the countable case, so we will handle the uncountable case first.

Many of the results which we will prove in Sections 6 and 7 will be used again in Sections 8 and 9, where the approximation property of certain quotients of $\mathbb{P}$ is verified. For this reason, it will be helpful to develop the notation and results of Sections 6 and 7 in great detail.

Let us give a brief outline of the main ideas presented in this section. The goal is to show that for any simple model $Q \in \mathcal{Y}$, the maximum condition in $\mathbb{P}$ is strongly Q -generic. This fact will imply that $\mathbb{P}$ is κ -c.c. Let D_Q denote the set of conditions $s \in \mathbb{P}$ such that for all $M \in A_s$, $M \cap Q \in A_s$. We will show that D_Q is dense in $\mathbb{P}$. For each $s \in D_Q$, we will define a condition $s \restriction Q$ in $Q \cap \mathbb{P}$. This condition will satisfy that for all $w \leq s \restriction Q$ in $Q \cap \mathbb{P}$, w and s are compatible. Since D_Q is dense, it will follow that the maximum condition of $\mathbb{P}$ is strongly Q -generic.

Lemma 6.1. *Let $q \in \mathbb{P}$ and $Q \in \mathcal{Y}$. Then there is $s \leq q$ such that for all $M \in A_s$, $M \cap Q \in A_s$. Moreover, $A_s = A_q \cup \{M \cap Q : M \in A_q\}$.*

Recall that by Assumption 2.5(2), if $M \in \mathcal{X}$ and $Q \in \mathcal{Y}$, then $M \cap Q \in \mathcal{X}$.

Proof. By Proposition 2.27, the set $A_q \cup \{M \cap Q : M \in A_q\}$ is adequate. Define

$$x_0 := r^*(A_q \cup \{M \cap Q : M \in A_q\}) \cap S,$$

and define

$$x := x_0 \setminus \text{dom}(f_q).$$

Let $r := q + x$. By Definition 4.7 and Lemma 4.8, r is a condition, $r \leq q$, and $A_r = A_q$. Also, $x_0 \subseteq \text{dom}(f_r)$.

Define s as follows. Let $f_s := f_r$, $g_s := g_r$, and

$$A_s := A_r \cup \{M \cap Q : M \in A_r\}.$$

We claim that s is as required. By Proposition 2.27, for all $M \in A_s$, $M \cap Q \in A_s$. It is trivial to check that if s is a condition, then $s \leq q$.

It remains to show that s is a condition. We verify requirements (1)–(7) of Definition 4.2. (1) follows from Proposition 2.27. Requirements (2)–(5) follow immediately from r being a condition, together with the fact that $f_s = f_r$, $g_s = g_r$, and $A_r \subseteq A_s$.

(6) Suppose that $\alpha \in \text{dom}(f_s) \cap S$, $M \in A_s$, and $\alpha \in M$. We will show that $M \cap \alpha \in f_s(\alpha)$. Since $f_r = f_s$, we have that $\alpha \in \text{dom}(f_r) \cap S$.

First, assume that $M \in A_r$. Then since r is a condition, $M \cap \alpha \in f_r(\alpha) = f_s(\alpha)$, and we are done. Secondly, assume that $M = M_1 \cap Q$ for some $M_1 \in A_r$. Then $\alpha \in M = M_1 \cap Q$. So $\alpha \in M_1$ and $\alpha < Q \cap \kappa$. Thus,

$$M \cap \alpha = M_1 \cap Q \cap \alpha = M_1 \cap \alpha.$$

Since $M_1 \in A_r$, $\alpha \in \text{dom}(f_r) \cap S$, and $\alpha \in M_1$, it follows that $M \cap \alpha = M_1 \cap \alpha \in f_r(\alpha) = f_s(\alpha)$.

(7) We need to show that $r^*(A_s) \cap S \subseteq \text{dom}(f_s)$. But since $A_r = A_q$, we have that

$$r^*(A_s) \cap S = r^*(A_q \cup \{M \cap Q : M \in A_q\}) \cap S = x_0 \subseteq \text{dom}(f_r) = \text{dom}(f_s).$$

□

Definition 6.2. For each $Q \in \mathcal{Y}$, let D_Q denote the set of conditions $q \in \mathbb{P}$ such that for all $M \in A_q$, $M \cap Q \in A_q$.

Lemma 6.3. Let $Q \in \mathcal{Y}$. Then D_Q is dense in $\mathbb{P}$.

Proof. Immediate from Lemma 6.1. □

Note that if $q \in Q \cap \mathbb{P}$, then $q \in D_Q$. Namely, for all $M \in A_q$, $M \in Q$, and therefore $M \cap Q = M \in A_q$.

Definition 6.4. Suppose that $Q \in \mathcal{Y}$ is simple and $q \in D_Q$. Let $q \restriction Q$ denote the triple (f, g, A) satisfying:

- (1) $\text{dom}(f) := \text{dom}(f_q) \cap Q$, and for all $x \in \text{dom}(f)$, $f(x) := f_q(x)$;
- (2) $\text{dom}(g) := \text{dom}(g_q) \cap Q$, and for all $(K, x) \in \text{dom}(g)$, $g(K, x) := g_q(K, x)$;
- (3) $A := A_q \cap Q$.

Note that in (1) above, if $x \in \text{dom}(f_q) \cap Q$, then $Sk(x) \subseteq Q$. So $f_q(x) = f(x)$ is a finite subset of Q , and therefore is in Q . In (2), if $K \in f_q(x)$ and K and x are in Q , then $g_q(K, x) \subseteq x \subseteq Q$. So $g_q(K, x)$ is a finite subset of Q , and hence is in Q . Similarly, $A = A_q \cap Q$ is in Q . It easily follows from these observations that $q \restriction Q \in Q$.

Lemma 6.5. Let $Q \in \mathcal{Y}$ be simple and $q \in D_Q$. Then $q \restriction Q$ is in $Q \cap \mathbb{P}$ and $q \leq q \restriction Q$.

Proof. Let $q \restriction Q = (f, g, A)$. We already observed that $q \restriction Q \in Q$. It is trivial to check that if $q \restriction Q$ is a condition, then $q \leq q \restriction Q$. So it suffices to show that $q \restriction Q$ is a condition. We verify requirements (1)–(7) of Definition 4.2. (1), (5), (6), and (7) are immediate. It remains to prove (2), (3), and (4).

(2) Obviously f is a function with a finite domain. Let $x \in \text{dom}(f) = \text{dom}(f_q) \cap Q$. Since $f(x) = f_q(x)$, it follows that $f(x)$ is a finite $\in$ -chain and a subset of $Sk(x) \setminus S$. We claim that either $x \in S$, or there is $M \in A$ and $\alpha \in (M \cap \text{dom}(f) \cap S) \cup \{\kappa\}$ such that $x = M \cap \alpha$.

Since $x \in \text{dom}(f_q)$ and q is a condition, we have that either $x \in S$, or there is $M_1 \in A_q$ and $\alpha \in (M_1 \cap \text{dom}(f_q) \cap S) \cup \{\kappa\}$ such that $x = M_1 \cap \alpha$. If $x \in S$, then we are done, so assume the second case.

Since $q \in D_Q$ and Q is simple,

$$M_1 \cap Q \in A_q \cap Q = A.$$

So it suffices to show that $x = M_1 \cap Q \cap \beta$, for some β with

$$\beta \in (M_1 \cap Q \cap \text{dom}(f) \cap S) \cup \{\kappa\}.$$

We split the proof into the cases of whether $\alpha < Q \cap \kappa$ or $Q \cap \kappa \leq \alpha$.

First, assume that $\alpha < Q \cap \kappa$. Then $\alpha \in \text{dom}(f_q) \cap S \cap Q = \text{dom}(f) \cap S$, and

$$x = M_1 \cap \alpha = M_1 \cap \alpha \cap Q = M_1 \cap Q \cap \alpha.$$

Hence, $x = M_1 \cap Q \cap \beta$, where $\beta = \alpha \in M_1 \cap Q \cap \text{dom}(f) \cap S$.

Secondly, assume that $Q \cap \kappa \leq \alpha$. Since $x \in Q$, $x = M_1 \cap \alpha \subseteq Q \cap \kappa$. Therefore,

$$x = M_1 \cap \alpha = (M_1 \cap \alpha) \cap (Q \cap \kappa) = M_1 \cap Q \cap \kappa.$$

Thus, $x = M_1 \cap Q \cap \beta$, where $\beta = \kappa$.

(3) Let $x \in \text{dom}(f) = \text{dom}(f_q) \cap Q$, and we will show that $f(x) \subseteq \text{dom}(f)$. We have that $f_q(x) \subseteq Sk(x) \subseteq Q$. And since q is a condition,

$$f(x) = f_q(x) \subseteq \text{dom}(f_q) \cap Q = \text{dom}(f).$$

So $f(x) \subseteq \text{dom}(f)$, as required. Now consider $K \in f(x) = f_q(x)$. Then since q is a condition,

$$f(K) = f_q(K) = f_q(x) \cap Sk(K) = f(x) \cap Sk(K).$$

(4) We have that $(K, x) \in \text{dom}(g)$ iff $(K, x) \in \text{dom}(g_q) \cap Q$ iff $(K$ and x are in Q and $K \in f_q(x))$ iff $K \in f(x)$. For each $(K, x) \in \text{dom}(g)$, $g(K, x) = g_q(K, x) \subseteq x \setminus \text{sup}(K)$. $\square$

The next lemma will not be used until Section 8.

Lemma 6.6. *Let $Q \in \mathcal{Y}$ be simple.*

- (1) *Suppose that $p \in Q \cap \mathbb{P}$, $q \in D_Q$, and $q \leq p$. Then $q \restriction Q \leq p$.*
- (2) *Suppose that q and r are in D_Q and $r \leq q \restriction Q$. Then $r \restriction Q \leq q \restriction Q$.*
- (3) *Suppose that p and q are in D_Q and $q \leq p$. Then $q \restriction Q \leq p \restriction Q$.*

Proof. (1) We verify properties (a)–(d) of Definition 4.2. (a) Since $p \in Q$, $A_p \subseteq Q$. As $q \leq p$, we have that

$$A_p \subseteq A_q \cap Q = A_{q \restriction Q}.$$

(b) Since $p \in Q$, $\text{dom}(f_p) \subseteq Q$. As $q \leq p$, we have that

$$\text{dom}(f_p) \subseteq \text{dom}(f_q) \cap Q = \text{dom}(f_{q \restriction Q}).$$

Let $x \in \text{dom}(f_p)$. Then

$$f_p(x) \subseteq f_q(x) = f_{q \restriction Q}(x).$$

(c) Let $(K, x) \in \text{dom}(g_p)$. Since $q \leq p$,

$$g_p(K, x) \subseteq g_q(K, x) = g_{q \restriction Q}(K, x).$$

(d) Assume that K and x are in $\text{dom}(f_p)$ and $K \in f_{q \restriction Q}(x)$. Then $K \in f_q(x) = f_p(x)$. Since $q \leq p$, it follows that $K \in f_p(x)$.

(2) We know that $q \restriction Q \in Q \cap \mathbb{P}$, $r \in D_Q$, and $r \leq q \restriction Q$. By (1), it follows that $r \restriction Q \leq q \restriction Q$.

(3) By Lemma 6.5, we know that $p \leq p \restriction Q$. So $q \leq p \leq p \restriction Q$. Hence, $q \leq p \restriction Q$. Thus, p and q are in D_Q and $q \leq p \restriction Q$. By (2), it follows that $q \restriction Q \leq p \restriction Q$. $\square$

We will now begin analyzing the situation where $q \in D_Q$ and $w \leq q \restriction Q$ is in $Q \cap \mathbb{P}$.

Lemma 6.7. *Let $Q \in \mathcal{Y}$ be simple and $q \in D_Q$. Suppose that $w \in Q \cap \mathbb{P}$ and $w \leq q \restriction Q$. Then:*

- (1) $A_q \cap Q \subseteq A_w$;
- (2) $\text{dom}(f_q) \cap Q \subseteq \text{dom}(f_w)$, and for all $x \in \text{dom}(f_q) \cap Q$, $f_q(x) \subseteq f_w(x)$;
- (3) $\text{dom}(g_q) \cap Q \subseteq \text{dom}(g_w)$, and for all $(K, x) \in \text{dom}(g_q) \cap Q$, $g_q(K, x) \subseteq g_w(K, x)$.

Proof. Immediate from the definition of $q \restriction Q$ and the fact that $w \leq q \restriction Q$. $\square$

As discussed at the beginning of the section, we are going to show that whenever $w \leq q \restriction Q$, where $q \in D_Q$ and $w \in Q \cap \mathbb{P}$, then w and q are compatible. We now begin the construction of a specific lower bound of w and q , which we will denote by $w \oplus_Q q$. In order to define the amalgam $w \oplus_Q q$, we will need to define the f , g , and A components of $w \oplus_Q q$. The amalgam of the A -components will be $A_w \cup A_q$. We handle the f -components next.

Definition 6.8. Let $Q \in \mathcal{Y}$ be simple and $q \in D_Q$. Suppose that $w \in Q \cap \mathbb{P}$ and $w \leq q \restriction Q$. Define $f_w \oplus_Q f_q = f$ as follows.

The domain of f is equal to $\text{dom}(f_w) \cup \text{dom}(f_q)$. The values of f are defined by the following cases:

- (1) for all $x \in \text{dom}(f_w)$, $f(x) := f_w(x)$;
- (2) for all $x \in \text{dom}(f_q) \setminus Q$, if $f_q(x) \cap Q = \emptyset$, then $f(x) := f_q(x)$;
- (3) for all $x \in \text{dom}(f_q) \setminus Q$, if $f_q(x) \cap Q \neq \emptyset$, then $f(x) := f_q(x) \cup f_w(M)$, where M is the membership largest element of $f_q(x) \cap Q$.

It is easy to see that cases 1–3 describe all of the possibilities for a set being in $\text{dom}(f)$, since $\text{dom}(f_q) \cap Q \subseteq \text{dom}(f_w)$ by Lemma 6.7(2). Moreover, cases 1–3 are obviously disjoint.

The next three lemmas describe some important properties of $f_w \oplus_Q f_q$. The first two lemmas are easy, but the third is quite involved.

Lemma 6.9. Let $Q \in \mathcal{Y}$ be simple and $q \in D_Q$. Suppose that $w \in Q \cap \mathbb{P}$ and $w \leq q \restriction Q$. Let $f := f_w \oplus_Q f_q$. Then:

- (1) if $x \in \text{dom}(f_w)$, then $f_w(x) = f(x)$;
- (2) if $x \in \text{dom}(f_q)$, then $f_q(x) \subseteq f(x)$.

Proof. (1) is by Definition 6.8(1), and (2) follows immediately from Definition 6.8(2,3). $\square$

Lemma 6.10. Let $Q \in \mathcal{Y}$ be simple and $q \in D_Q$. Suppose that $w \in Q \cap \mathbb{P}$ and $w \leq q \restriction Q$. Let $f := f_w \oplus_Q f_q$. Then:

- (1) $\text{dom}(f) \cap Q = \text{dom}(f_w)$;
- (2) if $K \in f(x)$ and K and x are in $\text{dom}(f_w)$, then $K \in f_w(x)$;
- (3) if $K \in f(x)$ and K and x are in $\text{dom}(f_q)$, then $K \in f_q(x)$.

Proof. (1) By Lemma 6.7(2), $\text{dom}(f_q) \cap Q \subseteq \text{dom}(f_w)$. Hence,

$$\text{dom}(f) \cap Q = (\text{dom}(f_w) \cup \text{dom}(f_q)) \cap Q = \text{dom}(f_w),$$

where the last equality follows from the fact that $\text{dom}(f_w) \cap Q = \text{dom}(f_w)$ and $\text{dom}(f_q) \cap Q \subseteq \text{dom}(f_w)$.

(2) Suppose that $K \in f(x)$ and K and x are in $\text{dom}(f_w)$. By Definition 6.8(1), $f(x) = f_w(x)$, so $K \in f_w(x)$.

(3) Assume that $K \in f(x)$ and K and x are in $\text{dom}(f_q)$. We will show that $K \in f_q(x)$. The proof splits into the three cases of Definition 6.8 for how $f(x)$ is defined. In case 2, $f(x) = f_q(x)$, so $K \in f_q(x)$.

In case 1, $f(x) = f_w(x)$. So $K \in f_w(x)$. In particular, K and x are in Q . So K and x are in $\text{dom}(f_q) \cap Q = \text{dom}(f_{q \upharpoonright Q})$. Since $w \leq q \upharpoonright Q$ and $K \in f_w(x)$, it follows that $K \in f_{q \upharpoonright Q}(x) = f_q(x)$.

In case 3, $f(x) = f_q(x) \cup f_w(M)$, where M is the membership largest element of $f_q(x) \cap Q$. So either $K \in f_q(x)$, or $K \in f_w(M)$. In the first case we are done, so assume that $K \in f_w(M)$. Then K and M are in $\text{dom}(f_q) \cap Q = \text{dom}(f_{q \upharpoonright Q})$. Since $w \leq q \upharpoonright Q$ and $K \in f_w(M)$, it follows that $K \in f_{q \upharpoonright Q}(M) = f_q(M)$. So $K \in f_q(M)$ and $M \in f_q(x)$. Therefore, $K \in f_q(x)$. $\square$

The next lemma will be used to verify that $f_w \oplus_Q f_q$ satisfies requirements (2) and (3) of Definition 4.2 for $w \oplus_Q q$.

Lemma 6.11. *Let $Q \in \mathcal{Y}$ be simple and $q \in D_Q$. Suppose that $w \in Q \cap \mathbb{P}$ and $w \leq q \upharpoonright Q$. Let $f := f_w \oplus_Q f_q$. Then:*

- (1) *f is a function with a finite domain, and for all $x \in \text{dom}(f)$, either $x \in S$, or there is $M \in A_w \cup A_q$ and*

$$\alpha \in (M \cap \text{dom}(f) \cap S) \cup \{\kappa\}$$

such that $x = M \cap \alpha$; moreover, for all $x \in \text{dom}(f)$, $f(x)$ is a finite $\in$ -chain and $f(x) \subseteq Sk(x) \setminus S$;

- (2) *if $x \in \text{dom}(f)$, then $f(x) \subseteq \text{dom}(f)$, and for all $K \in f(x)$, $f(K) = f(x) \cap Sk(K)$.*

Proof. (1) The domain of f is equal to $\text{dom}(f_w) \cup \text{dom}(f_q)$, which is finite. Let $x \in \text{dom}(f)$. Then either $x \in \text{dom}(f_w)$ or $x \in \text{dom}(f_q)$.

If $x \in \text{dom}(f_w)$, then either $x \in S$, or there is $M \in A_w$ and $\alpha \in (M \cap \text{dom}(f_w) \cap S) \cup \{\kappa\}$ such that $x = M \cap \alpha$. If $x \in \text{dom}(f_q)$, then either $x \in S$, or there is $M \in A_q$ and $\alpha \in (M \cap \text{dom}(f_q) \cap S) \cup \{\kappa\}$ such that $x = M \cap \alpha$. In either case, either $x \in S$, or there is $M \in A_w \cup A_q$ and $\alpha \in (M \cap \text{dom}(f) \cap S) \cup \{\kappa\}$ such that $x = M \cap \alpha$.

Let $x \in \text{dom}(f)$, and we will show that $f(x)$ is a finite $\in$ -chain and a subset of $Sk(x) \setminus S$. We consider the three cases in the definition of $f(x)$ given in Definition 6.8. In cases 1 and 2, $f(x)$ is equal to either $f_w(x)$ or $f_q(x)$. Since w and q are conditions, then in either case, $f(x)$ is a finite $\in$ -chain and a subset of $Sk(x) \setminus S$.

Consider case 3, which says that $x \in \text{dom}(f_q) \setminus Q$ and $f(x) = f_q(x) \cup f_w(M)$, where M is the membership largest element of $f_q(x) \cap Q$. Since w and q are conditions, it follows that

$$f(x) \subseteq (Sk(x) \cup Sk(M)) \setminus S.$$

But $M \in f_q(x)$ implies that $M \in Sk(x)$, and therefore $Sk(M) \subseteq Sk(x)$. Hence,

$$f(x) \subseteq Sk(x) \setminus S.$$

Since w and q are conditions, $f_q(x)$ and $f_w(M)$ are each finite $\in$ -chains. So to prove that $f(x)$ is a finite $\in$ -chain, it suffices to show that whenever $K \in f_w(M)$ and $L \in f_q(x) \setminus f_w(M)$, then $K \in Sk(L)$.

If $L = M$, then since $K \in f_w(M)$, $K \in Sk(M) = Sk(L)$, and we are done. Suppose that $L \neq M$. As L and M are different elements of $f_q(x)$, either $L \in f_q(M)$ or $M \in f_q(L)$. But $f_q(M) \subseteq f_w(M)$ by Lemma 6.7(2), and we assumed that $L \notin f_w(M)$. Thence, $M \in f_q(L)$. But $K \in f_w(M)$ implies that $K \in Sk(M)$, and $M \in f_q(L)$ implies that $M \in Sk(L)$. Therefore, $K \in Sk(M) \subseteq Sk(L)$, so $K \in Sk(L)$.

(2) Let $x \in \text{dom}(f)$. We claim that $f(x) \subseteq \text{dom}(f)$. In cases 1 and 2 of Definition 6.8, either $f(x) = f_w(x)$ or $f(x) = f_q(x)$. Since w and q are conditions, $f(x) \subseteq \text{dom}(f_w) \subseteq \text{dom}(f)$ in the first case, and $f(x) \subseteq \text{dom}(f_q) \subseteq \text{dom}(f)$ in the second case.

In the third case, $f(x) = f_q(x) \cup f_w(M)$, where M is the membership largest element of $f_q(x) \cap Q$. Since w and q are conditions, $f(x) \subseteq \text{dom}(f_q) \cup \text{dom}(f_w) = \text{dom}(f)$.

Assume that $K \in f(x)$, and we will show that $f(K) = f(x) \cap \text{Sk}(K)$. We split the proof into the three cases of Definition 6.8 for the definition of $f(x)$.

In case 1, $x \in \text{dom}(f_w)$ and $f(x) = f_w(x)$. So $K \in f_w(x) \subseteq \text{dom}(f_w)$. Hence, $f(K) = f_w(K)$. Since w is a condition,

$$f(K) = f_w(K) = f_w(x) \cap \text{Sk}(K) = f(x) \cap \text{Sk}(K).$$

In case 2, $x \in \text{dom}(f_q) \setminus Q$, $f_q(x) \cap Q = \emptyset$, and $f(x) = f_q(x)$. Since $K \in f(x) = f_q(x)$, we have that $f_q(K) = f_q(x) \cap \text{Sk}(K)$, since q is a condition. In particular, $K \in \text{dom}(f_q) \setminus Q$ and $f_q(K) \cap Q = \emptyset$. Therefore, by definition, $f(K) = f_q(K)$. So

$$f(K) = f_q(K) = f_q(x) \cap \text{Sk}(K) = f(x) \cap \text{Sk}(K).$$

In case 3, $x \in \text{dom}(f_q) \setminus Q$ and $f(x) = f_q(x) \cup f_w(M)$, where M is the membership largest element of $f_q(x) \cap Q$. Then either $K \in f_q(x)$, or $K \in f_w(M)$. Since M is the largest element of $f_q(x) \cap Q$ and q is a condition,

$$f_q(x) \cap Q = (f_q(x) \cap \text{Sk}(M)) \cup \{M\} = f_q(M) \cup \{M\} \subseteq f_w(M) \cup \{M\},$$

where the inclusion holds by Lemma 6.7(2). So $f_q(x) \cap Q \subseteq f_w(M) \cup \{M\}$. It easily follows that either $K \in f_q(x) \setminus Q$, $K = M$, or $K \in f_w(M)$.

First, assume that $K \in f_q(x) \setminus Q$. Then, since $f_q(K) = f_q(x) \cap \text{Sk}(K)$, M is the membership largest element of $f_q(K) \cap Q$. So by definition, $f(K) = f_q(K) \cup f_w(M)$. Since $f_w(M) \subseteq \text{Sk}(K)$, we have that

$$\begin{aligned} f(K) &= f_q(K) \cup f_w(M) = (f_q(x) \cap \text{Sk}(K)) \cup f_w(M) = \\ &= (f_q(x) \cup f_w(M)) \cap \text{Sk}(K) = f(x) \cap \text{Sk}(K). \end{aligned}$$

Secondly, assume that $K = M$. Then $f(K) = f(M) = f_w(M)$. Hence, it suffices to show that

$$f_w(M) = (f_q(x) \cup f_w(M)) \cap \text{Sk}(M).$$

The forward inclusion is immediate. For the reverse inclusion, let $J \in (f_q(x) \cup f_w(M)) \cap \text{Sk}(M)$, and we will show that $J \in f_w(M)$. So either $J \in f_q(x) \cap \text{Sk}(M)$ or $J \in f_w(M) \cap \text{Sk}(M)$. In the latter case, we are done. In the former case, by Lemma 6.7(2) we have that

$$J \in f_q(x) \cap \text{Sk}(M) = f_q(M) \subseteq f_w(M),$$

so $J \in f_w(M)$.

Thirdly, assume that $K \in f_w(M)$. Then $K \in \text{dom}(f_w)$, so $f(K) = f_w(K)$. Since w is a condition, $f_w(K) = f_w(M) \cap \text{Sk}(K)$. But by the case in the previous paragraph, $f_w(M) = f(M) = f(x) \cap \text{Sk}(M)$. And since $K \in \text{Sk}(M)$, $\text{Sk}(K) \subseteq \text{Sk}(M)$. Therefore,

$$f(K) = f_w(K) = f_w(M) \cap \text{Sk}(K) = (f(x) \cap \text{Sk}(M)) \cap \text{Sk}(K) = f(x) \cap \text{Sk}(K).$$

□

We now handle the amalgamation of the g -components of w and q .

Definition 6.12. Let $Q \in \mathcal{Y}$ be simple and $q \in D_Q$. Suppose that $w \in Q \cap \mathbb{P}$ and $w \leq q \restriction Q$. Let $f := f_w \oplus_Q f_q$.

Define $g_w \oplus_Q g_q$ as the function g with domain equal to the set of pairs (K, x) such that $K \in f(x)$, such that for all $(K, x) \in \text{dom}(g)$,

$$g(K, x) := \bigcup \{g_w(K, y) \cup g_q(K, y) : x = y, \text{ or } x \in f(y)\}^4$$

Note that $g(K, x)$ is finite.

The next lemma will be used to show that $g_w \oplus_Q g_q$ satisfies requirement (4) of Definition 4.2 for $w \oplus_Q q$.

Lemma 6.13. Let $Q \in \mathcal{Y}$ be simple and $q \in D_Q$. Suppose that $w \in Q \cap \mathbb{P}$ and $w \leq q \restriction Q$. Let $g := g_w \oplus_Q g_q$. Then for any $(K, x) \in \text{dom}(g)$,

$$g(K, x) \subseteq x \setminus \text{sup}(K).$$

Proof. Let $(K, x) \in \text{dom}(g)$, which means that $K \in f(x)$. By definition, any ordinal in $g(K, x)$ is either in $g_w(K, x) \cup g_q(K, x)$, or in $g_w(K, y) \cup g_q(K, y)$ for some y with $x \in f(y)$. In the first case, the ordinal is in $x \setminus \text{sup}(K)$, since w and q are conditions. In the second case, the ordinal is not less than $\text{sup}(K)$, because w and q are conditions. Thus, it suffices to show that whenever $x \in f(y)$, then

$$g_w(K, y) \cup g_q(K, y) \subseteq x.$$

We split the proof into the three cases of Definition 6.8 for how $f(y)$ is defined. Note that by Lemma 6.11(2), $K \in f(x)$ and $x \in f(y)$ implies that $K \in f(y)$.

(1) Suppose that $y \in \text{dom}(f_w)$, so that $f(y) = f_w(y)$. Then $x \in f(y) = f_w(y) \subseteq \text{dom}(f_w)$. So $x \in \text{dom}(f_w)$, and therefore $f(x) = f_w(x)$. Hence, $K \in f_w(x)$ and $x \in f_w(y)$. By Definition 4.2(4,5),

$$g_w(K, y) \subseteq g_w(K, x) \subseteq x.$$

Now assume that $(K, y) \in \text{dom}(g_q)$, and we will show that $g_q(K, y) \subseteq x$. Then $(K, y) \in \text{dom}(g_q) \cap Q$, so by Lemma 6.7(3), $g_q(K, y) \subseteq g_w(K, y)$. But we just proved that $g_w(K, y) \subseteq x$. So $g_q(K, y) \subseteq x$.

(2) Suppose that $y \in \text{dom}(f_q) \setminus Q$ and $f_q(y) \cap Q = \emptyset$. Then $f(y) = f_q(y)$, so $x \in f_q(y)$. And since K and x are in $f(y) = f_q(y)$, K and x are not in Q .

Since $f_q(x) = f_q(y) \cap \text{Sk}(x)$, we have that $f_q(x) \cap Q = \emptyset$. Therefore, by definition, $f(x) = f_q(x)$. Hence, $K \in f_q(x)$ and $x \in f_q(y)$. By Definition 4.2(4,5), it follows that

$$g_q(K, y) \subseteq g_q(K, x) \subseteq x.$$

On the other hand, since y is not in Q , (K, y) is not in the domain of g_w . So the inclusion $g_w(K, y) \subseteq x$ is trivial.

(3) Suppose that $y \in \text{dom}(f_q) \setminus Q$, and $f(y) = f_q(y) \cup f_w(M)$, where M is the membership largest element of $f_q(y) \cap Q$. Since $y \notin Q$, (K, y) is not in $\text{dom}(g_w)$, and therefore $g_w(K, y) = \emptyset$, which is a subset of x . It remains to show that $g_q(K, y) \subseteq x$.

⁴When working with the g -components of a condition, we will adopt the convention that when (K, x) is not a member of the domain of g , then $g(K, x)$ will denote the empty set. In particular, when verifying an inclusion of the form $g(K, x) \subseteq y$, the inclusion is trivial in the case that (K, x) is not in $\text{dom}(g)$.

This is trivial if $(K, y) \notin \text{dom}(g_q)$, so assume that $(K, y) \in \text{dom}(g_q)$, which means that $K \in f_q(y)$.

Since $f(y) = f_q(y) \cup f_w(M)$, either $x \in f_q(y)$, or $x \in f_w(M)$. First, assume that $x \in f_q(y)$. Then $K \in f_q(y)$, $x \in f_q(y)$, and $K \in Sk(x)$, which implies that $K \in f_q(x)$, since q is a condition. Therefore,

$$g_q(K, y) \subseteq g_q(K, x) \subseteq x,$$

since q is a condition.

Secondly, assume that $x \in f_w(M)$. So $x \in Q$, and since $K \in Sk(x)$, it follows that $K \in Q$ as well. Now $x \in f_w(M)$ implies that x , and hence K , are in $Sk(M)$. Also, K and M are both in $f_q(y)$. So $K \in f_q(M)$. By Lemma 6.7(2), $f_q(M) \subseteq f_w(M)$, so $K \in f_w(M)$.

Since K and M are in $f_q(y) \cap Q$ and $K \in f_q(M)$,

$$g_q(K, y) \subseteq g_q(K, M) \subseteq g_w(K, M),$$

where the last inclusion holds by Lemma 6.7(3). Also, $K \in f_w(M)$, $x \in f_w(M)$, and $K \in Sk(x)$ imply that $K \in f_w(x)$, since w is a condition. So

$$g_w(K, M) \subseteq g_w(K, x) \subseteq x,$$

since w is a condition. Thence, $g_q(K, y) \subseteq g_w(K, M) \subseteq x$. $\square$

We are ready to define the amalgam $w \oplus_Q q$.

Definition 6.14. Let $Q \in \mathcal{Y}$ be simple and $q \in D_Q$. Suppose that $w \in Q \cap \mathbb{P}$ and $w \leq q \restriction Q$. Let $w \oplus_Q q$ be the triple (f, g, A) defined by:

- (1) $f := f_w \oplus_Q f_q$;
- (2) $g := g_w \oplus_Q g_q$;
- (3) $A := A_w \cup A_q$.

We will now show that $w \oplus_Q q$ is a condition below w and q . We have done most of the work of the proof already.

Proposition 6.15. Let $Q \in \mathcal{Y}$ be simple and $q \in D_Q$. Suppose that $w \in Q \cap \mathbb{P}$ and $w \leq q \restriction Q$. Then w and q are compatible. In fact, $w \oplus_Q q$ is in $\mathbb{P}$ and $w \oplus_Q q \leq w, q$.

Proof. We will prove that $w \oplus_Q q$ is a condition and that $w \oplus_Q q$ is below w and q . Let $w \oplus_Q q = (f, g, A)$.

To show that $w \oplus_Q q$ is a condition, we verify requirements (1)–(7) of Definition 4.2.

(1) We apply Proposition 2.28. Since $q \in D_Q$, we have that for all $M \in A_q$, $M \cap Q \in A_q$. Also, A_w is adequate, and by Lemma 6.7(1),

$$A_q \cap Q \subseteq A_w \subseteq Q.$$

By Proposition 2.28, $A_w \cup A_q = A$ is adequate.

(2,3) These statements are immediate from Lemma 6.11.

(4) By Definition 6.12, g is a function whose domain is the set of pairs (K, x) such that $K \in f(x)$. And by Lemma 6.13, for all $(K, x) \in \text{dom}(g)$, $g(K, x) \subseteq x \setminus \text{sup}(K)$. Also $g(K, x)$ is finite, by Definition 6.12.

(5) Let $K \in f(L)$ and $L \in f(x)$, and we will show that $g(K, x) \subseteq g(K, L)$. Let $\xi \in g(K, x)$. Then by Definition 6.12, either $\xi \in g_w(K, x) \cup g_q(K, x)$, or for some y with $x \in f(y)$, $\xi \in g_w(K, y) \cup g_q(K, y)$. In the second case, $L \in f(x)$ and $x \in f(y)$ imply by requirement (3) that $L \in f(y)$. So letting $z := x$ in the first case, and $z := y$ in the second case, we have that $L \in f(z)$ and $\xi \in g_w(K, z) \cup g_q(K, z)$. By Definition 6.12, it follows that $\xi \in g(K, L)$.

(6) Let $\alpha \in \text{dom}(f) \cap S$, $M \in A$, and suppose that $\alpha \in M$. We will show that $M \cap \alpha \in f(\alpha)$. Since $\text{dom}(f) = \text{dom}(f_w) \cup \text{dom}(f_q)$, either $\alpha \in \text{dom}(f_w)$ or $\alpha \in \text{dom}(f_q)$. As $A = A_w \cup A_q$, either $M \in A_w$ or $M \in A_q$.

First, assume that $\alpha \in \text{dom}(f_w)$. Then $f(\alpha) = f_w(\alpha)$ by Definition 6.8. If $M \in A_w$, then $M \cap \alpha \in f_w(\alpha)$, since w is a condition. But $f(\alpha) = f_w(\alpha)$, so $M \cap \alpha \in f(\alpha)$.

Suppose that $M \in A_q$. Since $q \in D_Q$ and Q is simple, $M \cap Q \in A_q \cap Q \subseteq A_w$, by Lemma 6.7(1). So $M \cap Q \in A_w$. As $\alpha \in \text{dom}(f_w)$, $\alpha \in Q$. Thus, $\alpha \in (M \cap Q) \cap \text{dom}(f_w)$. Since w is a condition, it follows that $(M \cap Q) \cap \alpha \in f_w(\alpha) = f(\alpha)$. But $\alpha \in Q$ implies that $M \cap Q \cap \alpha = M \cap \alpha$. So $M \cap \alpha \in f(\alpha)$, as required.

Secondly, assume that $\alpha \in \text{dom}(f_q) \setminus \text{dom}(f_w)$. Then $\alpha \notin Q$, for otherwise $\alpha \in \text{dom}(f_q) \cap Q \subseteq \text{dom}(f_w)$ by Lemma 6.7(2). Note that this implies that $M \notin A_w$. For otherwise, $\alpha \in M \in A_w \subseteq Q$, which implies that $\alpha \in Q$. So $M \in A_q$. Since q is a condition, $M \cap \alpha \in f_q(\alpha)$. But $f_q(\alpha) \subseteq f(\alpha)$, by cases 2 and 3 of Definition 6.8. So $M \cap \alpha \in f(\alpha)$.

(7) As in (1) above, the assumptions of Proposition 3.8 hold for A_q and A_w . Therefore,

$$r^*(A) = r^*(A_w \cup A_q) = r^*(A_w) \cup r^*(A_q).$$

As w and q are conditions,

$$r^*(A_w) \cap S \subseteq \text{dom}(f_w), \quad r^*(A_q) \cap S \subseteq \text{dom}(f_q).$$

But $\text{dom}(f_w) \subseteq \text{dom}(f)$ and $\text{dom}(f_q) \subseteq \text{dom}(f)$. Hence,

$$r^*(A) \cap S = (r^*(A_w) \cap S) \cup (r^*(A_q) \cap S) \subseteq \text{dom}(f).$$

This completes the proof that $w \oplus_Q q$ is a condition. Now we show that $w \oplus_Q q \leq w, q$. First, we prove that $w \oplus_Q q \leq w$ by verifying properties (a)–(d) of Definition 4.2 for w . (a) Since $A = A_w \cup A_q$, clearly $A_w \subseteq A$. (b) follows from Lemma 6.9(1), (c) is immediate from Definition 6.12, and (d) was proved in Lemma 6.10(2).

Secondly, we prove that $w \oplus_Q q \leq q$ by verifying properties (a)–(d) of Definition 4.2 for q . (a) Since $A = A_w \cup A_q$, clearly $A_q \subseteq A$. (b) was proved in Lemma 6.9(2), (c) is immediate from Definition 6.12, and (d) was proved in Lemma 6.10(3). $\square$

Corollary 6.16. *The forcing poset $\mathbb{P}$ is κ -c.c.*

Proof. Let A be an antichain of $\mathbb{P}$, and suppose for a contradiction that A has size at least κ . Without loss of generality, assume that A is maximal. By Assumption 2.23, there are stationarily many simple models in $\mathcal{V}$, so we can fix a simple model $Q \in \mathcal{V}$ such that $Q \prec (H(\lambda), \in, \mathbb{P}, A)$. As A has size at least κ and $|Q| < \kappa$, we can fix $s \in A \setminus Q$.

By Lemma 6.3, fix $q \leq s$ such that $q \in D_Q$. Then $q \restriction Q$ is a condition in $Q \cap \mathbb{P}$. By the elementarity of Q and the maximality of A , there is $t \in A \cap Q$ which is compatible with $q \restriction Q$. By elementarity, fix $w \in Q \cap \mathbb{P}$ such that $w \leq q \restriction Q, t$.

By Proposition 6.15, w and q are compatible, so fix $v \leq w, q$. Then $v \leq w \leq t$, and $v \leq q \leq s$. Hence, s and t are compatible. But s and t are in A and A is an antichain. Therefore, $s = t$. This is impossible, since $t \in Q$ and $s \notin Q$. $\square$

§7. Amalgamation over countable models

In this section we will prove that the forcing poset $\mathbb{P}$ is strongly proper on a stationary set. We will show that for any simple model $N \in \mathcal{X}$, for any $p \in N \cap \mathbb{P}$, there is $q \leq p$ which is strongly N -generic.

For each $p \in N \cap \mathbb{P}$, we will show that there is $q \leq p$ such that $N \in A_q$. We will argue that q is strongly N -generic as follows. We will define a set D_N which satisfies, among other things, that for all $r \in D_N$, if $M \in A_r$ and $M < N$, then $M \cap N \in A_r$. The set D_N will be dense below q . For each $r \in D_N$, we will define a condition $r \restriction N$ in N satisfying that for all $w \in N \cap \mathbb{P}$, if $w \leq r \restriction N$, then w and r are compatible.

The arguments given in this section are very similar to those in the previous section. However, since $N \cap \kappa$ is a countable set, rather than an ordinal as in the uncountable case, the arguments given in this section are more complicated.

The first thing we will prove is that any condition $p \in N$ can be extended to a condition containing N .

Lemma 7.1. *Let $p \in \mathbb{P}$, $N \in \mathcal{X}$, and suppose that $p \in N$. Then there is $q \leq p$ such that $N \in A_q$.*

Proof. Define q as follows. Let $A_q := A_p \cup \{N\}$. Define f_q by letting

$$\text{dom}(f_q) := \text{dom}(f_p) \cup \{N \cap \alpha : \alpha \in \text{dom}(f_p) \cap S\}.$$

For each $\alpha \in \text{dom}(f_p) \cap S$, define

$$f_q(\alpha) := f_p(\alpha) \cup \{N \cap \alpha\}.$$

For each $M \in \text{dom}(f_p) \setminus S$, define $f_q(M) := f_p(M)$. Finally, for each $\alpha \in \text{dom}(f_p) \cap S$, define

$$f_q(N \cap \alpha) := f_p(\alpha).$$

For $K \in f_p(x)$, define $g_q(K, x) := g_p(K, x)$. Let $\alpha \in \text{dom}(f_p) \cap S$. For $K \in f_p(\alpha)$, let $g_q(K, N \cap \alpha) := g_p(K, \alpha)$ and $g_q(N \cap \alpha, \alpha) := \emptyset$.

It is easy to verify that if q is a condition, then $q \leq p$. Also, $N \in A_q$ by definition. It remains to prove that q is a condition. We verify requirements (1)–(7) of Definition 4.2. For (1), A_q is adequate by Lemma 2.16. (4), (5), and (6) are easy. It remains to prove (2), (3), and (7).

(2) Clearly f_q is a function, and every member of $\text{dom}(f_q)$ is of the required form. Let $x \in \text{dom}(f_q)$, and we will show that $f_q(x)$ is a finite $\in$ -chain and a subset of $Sk(x) \setminus S$. If $x \in \text{dom}(f_p) \setminus S$, then $f_q(x) = f_p(x)$, so we are done since p is a condition.

Suppose that $x = \alpha \in \text{dom}(f_p) \cap S$. Then $f_q(\alpha) = f_p(\alpha) \cup \{N \cap \alpha\}$. Since p is a condition, $f_p(\alpha)$ is a finite $\in$ -chain and a subset of $Sk(\alpha) \setminus S$. So it suffices to show that $f_p(\alpha) \subseteq Sk(N \cap \alpha)$ and $N \cap \alpha \in Sk(\alpha)$.

By Assumption 2.6, $N \cap \alpha \in Sk(\alpha)$. Let $K \in f_p(\alpha)$, and we will show that $K \in Sk(N \cap \alpha)$. By Definition 4.2(2), fix $K_1 \in A_p$ and $\beta \in (K_1 \cap \text{dom}(f_p) \cap S) \cup \{\kappa\}$

such that $K = K_1 \cap \beta$. Since $p \in N$, we have that K_1 and α are in N . Therefore, $K_1 \cap \alpha \in N$. By Lemma 2.8(1), $K_1 \cap \alpha \in Sk(N \cap \alpha)$.

Note that $K = K_1 \cap \beta$ is an initial segment of $K_1 \cap \alpha$ if $\beta \leq \alpha$, and since $K \subseteq \alpha$, $K = K_1 \cap \alpha$ if $\alpha < \beta$. In either case, K is an initial segment of $K_1 \cap \alpha$. Since $K_1 \cap \alpha$ is in $Sk(N \cap \alpha)$, by elementarity so is K .

Finally, assume that $x = N \cap \alpha$, where $\alpha \in \text{dom}(f_p) \cap S$. Then $f_q(x) = f_p(\alpha)$. We just showed that $f_p(\alpha)$ is a subset of $Sk(N \cap \alpha)$, and since p is a condition, it is a finite $\in$ -chain disjoint from S .

(3) Let $x \in \text{dom}(f_q)$. It is easy to check by cases that $f_q(x) \subseteq \text{dom}(f_q)$. Let $K \in f_q(x)$, and we will show that $f_q(K) = f_q(x) \cap Sk(K)$.

First, assume that $x \in \text{dom}(f_p) \setminus S$. Then $f_q(x) = f_p(x)$, so $K \in f_p(x) \subseteq \text{dom}(f_p) \setminus S$. So by definition, $f_q(K) = f_p(K)$. Therefore,

$$f_q(K) = f_p(K) = f_p(x) \cap Sk(K) = f_q(x) \cap Sk(K).$$

Secondly, assume that $x = \alpha \in \text{dom}(f_p) \cap S$. Then $f_q(\alpha) = f_p(\alpha) \cup \{N \cap \alpha\}$. If $K \in f_p(\alpha)$, then $K \in \text{dom}(f_p) \setminus S$, so by definition, $f_q(K) = f_p(K)$. Thence,

$$f_q(K) = f_p(K) = f_p(\alpha) \cap Sk(K) = f_q(\alpha) \cap Sk(K),$$

where the last equality follows from the fact that $N \cap \alpha \notin Sk(K)$. If $K = N \cap \alpha$, then

$$f_q(K) = f_q(N \cap \alpha) = f_p(\alpha) = f_q(\alpha) \cap Sk(N \cap \alpha),$$

where the last equality follows from the fact that $f_p(\alpha) \subseteq Sk(N \cap \alpha)$, as shown above.

Thirdly, assume that $x = N \cap \alpha$. Then $f_q(N \cap \alpha) = f_p(\alpha)$. Hence, $K \in f_p(\alpha) \setminus S$. So by definition, $f_q(K) = f_p(K)$. Thus,

$$f_q(K) = f_p(K) = f_p(\alpha) \cap Sk(K) = f_q(N \cap \alpha) \cap Sk(K).$$

(7) Note that $r^*(A_p \cup \{N\}) = r^*(A_p)$. Namely, if $\gamma \in r^*(A_p \cup \{N\})$, then $\gamma \in r^*(\{K, M\})$ for some distinct K and M in $A_p \cup \{N\}$ such that $K \sim M$. But for all $K \in A_p$, $K \not\sim N$. Hence, K and M are in A_p , and $\gamma \in r^*(A_p)$. It follows that

$$r^*(A_q) \cap S = r^*(A_p \cup \{N\}) \cap S = r^*(A_p) \cap S \subseteq \text{dom}(f_p) \subseteq \text{dom}(f_q).$$

□

The next lemma will be used to show that the set D_N , which we will define shortly, is dense below any condition which contains N .

Lemma 7.2. *Let $q \in \mathbb{P}$ and let $N \in A_q$. Then there is $s \leq q$ such that for all $M \in A_s$, if $M < N$ then $M \cap N \in A_s$.*

Recall that if $M < N$ are in $\mathcal{X}$, then $\{M, N\}$ is adequate, and therefore $M \cap N \in \mathcal{X}$ by Assumption 2.19.

Proof. By Proposition 2.24, the set $A_q \cup \{M \cap N : M \in A_q, M < N\}$ is adequate. Define

$$x_0 := r^*(A_q \cup \{M \cap N : M \in A_q, M < N\}) \cap S,$$

and define

$$x := x_0 \setminus \text{dom}(f_q).$$

Let $r := q + x$. By Lemma 4.8, r is a condition and $r \leq q$. By Definition 4.7, $A_r = A_q$, and easily, $x_0 \subseteq \text{dom}(f_r)$.

Define s as follows. Let $f_s := f_r$, $g_s := g_r$, and

$$A_s := A_r \cup \{M \cap N : M \in A_r, M < N\}.$$

We claim that s is as required.

By Proposition 2.24, for all $M \in A_s$, if $M < N$ then $M \cap N \in A_s$. It is trivial to check that if s is a condition, then $s \leq r$, and therefore $s \leq q$.

It remains to prove that s is a condition. We verify requirements (1)–(7) of Definition 4.2. (1) follows from Proposition 2.24. (2)–(5) follow immediately from r being a condition, together with the fact that $f_s = f_r$, $g_s = g_r$, and $A_r \subseteq A_s$.

(6) Suppose that $\alpha \in \text{dom}(f_s) \cap S$, $M \in A_s$, and $\alpha \in M$. We will show that $M \cap \alpha \in f_s(\alpha)$. Since $f_s = f_r$, we have that $\alpha \in \text{dom}(f_r) \cap S$. So if $M \in A_r$, then $M \cap \alpha \in f_r(\alpha) = f_s(\alpha)$, since r is a condition.

Assume that $M \in A_s \setminus A_r$, which means that $M = M_1 \cap N$ for some $M_1 \in A_r$ with $M_1 < N$. Then $\alpha \in M \cap \kappa = M_1 \cap N \cap \kappa$. By Proposition 2.11, it follows that $\alpha < \beta_{M_1, N}$. Since $M_1 < N$, $M_1 \cap \beta_{M_1, N} \in N$, so $M_1 \cap \alpha \in N$. In particular, $M_1 \cap \alpha \subseteq N$. So

$$M \cap \alpha = M_1 \cap N \cap \alpha = M_1 \cap \alpha.$$

But $M_1 \in A_r$ and $\alpha \in M_1$. Since r is a condition,

$$M \cap \alpha = M_1 \cap \alpha \in f_r(\alpha) = f_s(\alpha).$$

(7) We need to show that $r^*(A_s) \cap S \subseteq \text{dom}(f_s)$. Since $f_s = f_r$, it suffices to show that $r^*(A_s) \cap S \subseteq \text{dom}(f_r)$. But by the definition of A_s and since $A_r = A_q$, we have that

$$r^*(A_s) \cap S = r^*(A_q \cup \{M \cap N : M \in A_q, M < N\}) \cap S = x_0,$$

and as noted above, $x_0 \subseteq \text{dom}(f_r)$. So $r^*(A_s) \cap S \subseteq \text{dom}(f_r)$. $\square$

Definition 7.3. For any $N \in \mathcal{X}$, let D_N denote the set of conditions $r \in \mathbb{P}$ satisfying:

- (1) $N \in A_r$;
- (2) for all $M \in A_r$, if $M < N$ then $M \cap N \in A_r$;
- (3) whenever $K \in f_r(x)$ and $x \in f_r(y)$, then

$$g_r(K, x) \subseteq g_r(K, y).$$

Note that by Definition 4.2(5), the conclusion of (3) is equivalent to $g_r(K, x) = g_r(K, y)$.

The next lemma says that D_N is dense below any condition which contains N .

Lemma 7.4. Let $N \in \mathcal{X}$. Then for any condition $q \in \mathbb{P}$, if $N \in A_q$, then there is $s \leq q$ such that $s \in D_N$.

Proof. Let $q \in \mathbb{P}$ be such that $N \in A_q$. By Lemma 7.2, there is $r \leq q$ such that for all $M \in A_r$, if $M < N$ then $M \cap N \in A_r$. By Lemma 4.9, there is $s \leq r$ such that $f_s = f_r$, $A_s = A_r$, and whenever $K \in f_s(x)$ and $x \in f_s(y)$, then $g_s(K, x) \subseteq g_s(K, y)$. Then $s \leq q$ and $s \in D_N$. $\square$

Definition 7.5. Suppose that $N \in \mathcal{X}$ is simple and $r \in D_N$. Define $r \restriction N$ as the triple (f, g, A) satisfying:

- (1) $\text{dom}(f) = \text{dom}(f_r) \cap N$, and for all $x \in \text{dom}(f)$, $f(x) := f_r(x) \cap N$;

- (2) $\text{dom}(g) = \text{dom}(g_r) \cap N$, and for all $(K, x) \in \text{dom}(g)$, $g(K, x) := g_r(K, x)$;
- (3) $A := A_r \cap N$.

Observe that in (1), if $x \in \text{dom}(f) \setminus S$, then $x \in N$ implies that $Sk(x) \subseteq N$. Therefore, $f_r(x) \subseteq N$. So in this case, $f(x) = f_r(x) \cap N$ is equal to $f_r(x)$.

Let us prove that $r \upharpoonright N$ is in N . Obviously $A = A_r \cap N$ and $\text{dom}(f) = \text{dom}(f_r) \cap N$ are in N , and for all $x \in \text{dom}(f)$, $f(x) = f_r(x) \cap N$ is in N . Consider $K \in f(x)$. Then K and x are in N . If $x \notin S$, then $x \subseteq N$. Therefore, $g(K, x) = g_r(K, x) \subseteq x \subseteq N$. So $g(K, x)$ is a finite subset of N , and hence is in N .

Finally, suppose that $x = \alpha \in S$. Then $\alpha \in N$, and as $N \in A_r$, $N \cap \alpha$ is in $f_r(\alpha)$ by Definition 4.2(6). Also,

$$g_r(K, \alpha) \subseteq g_r(K, N \cap \alpha) \subseteq N \cap \alpha$$

by Definition 4.2(4,5). Hence, $g(K, \alpha) = g_r(K, \alpha)$ is a finite subset of N , and hence is in N .

We have proven that all of the components of $r \upharpoonright N$ are in N . Therefore, $r \upharpoonright N$ is in N .

Lemma 7.6. *Suppose that $N \in \mathcal{X}$ is simple and $r \in D_N$. Then $r \upharpoonright N$ is in $N \cap \mathbb{P}$ and $r \leq r \upharpoonright N$.*

Proof. Let $r \upharpoonright N = (f, g, A)$. We have already observed that $r \upharpoonright N \in N$. It is trivial to check that if $r \upharpoonright N$ is a condition, then $r \leq r \upharpoonright N$. So it suffices to show that $r \upharpoonright N$ is a condition. We verify requirements (1)–(7) of Definition 4.2. (1), (4), (5), (6), and (7) are easy to check. It remains to prove (2) and (3).

(2) Obviously f is a function with a finite domain. Let $x \in \text{dom}(f) = \text{dom}(f_r) \cap N$. Then $x \in N$. We will show that either $x \in S$, or there is $M \in A$ and $\alpha \in (M \cap \text{dom}(f) \cap S) \cup \{\kappa\}$ such that $x = M \cap \alpha$, and moreover, $f(x)$ is a finite $\in$ -chain and $f(x) \subseteq Sk(x) \setminus S$.

We begin by showing that $f(x)$ is a finite $\in$ -chain and $f(x) \subseteq Sk(x) \setminus S$. Since r is a condition, $f_r(x)$ is a finite $\in$ -chain and $f_r(x) \subseteq Sk(x) \setminus S$. But $f(x) = f_r(x) \cap N \subseteq f_r(x)$. Therefore, $f(x)$ is a finite $\in$ -chain and $f(x) \subseteq Sk(x) \setminus S$.

Now we show that either $x \in S$, or there is $M \in A$ and $\alpha \in (M \cap \text{dom}(f) \cap S) \cup \{\kappa\}$ such that $x = M \cap \alpha$. Since $x \in \text{dom}(f_r)$ and r is a condition, we have that either $x \in S$, or there is $M_1 \in A_r$ and $\alpha \in (M_1 \cap \text{dom}(f_r) \cap S) \cup \{\kappa\}$ such that $x = M_1 \cap \alpha$. In the first case, we are done, so assume the second case.

Since $x = M_1 \cap \alpha \in N$, clearly $M_1 \cap \omega_1 \in N$. By Lemma 2.17(1), it follows that $M_1 < N$. Therefore, as $r \in D_N$, we have that $M_1 \cap N \in A_r$. But N is simple, so also $M_1 \cap N \in N$. So $M_1 \cap N \in A_r \cap N = A$. Hence, to complete the proof, it suffices to show that $x = M_1 \cap N \cap \beta$, for some $\beta \in (M_1 \cap N \cap \text{dom}(f) \cap S) \cup \{\kappa\}$.

If $\alpha = \kappa$, then $x = M_1 \cap \kappa \in N$, so $M_1 \cap \kappa \subseteq N$. Hence

$$x = M_1 \cap \kappa = M_1 \cap N \cap \kappa,$$

and we are done.

Suppose that $\alpha < \kappa$. First, assume that $\alpha \in N$. Then $\alpha \in \text{dom}(f_r) \cap S \cap N = \text{dom}(f) \cap S$. Also, $\alpha \in M_1 \cap N \cap \kappa$, which implies that $\alpha < \beta_{M_1, N}$ by Proposition 2.11. Since $M_1 < N$, it follows that $M_1 \cap \alpha \subseteq N$. Hence,

$$x = M_1 \cap \alpha = M_1 \cap N \cap \alpha.$$

As $\alpha \in M_1 \cap N \cap \text{dom}(f) \cap S$, we are done.

Secondly, assume that $\alpha \notin N$. We claim that

$$M_1 \cap \alpha = M_1 \cap N \cap \kappa,$$

which will finish the proof. Since $x = M_1 \cap \alpha$ is in N , the forward inclusion is immediate. For the reverse inclusion, let $\gamma \in M_1 \cap N \cap \kappa$, and we will show that $\gamma \in M_1 \cap \alpha$. Then $\gamma < \beta_{M_1, N}$ by Proposition 2.11, and therefore, since $M_1 < N$, we have that

$$M_1 \cap \gamma \subseteq M_1 \cap \beta_{M_1, N} \subseteq N.$$

Since $\alpha \notin N$, it follows that $\alpha \notin M_1 \cap \gamma$. But $\alpha \in M_1$. Therefore, $\gamma \leq \alpha$. Also, $\gamma \in N$ and $\alpha \notin N$ implies that $\gamma \neq \alpha$, so $\gamma < \alpha$. Thus, $\gamma \in M_1 \cap \alpha$, completing the proof.

(3) Let $x \in \text{dom}(f) = \text{dom}(f_r) \cap N$. Since r is a condition, we have that

$$f(x) = f_r(x) \cap N \subseteq \text{dom}(f_r) \cap N = \text{dom}(f).$$

Thus, $f(x) \subseteq \text{dom}(f)$.

Let $K \in f(x) = f_r(x) \cap N$, and we will show that $f(K) = f(x) \cap Sk(K)$. Since $K \in f_r(x)$, $K \notin S$, so $Sk(K) \subseteq N$. Therefore, $f_r(K) \subseteq Sk(K) \subseteq N$. So

$$f(K) = f_r(K) \cap N = f_r(K).$$

As r is a condition, we have that $f(K) = f_r(K) = f_r(x) \cap Sk(K)$. But $Sk(K) \subseteq N$ implies that

$$f_r(x) \cap Sk(K) = f_r(x) \cap Sk(K) \cap N = f(x) \cap Sk(K).$$

Thus, $f(K) = f(x) \cap Sk(K)$. □

The next lemma will not be used until Section 8.

Lemma 7.7. *Suppose that $N \in \mathcal{X}$ is simple and $r \in D_N$. Assume that $p \in N \cap \mathbb{P}$ and $r \leq p$. Then $r \restriction N \leq p$.*

Proof. We verify properties (a)–(d) of Definition 4.2. (a) Since $p \in N$, $A_p \subseteq N$. As $r \leq p$, we have that

$$A_p \subseteq A_r \cap N = A_{r \restriction N}.$$

(b) Since $p \in N$, $\text{dom}(f_p) \subseteq N$ and for all $x \in \text{dom}(f_p)$, $f_p(x) \subseteq N$. As $r \leq p$, we have that

$$\text{dom}(f_p) \subseteq \text{dom}(f_r) \cap N = \text{dom}(f_{r \restriction N}).$$

Let $x \in \text{dom}(f_p)$. Then

$$f_p(x) \subseteq f_r(x) \cap N = f_{r \restriction N}(x).$$

(c) Let $(K, x) \in \text{dom}(g_p)$. Then since $r \leq p$,

$$g_p(K, x) \subseteq g_r(K, x) = g_{r \restriction N}(K, x).$$

(d) Assume that K and x are in $\text{dom}(f_p)$ and $K \in f_{r \restriction N}(x)$. We claim that $K \in f_p(x)$. But

$$K \in f_{r \restriction N}(x) = f_r(x) \cap N \subseteq f_r(x).$$

So $K \in f_r(x)$. Since $r \leq p$, it follows that $K \in f_p(x)$. □

We will now begin analyzing the situation where $r \in D_N$ and $w \leq r \restriction N$ is in $N \cap \mathbb{P}$.

Lemma 7.8. *Let $N \in \mathcal{X}$ be simple and $r \in D_N$. Suppose that $w \in N \cap \mathbb{P}$ and $w \leq r \restriction N$. Then:*

- (1) $A_r \cap N \subseteq A_w$;
- (2) $\text{dom}(f_r) \cap N \subseteq \text{dom}(f_w)$, and for all $x \in \text{dom}(f_r) \cap N$, $f_r(x) \cap N \subseteq f_w(x)$;
- (3) $\text{dom}(g_r) \cap N \subseteq \text{dom}(g_w)$, and for all $(K, x) \in \text{dom}(g_r) \cap N$, $g_r(K, x) \subseteq g_w(K, x)$.

Proof. Immediate from the definition of $r \restriction N$ and the fact that $w \leq r \restriction N$. $\square$

As we discussed at the beginning of the section, we are going to show that whenever $w \leq r \restriction N$, where $r \in D_N$ and $w \in N \cap \mathbb{P}$, then w and r are compatible. As in the previous section, we will construct a specific lower bound $w \oplus_N r$ of w and r . We will describe separately the f , g , and A components of $w \oplus_N r$. The A -component of $w \oplus_N r$ will be defined as $A_w \cup A_r$.

We handle the f -component next. Unfortunately, the definition of $f_w \oplus_N f_r$ is much more complicated than in the previous section. The domain of $f_w \oplus_N f_r$ will include not only $\text{dom}(f_w) \cup \text{dom}(f_r)$, but also some additional sets.

Before defining $f_w \oplus_N f_r$, we prove two lemmas which will help us handle its domain.

Lemma 7.9. *Let $N \in \mathcal{X}$ be simple and $r \in D_N$. Suppose that $w \in N \cap \mathbb{P}$ and $w \leq r \restriction N$. Then the set*

$$\{M \cap \alpha : M \in A_r, N \leq M, \alpha \in (M \cap \text{dom}(f_w) \cap S) \setminus \text{dom}(f_r)\}$$

is disjoint from N , $\text{dom}(f_w)$, and $\text{dom}(f_r)$.

Proof. It is clear that any member $M \cap \alpha$ of the displayed set is not in N , since $N \leq M$ implies that $N \cap \omega_1 \leq M \cap \omega_1 = M \cap \alpha \cap \omega_1$. Since $w \in N$, and hence $\text{dom}(f_w) \subseteq N$, it follows that the displayed set is disjoint from $\text{dom}(f_w)$.

Suppose for a contradiction that for some $M \in A_r$ with $N \leq M$ and some $\alpha \in (M \cap \text{dom}(f_w) \cap S) \setminus \text{dom}(f_r)$, $M \cap \alpha$ is in $\text{dom}(f_r)$. By Definition 4.2(2), fix $M_1 \in A_r$ and $\beta \in (M_1 \cap \text{dom}(f_r) \cap S) \cup \{\kappa\}$ such that $M \cap \alpha = M_1 \cap \beta$. Since $\alpha \in \kappa \setminus \text{dom}(f_r)$ and $\beta \in \text{dom}(f_r) \cup \{\kappa\}$, $\alpha \neq \beta$.

Applying Lemma 3.2 to M , M_1 , α , and β , we get that $M \sim M_1$ and $\alpha = \min((M \cap \kappa) \setminus \beta_{M, M_1})$. Since M and M_1 are in A_r and $\alpha \in S$, we have that $\alpha \in r^*(A_r) \cap S$. By Definition 4.2(7), it follows that $\alpha \in \text{dom}(f_r)$. But this contradicts the choice of α . $\square$

Lemma 7.10. *Suppose that $r \in \mathbb{P}$, $N \in A_r$, and $x \in \text{dom}(f_r)$. Then there is at most one ordinal α such that $\alpha \in \text{dom}(f_r) \cap S \cap N$ and $N \cap \alpha \in f_r(x) \cup \{x\}$.*

Proof. Suppose for a contradiction that $\alpha < \beta$ are in $\text{dom}(f_r) \cap S \cap N$, and $N \cap \alpha$ and $N \cap \beta$ are both in $f_r(x) \cup \{x\}$. Then $N \cap \alpha$ and $N \cap \beta$ are membership comparable. Since $N \cap \alpha$ and $N \cap \beta$ have the same intersection with ω_1 , they must be equal. But then $\alpha \in N \cap \beta = N \cap \alpha$, which is impossible. $\square$

We are ready to define $f_w \oplus_N f_r$.

Definition 7.11. *Let $N \in \mathcal{X}$ be simple and $r \in D_N$. Suppose that $w \in N \cap \mathbb{P}$ and $w \leq r \restriction N$. Define $f_w \oplus_N f_r = f$ as follows.*

The domain of f is equal to the union of $\text{dom}(f_w)$, $\text{dom}(f_r)$, and the set

$$\{M \cap \alpha : M \in A_r, N \leq M, \alpha \in (M \cap \text{dom}(f_w) \cap S) \setminus \text{dom}(f_r)\}.$$

The values of f are defined by the following cases:

- (1) for all $x \in \text{dom}(f_w) \setminus S$,

$$f(x) := f_w(x);$$
- (2) for all $\alpha \in \text{dom}(f_w) \cap S \cap \text{dom}(f_r)$,

$$f(\alpha) := f_w(\alpha) \cup f_r(\alpha);$$
- (3) for all $\alpha \in (\text{dom}(f_w) \cap S) \setminus \text{dom}(f_r)$,

$$f(\alpha) := f_w(\alpha) \cup \{M \cap \alpha : M \in A_r, N \leq M, \alpha \in M\};$$
- (4) if $x \in \text{dom}(f_r) \setminus N$, and for some $\alpha \in \text{dom}(f_r) \cap S \cap N$, $N \cap \alpha \in f_r(x) \cup \{x\}$,
then

$$f(x) := f_w(\alpha) \cup f_r(x);$$
- (5) for all $x \in \text{dom}(f_r) \setminus N$ such that (4) fails, if $f_r(x) \cap N = \emptyset$, then

$$f(x) := f_r(x);$$
- (6) for all $x \in \text{dom}(f_r) \setminus N$ such that (4) fails, if $f_r(x) \cap N \neq \emptyset$, then

$$f(x) := f_r(x) \cup f_w(M),$$

where M is the membership largest element of $f_r(x) \cap N$;
- (7) for a set of the form $M \cap \alpha$, where $M \in A_r$, $N \leq M$, and $\alpha \in (M \cap \text{dom}(f_w) \cap S) \setminus \text{dom}(f_r)$,

$$f(M \cap \alpha) := f(\alpha) \cap \text{Sk}(M \cap \alpha),$$

where $f(\alpha)$ was defined in (3).

It is easy to see that cases 1–7 describe all of the possibilities for a set being in $\text{dom}(f)$, using the fact that $\text{dom}(f_r) \cap N \subseteq \text{dom}(f_w)$ by Lemma 7.8(2). Moreover, cases 1–6 are obviously disjoint, and they are also disjoint from case 7 by Lemma 7.9. Finally, note that the ordinal α in case 4 is unique by Lemma 7.10, so $f(x)$ is well-defined in this case.

The next four lemmas describe some important properties of $f_w \oplus_N f_r$. Lemmas 7.12 and 7.13 will be used to show that $w \oplus_N r$ is below w and r in $\mathbb{P}$.

Lemma 7.12. *Let $N \in \mathcal{X}$ be simple and $r \in D_N$. Suppose that $w \in N \cap \mathbb{P}$ and $w \leq r \restriction N$. Let $f := f_w \oplus_N f_r$. Then:*

- (1) $\text{dom}(f_w) \cup \text{dom}(f_r) \subseteq \text{dom}(f)$;
- (2) if $x \in \text{dom}(f_w)$, then $f_w(x) \subseteq f(x)$;
- (3) if $x \in \text{dom}(f_r)$, then $f_r(x) \subseteq f(x)$.

Proof. (1,2) By the definition of f in Definition 7.11, $\text{dom}(f_w)$ and $\text{dom}(f_r)$ are subsets of $\text{dom}(f)$. By Cases 1, 2, and 3 of Definition 7.11, for all $x \in \text{dom}(f_w)$, $f_w(x) \subseteq f(x)$.

(3) Suppose that $x \in \text{dom}(f_r)$, and we will show that $f_r(x) \subseteq f(x)$. We consider each of the cases 1–7 of Definition 7.11 in the definition of $f(x)$.

If $f(x)$ is defined by cases 2, 4, 5, or 6, then $f_r(x) \subseteq f(x)$ by definition. Since $x \in \text{dom}(f_r)$, case 3 does not hold, and case 7 does not hold by Lemma 7.9. It remains to consider case 1.

For case 1, suppose that $x \in \text{dom}(f_w) \setminus S$ and $f(x) = f_w(x)$. Then $x \in N$. So $f_r(x) \subseteq \text{Sk}(x) \subseteq N$. Hence, by Lemma 7.8(2), $f_r(x) = f_r(x) \cap N \subseteq f_w(x) = f(x)$. $\square$

Lemma 7.13. *Let $N \in \mathcal{X}$ be simple and $r \in D_N$. Suppose that $w \in N \cap \mathbb{P}$ and $w \leq r \restriction N$. Let $f := f_w \oplus_N f_r$. Then:*

- (1) $\text{dom}(f) \cap N = \text{dom}(f_w)$;
- (2) *if $K \in f(x)$ and K and x are in $\text{dom}(f_w)$, then $K \in f_w(x)$;*
- (3) *if $K \in f(x)$ and K and x are in $\text{dom}(f_r)$, then $K \in f_r(x)$.*

Proof. (1) The inclusion $\text{dom}(f_w) \subseteq \text{dom}(f) \cap N$ is immediate, so it suffices to show that if $x \in \text{dom}(f) \cap N$, then $x \in \text{dom}(f_w)$. Note that x is not equal to $M \cap \alpha$, for any $M \in A_r$ with $N \leq M$ and $\alpha \in (M \cap \text{dom}(f_w) \cap S) \setminus \text{dom}(f_r)$, since such a set is not in N by Lemma 7.9. So by the definition of $\text{dom}(f)$, we have that either $x \in \text{dom}(f_w)$, in which case we are done, or $x \in \text{dom}(f_r)$. In the second case, $x \in \text{dom}(f_r) \cap N \subseteq \text{dom}(f_w)$ by Lemma 7.8(2).

(2) Suppose that $K \in f(x)$ and K and x are in $\text{dom}(f_w)$. We will show that $K \in f_w(x)$. Since K and x are in $\text{dom}(f_w)$, they are in N . Hence, we are in cases 1, 2, or 3 of Definition 7.11. In case 1, $f(x) = f_w(x)$, so $K \in f_w(x)$. In case 2, $f(x) = f_w(\alpha) \cup f_r(\alpha)$, where $x = \alpha \in \text{dom}(f_w) \cap S \cap \text{dom}(f_r)$. So either $K \in f_w(\alpha)$, in which case we are done, or $K \in f_r(\alpha)$. In the second case, by Lemma 7.8(2) we have that

$$K \in f_r(\alpha) \cap N \subseteq f_w(\alpha) = f_w(x).$$

In case 3, $f(x)$ is equal to the union of $f_w(x)$ together with a collection of sets which are not members of N . Since $K \in N$, $K \in f_w(x)$.

(3) Suppose that $K \in f(x)$ and K and x are in $\text{dom}(f_r)$. We will show that $K \in f_r(x)$. First, assume that K and x are both in N . Then K and x are in $\text{dom}(f_r) \cap N \subseteq \text{dom}(f_w)$ by Lemma 7.8(2). By (2) just proven, $K \in f_w(x)$. Yet K and x are in $\text{dom}(f_r) \cap N = \text{dom}(f_r \restriction N)$. Since $w \leq r \restriction N$, it follows that $K \in f_{r \restriction N}(x) \subseteq f_r(x)$. So $K \in f_r(x)$.

Next, we consider each of the cases 1–7 of Definition 7.11 for $f(x)$. Case 1 is immediate, since it implies that K and x are in N . In case 2, $x = \alpha$ is in N and $f(x) = f_w(\alpha) \cup f_r(\alpha)$. If $K \in f_w(\alpha)$, then K and x are both in N , and we are done. Otherwise $K \in f_r(\alpha)$, and we are also done.

Case 3 does not apply, since it says that $x = \alpha$ is not in $\text{dom}(f_r)$. In case 5, $f(x) = f_r(x)$, so $K \in f_r(x)$, and we are done. Case 7 does not apply, since any set of the form described there is not in $\text{dom}(f_r)$ by Lemma 7.9. It remains to consider cases 4 and 6.

In case 4, there is $\alpha \in \text{dom}(f_r) \cap S \cap N$ such that $N \cap \alpha \in f_r(x) \cup \{x\}$, and $f(x) = f_w(\alpha) \cup f_r(x)$. So either $K \in f_w(\alpha)$, or $K \in f_r(x)$. In the second case we are done, so assume that $K \in f_w(\alpha)$. By Lemma 7.12(2), $f_w(\alpha) \subseteq f(\alpha)$, so $K \in f(\alpha)$. Since K and α are in $\text{dom}(f_r)$, it follows by case 2 just handled that $K \in f_r(\alpha)$. But also $K \in f_w(\alpha)$ means that $K \in N$. Since $N \cap \alpha \in f_r(\alpha)$ by Definition 4.2(6) and $K \in N$, we must have that $K \in f_r(N \cap \alpha)$. But $N \cap \alpha \in f_r(x) \cup \{x\}$, so $K \in f_r(x)$.

In case 6, $f(x) = f_r(x) \cup f_w(M)$, where M is the membership largest element of $f_r(x) \cap N$. If $K \in f_r(x)$ then we are done, so assume that $K \in f_w(M)$. Since M is in $\text{dom}(f_r) \cap N \subseteq \text{dom}(f_w)$ by Lemma 7.8(2), M is in $\text{dom}(f_w) \setminus S$. So $f(M)$ is defined as in case 1 of Definition 7.11. Hence, $f(M) = f_w(M)$. Therefore, $K \in f(M)$. Since K and M are in $\text{dom}(f_r)$, it follows by case 1 handled above that $K \in f_r(M)$. Thus, $K \in f_r(M)$ and $M \in f_r(x)$, so $K \in f_r(x)$. $\square$

The next lemma will be used to show that $f_w \oplus_N f_r$ satisfies requirement (2) of Definition 4.2 for $w \oplus_N r$.

Lemma 7.14. *Let $N \in \mathcal{X}$ be simple and $r \in D_N$. Suppose that $w \in N \cap \mathbb{P}$ and $w \leq r \restriction N$. Let $f := f_w \oplus_N f_r$. Then f is a function with a finite domain, and for all $x \in \text{dom}(f)$, either $x \in S$, or there is $M \in A_w \cup A_r$ and*

$$\alpha \in (M \cap \text{dom}(f) \cap S) \cup \{\kappa\}$$

satisfying that $x = M \cap \alpha$; moreover, for all $x \in \text{dom}(f)$, $f(x)$ is a finite $\in$ -chain and $f(x) \subseteq Sk(x) \setminus S$.

Proof. It is immediate that f is a function with a finite domain. By the definition of the domain of f in Definition 7.11, together with the fact that $\text{dom}(f_w) \cup \text{dom}(f_r) \subseteq \text{dom}(f)$ and w and r are conditions, it is easy to see that if $x \in \text{dom}(f)$, then either $x \in S$, or there is $M \in A_w \cup A_r$ and $\alpha \in (M \cap \text{dom}(f) \cap S) \cup \{\kappa\}$ satisfying that $x = M \cap \alpha$.

Let $x \in \text{dom}(f)$, and we will show that $f(x)$ is a finite $\in$ -chain and $f(x) \subseteq Sk(x) \setminus S$. We will consider each of the cases 1–7 of Definition 7.11. We are done if either $f(x) = f_w(x)$ or $f(x) = f_r(x)$ as in cases 1 and 5, since w and r are conditions. In case 7, $f(x) \subseteq Sk(x)$ by definition, and $f(x)$ is a finite $\in$ -chain and disjoint from S provided that the result is true for case 3. It remains to handle cases 2, 3, 4, and 6.

Case 2: $x = \alpha \in \text{dom}(f_w) \cap S \cap \text{dom}(f_r)$ and $f(\alpha) = f_w(\alpha) \cup f_r(\alpha)$. Since w and r are conditions, $f_w(\alpha)$ and $f_r(\alpha)$ are themselves finite $\in$ -chains and subsets of $Sk(\alpha) \setminus S$. So it suffices to show that if $K \in f_w(\alpha)$ and $M \in f_r(\alpha) \setminus f_w(\alpha)$, then $K \in Sk(M)$.

Since $\alpha \in \text{dom}(f_w)$, α is in N . Thus, $\alpha \in \text{dom}(f_r) \cap S \cap N$. Since $N \in A_r$, we have that $N \cap \alpha \in f_r(\alpha)$, because r is a condition. As M is also in $f_r(\alpha)$, M and $N \cap \alpha$ are membership comparable. We claim that M is not in $Sk(N \cap \alpha)$. Otherwise, $M \in N$, so $M \in f_r(\alpha) \cap N \subseteq f_w(\alpha)$ by Lemma 7.8(2). Hence, $M \in f_w(\alpha)$, which contradicts the choice of M .

Thus, either $M = N \cap \alpha$, or $N \cap \alpha \in Sk(M)$. In either case, $Sk(N \cap \alpha) \subseteq Sk(M)$. Since $K \in f_w(\alpha)$, $K \subseteq \alpha$. Therefore, $K \in N$ implies that $K \in Sk(N \cap \alpha)$ by Lemma 4.3(3). Hence, $K \in Sk(N \cap \alpha) \subseteq Sk(M)$, so $K \in Sk(M)$.

Case 3: $x = \alpha \in (\text{dom}(f_w) \cap S) \setminus \text{dom}(f_r)$ and

$$f(\alpha) = f_w(\alpha) \cup \{M \cap \alpha : M \in A_r, N \leq M, \alpha \in M\}.$$

Since w is a condition, $f_w(\alpha)$ is a finite $\in$ -chain and a subset of $Sk(\alpha) \setminus S$. By Lemma 2.33, the second set in the above union is also a finite $\in$ -chain and a subset of $Sk(\alpha)$, and it is obviously disjoint from S . So it suffices to show that if $K \in f_w(\alpha)$, $M \in A_r$, $N \leq M$, and $\alpha \in M$, then $K \in Sk(M \cap \alpha)$.

Since $K \in f_w(\alpha)$, we have that $K \in N$ and $K \subseteq \alpha$. By Lemma 4.3(3), $K \in Sk(N \cap \alpha)$. Since $\alpha \in M \cap N$, $\alpha < \beta_{M,N}$ by Proposition 2.11. As $N \leq M$, we have that $N \cap \alpha \subseteq M \cap \alpha$. Thus, $Sk(N \cap \alpha) \subseteq Sk(M \cap \alpha)$. So $K \in Sk(M \cap \alpha)$.

Case 4: $x \in \text{dom}(f_r) \setminus N$, $\alpha \in \text{dom}(f_r) \cap S \cap N$, and $N \cap \alpha \in f_r(x) \cup \{x\}$. Then $f(x) = f_w(\alpha) \cup f_r(x)$. Since w and r are conditions, $f_w(\alpha)$ and $f_r(x)$ are themselves finite $\in$ -chains, and subsets of $Sk(\alpha) \setminus S$ and $Sk(x) \setminus S$ respectively. Consider $K \in f_w(\alpha)$, and we will show that $K \in Sk(x)$. So $K \in N$ and $K \subseteq \alpha$. By Lemma

4.3(3), $K \in Sk(N \cap \alpha)$. But $N \cap \alpha \in f_r(x) \cup \{x\}$, so in particular, $Sk(N \cap \alpha) \subseteq Sk(x)$. Thus, $K \in Sk(x)$. This completes the proof that $f(x) \subseteq Sk(x) \setminus S$.

Since $f_w(\alpha)$ and $f_r(x)$ are themselves finite $\in$ -chains, in order to show that $f(x)$ is a finite $\in$ -chain, it suffices to show that if $K \in f_w(\alpha)$ and $L \in f_r(x) \setminus f_w(\alpha)$, then $K \in Sk(L)$.

Since $K \in N$ and $K \subseteq \alpha$, $K \in Sk(N \cap \alpha)$ by Lemma 4.3(3). As $L \in f_r(x)$ and $N \cap \alpha \in f_r(x) \cup \{x\}$, L and $N \cap \alpha$ are membership comparable. We claim that L is not in $Sk(N \cap \alpha)$. Suppose for a contradiction that $L \in Sk(N \cap \alpha)$. Then $L \in N$, and $L \in f_r(N \cap \alpha) \subseteq f_r(\alpha)$. So $L \in f_r(\alpha) \cap N \subseteq f_w(\alpha)$ by Lemma 7.8(2). Hence, $L \in f_w(\alpha)$, which contradicts the choice of L . Thence, either $L = N \cap \alpha$ or $N \cap \alpha \in Sk(L)$. In either case, $Sk(N \cap \alpha) \subseteq Sk(L)$, and in particular, $K \in Sk(L)$.

Case 6: $x \in \text{dom}(f_r) \setminus N$ and $f(x) = f_r(x) \cup f_w(M)$, where M is the membership largest element of $f_r(x) \cap N$. Since r and w are conditions, $f_r(x)$ is a finite $\in$ -chain and a subset of $Sk(x) \setminus S$, and $f_w(M)$ is a finite $\in$ -chain and a subset of $Sk(M) \setminus S$. Since $M \in f_r(x)$, $M \in Sk(x)$. Therefore, $f_w(M) \subseteq Sk(M) \subseteq Sk(x)$. Hence, both $f_r(x)$ and $f_w(M)$ are finite $\in$ -chains and subsets of $Sk(x) \setminus S$. So it suffices to show that if $K \in f_w(M)$ and $L \in f_r(x) \setminus f_w(M)$, then $K \in Sk(L)$.

Since L and M are both in $f_r(x)$, they are membership comparable. But if $L \in Sk(M)$, then by Lemma 7.8(2),

$$L \in (f_r(x) \cap Sk(M)) = f_r(M) \subseteq f_w(M).$$

So $L \in f_w(M)$, which contradicts the choice of L . Therefore, either $L = M$, or $M \in Sk(L)$. In either case, $Sk(M) \subseteq Sk(L)$. As $K \in f_w(M)$, $K \in Sk(M)$, so $K \in Sk(L) \subseteq Sk(L)$. $\square$

The next lemma will be used to show that $f_w \oplus_N f_r$ satisfies requirement (3) of Definition 4.2 for $w \oplus_N r$.

Lemma 7.15. *Let $N \in \mathcal{X}$ be simple and $r \in D_N$. Suppose that $w \in N \cap \mathbb{P}$ and $w \leq r \restriction N$. Let $f := f_w \oplus_N f_r$. If $x \in \text{dom}(f)$, then $f(x) \subseteq \text{dom}(f)$, and for all $K \in f(x)$, $f(K) = f(x) \cap Sk(K)$.*

Proof. We begin by proving that $f(x) \subseteq \text{dom}(f)$. This is immediate in cases 1, 2, 4, 5, and 6 of Definition 7.11, since the fact that w and r are conditions implies that for any $y \in \text{dom}(f_w)$ and $z \in \text{dom}(f_r)$, $f_w(y) \subseteq \text{dom}(f_w) \subseteq \text{dom}(f)$ and $f_r(z) \subseteq \text{dom}(f_r) \subseteq \text{dom}(f)$. Also, case 7 follows from case 3. It remains to consider case 3.

In case 3, α is in $(\text{dom}(f_w) \cap S) \setminus \text{dom}(f_r)$ and $f(\alpha) = f_w(\alpha) \cup \{M \cap \alpha : M \in A_r, N \leq M, \alpha \in M\}$. Again, we know that $f_w(\alpha) \subseteq \text{dom}(f_w) \subseteq \text{dom}(f)$. And if $M \in A_r$, $N \leq M$, and $\alpha \in M$, then $M \cap \alpha$ is in $\text{dom}(f)$ by Definition 7.11.

Let $K \in f(x)$, and we will prove that $f(K) = f(x) \cap Sk(K)$. The proof splits into the seven cases of Definition 7.11 in the definition of $f(x)$. Note that since $K \in f(x)$, it follows that $K \notin S$ by Lemma 7.14.

Case 1: $x \in \text{dom}(f_w) \setminus S$ and $f(x) = f_w(x)$. Then $K \in f_w(x)$. So $K \in \text{dom}(f_w) \setminus S$ as well. Hence, $f(K) = f_w(K)$. Therefore,

$$f(K) = f_w(K) = f_w(x) \cap Sk(K) = f(x) \cap Sk(K).$$

Case 2: $x = \alpha \in \text{dom}(f_w) \cap S \cap \text{dom}(f_r)$ and $f(\alpha) = f_w(\alpha) \cup f_r(\alpha)$. So either $K \in f_w(\alpha)$ or $K \in f_r(\alpha)$.

First, assume that $K \notin N$. Then $K \in f_r(\alpha) \setminus N$. By Definition 4.2(6), $N \cap \alpha \in f_r(\alpha)$. So both K and $N \cap \alpha$ are in $f_r(\alpha)$. As $K \notin N$, $N \cap \alpha \in f_r(K) \cup \{K\}$. So we are in case 4 of Definition 7.11 in the definition of $f(K)$. Therefore, $f(K) = f_w(\alpha) \cup f_r(K)$. So it suffices to show that

$$f_w(\alpha) \cup f_r(K) = (f_w(\alpha) \cup f_r(\alpha)) \cap Sk(K).$$

Since $w \in N$, $f_w(\alpha) \subseteq N \cap Sk(\alpha) = Sk(N \cap \alpha)$ by Lemma 2.7(1). Since $N \cap \alpha \in f_r(K) \cup \{K\}$, $N \cap \alpha \subseteq K$, so $Sk(N \cap \alpha) \subseteq Sk(K)$. Hence, $f_w(\alpha) \subseteq Sk(K)$. Also, as $K \in f_r(\alpha)$, $f_r(K) = f_r(\alpha) \cap Sk(K)$. Thence,

$$\begin{aligned} (f_w(\alpha) \cup f_r(\alpha)) \cap Sk(K) &= \\ &= (f_w(\alpha) \cap Sk(K)) \cup (f_r(\alpha) \cap Sk(K)) = f_w(\alpha) \cup f_r(K). \end{aligned}$$

Secondly, assume that $K \in N$. Then either $K \in f_w(\alpha)$, or $K \in f_r(\alpha) \cap N \subseteq f_w(\alpha)$ by Lemma 7.8(2). In either case, $K \in f_w(\alpha)$. So $K \in \text{dom}(f_w) \setminus S$, and therefore, by definition, $f(K) = f_w(K)$. So

$$f(K) = f_w(K) = f_w(\alpha) \cap Sk(K).$$

Since K and w are N , and $f_r(\alpha) \cap N \subseteq f_w(\alpha)$ by Lemma 7.8(2), we have that

$$\begin{aligned} f(\alpha) \cap Sk(K) &\subseteq f(\alpha) \cap N = \\ &= (f_w(\alpha) \cup f_r(\alpha)) \cap N \subseteq f_w(\alpha) \cup (f_r(\alpha) \cap N) \subseteq f_w(\alpha). \end{aligned}$$

So $f(\alpha) \cap Sk(K) \subseteq f_w(\alpha)$. On the other hand, by Lemma 7.12(2), $f_w(\alpha) \subseteq f(\alpha)$. Therefore,

$$f(\alpha) \cap Sk(K) = f_w(\alpha) \cap Sk(K) = f(K).$$

Case 3: $x = \alpha \in (\text{dom}(f_w) \cap S) \setminus \text{dom}(f_r)$ and

$$f(\alpha) = f_w(\alpha) \cup \{M \cap \alpha : M \in A_r, N \leq M, \alpha \in M\}.$$

First, assume that $K \in f_w(\alpha)$. Then $f(K)$ is defined as in case 1 of Definition 7.11, so $f(K) = f_w(K)$. Since $K \in N$, and the members of the second set in the displayed union are not in N by Lemma 7.9, and hence not in $Sk(K)$, we have that

$$f(\alpha) \cap Sk(K) = f_w(\alpha) \cap Sk(K) = f_w(K) = f(K).$$

Secondly, assume that $K = M \cap \alpha$, where $M \in A_r$, $N \leq M$, and $\alpha \in M$. Then $f(K)$ is defined as in case 7 of Definition 7.11, namely,

$$f(K) = f(M \cap \alpha) = f(\alpha) \cap Sk(M \cap \alpha) = f(\alpha) \cap Sk(K).$$

Case 4: $x \in \text{dom}(f_r) \setminus N$, $\alpha \in \text{dom}(f_r) \cap S \cap N$, and $N \cap \alpha \in f_r(x) \cup \{x\}$. Then $f(x) = f_w(\alpha) \cup f_r(x)$. Note that by Definition 4.2(6), $N \cap \alpha \in f_r(\alpha)$.

First, assume that $K \in N$. Then either $K \in f_w(\alpha)$, or $K \in f_r(x) \cap N$. Consider the second case. Since K and $N \cap \alpha$ are both in $f_r(x) \cup \{x\}$ and $K \in N$, we must have that $K \in f_r(N \cap \alpha)$. Then $K \in f_r(N \cap \alpha) \subseteq f_r(\alpha)$. So $K \in f_r(\alpha) \cap N \subseteq f_w(\alpha)$ by Lemma 7.8(2). Hence, in either case, $K \in f_w(\alpha)$.

By Definition 7.11(1), $f(K) = f_w(K)$. Since $K \in f_w(\alpha)$, $f(K) = f_w(K) = f_w(\alpha) \cap Sk(K)$. Therefore,

$$\begin{aligned} f(x) \cap Sk(K) &= (f_w(\alpha) \cup f_r(x)) \cap Sk(K) = \\ &= (f_w(\alpha) \cap Sk(K)) \cup (f_r(x) \cap Sk(K)) = f_w(K) \cup (f_r(x) \cap Sk(K)). \end{aligned}$$

Since $f(K) = f_w(K)$, it suffices to show that $f_r(x) \cap Sk(K) \subseteq f_w(K)$, for then

$$f(x) \cap Sk(K) = f_w(K) \cup (f_r(x) \cap Sk(K)) = f_w(K) = f(K).$$

Let $z \in f_r(x) \cap Sk(K)$, and we will show that $z \in f_w(K)$. Since $K \in N$, $z \in N$. As $z \in f_r(x) \cap N$ and $N \cap \alpha \in f_r(x) \cup \{x\}$, we have that $z \in f_r(N \cap \alpha)$. So $z \in f_r(N \cap \alpha) \cap N \subseteq f_r(\alpha) \cap N \subseteq f_w(\alpha)$ by Lemma 7.8(2). So $z \in f_w(\alpha)$. Hence, $z \in f_w(\alpha) \cap Sk(K) = f_w(K)$, which completes the proof.

Secondly, assume that $K \notin N$. Then since $K \in f(x) = f_w(\alpha) \cup f_r(x)$, and $w \in N$, we must have that $K \in f_r(x)$. As $N \cap \alpha \in f_r(x) \cup \{x\}$ and $K \notin N$, it follows that $N \cap \alpha \in f_r(K) \cup \{K\}$. So we are in case 4 of Definition 7.11 in the definition of $f(K)$. That means that $f(K) = f_w(\alpha) \cup f_r(K)$. We have that

$$\begin{aligned} f(x) \cap Sk(K) &= (f_w(\alpha) \cup f_r(x)) \cap Sk(K) = \\ &= (f_w(\alpha) \cap Sk(K)) \cup (f_r(x) \cap Sk(K)) = (f_w(\alpha) \cap Sk(K)) \cup f_r(K). \end{aligned}$$

So $f(x) \cap Sk(K) = (f_w(\alpha) \cap Sk(K)) \cup f_r(K)$. On the other hand,

$$\begin{aligned} f(K) &= f(K) \cap Sk(K) = (f_w(\alpha) \cup f_r(K)) \cap Sk(K) = \\ &= (f_w(\alpha) \cap Sk(K)) \cup (f_r(K) \cap Sk(K)) = (f_w(\alpha) \cap Sk(K)) \cup f_r(K). \end{aligned}$$

So $f(K) = (f_w(\alpha) \cap Sk(K)) \cup f_r(K) = f(x) \cap Sk(K)$.

Case 5: $x \in \text{dom}(f_r) \setminus N$, case 4 fails for x , $f_r(x) \cap N = \emptyset$, and $f(x) = f_r(x)$. Then $K \in f_r(x)$. Since $f_r(x) \cap N = \emptyset$, $K \notin N$. Also, since $f_r(K) = f_r(x) \cap Sk(K)$, we have that $f_r(K) \cap N = \emptyset$. And as $f_r(K) \cup \{K\} \subseteq f_r(x)$, we cannot have that $N \cap \alpha \in f_r(K) \cup \{K\}$ for any $\alpha \in \text{dom}(f_r) \cap S \cap N$, since otherwise case 4 would be true for x . So we are in case 5 of Definition 7.11 in the definition of $f(K)$, and therefore $f(K) = f_r(K)$. So

$$f(K) = f_r(K) = f_r(x) \cap Sk(K) = f(x) \cap Sk(K).$$

Case 6: $x \in \text{dom}(f_r) \setminus N$, case 4 fails, M is the largest member of $f_r(x) \cap N$, and $f(x) = f_r(x) \cup f_w(M)$. By the maximality of M , and since $f_r(M) = f_r(x) \cap Sk(M)$, we have that

$$f_r(x) \cap N = f_r(M) \cup \{M\} \subseteq f_w(M) \cup \{M\},$$

where the inclusion holds by Lemma 7.8(2) and the fact that $f_r(M) = f_r(M) \cap N$. Therefore,

$$f(x) \cap N = (f_r(x) \cup f_w(M)) \cap N = f_w(M) \cup \{M\}.$$

Also, since $M \in N$, clearly $f_r(M) \setminus N = \emptyset$, and therefore,

$$f(x) \setminus N = (f_r(x) \cup f_w(M)) \setminus N = f_r(x) \setminus N.$$

To summarize,

$$f(x) \cap N = f_w(M) \cup \{M\}, \quad f(x) \setminus N = f_r(x) \setminus N.$$

We split into the two cases of whether $K \in N$ or $K \notin N$. First, assume that $K \in N$. Then $K \in f_w(M) \cup \{M\}$. So $f(K)$ is defined as in case 1 of Definition 7.11 and $f(K) = f_w(K)$. So it suffices to show that

$$f_w(K) = f(x) \cap Sk(K).$$

For the forward inclusion, $K \in f_w(M) \cup \{M\}$ implies that

$$f_w(K) \subseteq f_w(M) \subseteq f_r(x) \cup f_w(M) = f(x).$$

So $f_w(K) \subseteq f(x) \cap Sk(K)$. For the reverse inclusion, let $J \in f(x) \cap Sk(K)$, and we will show that $J \in f_w(K)$. Then

$$J \in f(x) \cap N = f_w(M) \cup \{M\}.$$

Since $K \in f_w(M) \cup \{M\}$ and $J \in Sk(K)$, clearly $J \neq M$. Therefore, $J \in f_w(M)$. Since w is a condition,

$$J \in f_w(M) \cap Sk(K) = f_w(K).$$

Secondly, assume that $K \notin N$. Then $K \in f(x) \setminus N = f_r(x) \setminus N$. Hence, $f_r(K) = f_r(x) \cap Sk(K)$, since r is a condition. As K and M are both in $f_r(x)$, they are membership comparable. But as $M \in N$ and $K \notin N$, clearly we must have that $M \in Sk(K)$. Since M is the membership largest element of $f_r(x) \cap N$ and $M \in f_r(K)$, M is the membership largest element of $f_r(K) \cap N$. Moreover, since $K \in f_r(x)$, $f_r(K) \cup \{K\} \subseteq f_r(x)$, so there is no $\alpha \in \text{dom}(f_r) \cap S \cap N$ with $N \cap \alpha \in f_r(K) \cup \{K\}$. So $f(K)$ is defined as in case 6 of Definition 7.11, which means that $f(K) = f_r(K) \cup f_w(M)$. So

$$\begin{aligned} f(K) &= f_r(K) \cup f_w(M) = (f_r(x) \cap Sk(K)) \cup f_w(M) = \\ &= (f_r(x) \cup f_w(M)) \cap (Sk(K) \cup f_w(M)) = f(x) \cap Sk(K), \end{aligned}$$

where the last equality follows from the definition of $f(x)$ and the fact that $f_w(M) \subseteq Sk(M) \subseteq Sk(K)$. So $f(K) = f(x) \cap Sk(K)$.

Case 7: $x = M \cap \alpha$, where $M \in A_r$, $N \leq M$, and $\alpha \in (M \cap \text{dom}(f_w) \cap S) \setminus \text{dom}(f_r)$. Then

$$f(x) = f(\alpha) \cap Sk(M \cap \alpha),$$

where $f(\alpha)$ is defined as in case 3 of Definition 7.11. So $K \in f(\alpha)$. Also, $x = M \cap \alpha \in f(\alpha)$ by the definition of $f(\alpha)$. By case 3 handled above,

$$f(K) = f(\alpha) \cap Sk(K), \quad f(x) = f(\alpha) \cap Sk(x).$$

We also know from Lemma 7.14 that $K \in f(x)$ implies that $K \in Sk(x)$, and hence $Sk(K) \subseteq Sk(x)$. Therefore, $Sk(x) \cap Sk(K) = Sk(K)$. Consequently,

$$f(K) = f(\alpha) \cap Sk(K) = f(\alpha) \cap Sk(x) \cap Sk(K) = f(x) \cap Sk(K).$$

□

This completes our analysis of $f_w \oplus_N f_r$. We now turn to amalgamating the g -components of w and r .

Definition 7.16. Let $N \in \mathcal{X}$ be simple and $r \in D_N$. Suppose that $w \in N \cap \mathbb{P}$ and $w \leq r \restriction N$. Let $f := f_w \oplus_N f_r$. Define $g_w \oplus_N g_r$ as the function g with domain equal to the set of pairs (K, x) such that $K \in f(x)$, such that for all $(K, x) \in \text{dom}(g)$,

$$g(K, x) := \bigcup \{g_w(K, y) \cup g_r(K, y) : x = y, \text{ or } x \in f(y)\}^5$$

The next lemma will be used to show that $g_w \oplus_N g_r$ satisfies requirement (4) of Definition 4.2 for $w \oplus_N r$.

Lemma 7.17. Let $N \in \mathcal{X}$ be simple and $r \in D_N$. Suppose that $w \in N \cap \mathbb{P}$ and $w \leq r \restriction N$. Let $g := g_w \oplus_N g_r$. Then for any (K, x) in $\text{dom}(g)$,

$$g(K, x) \subseteq x \setminus \text{sup}(K).$$

Proof. Let $(K, x) \in \text{dom}(g)$. Then $K \in f(x)$. By definition, we have that

$$g(K, x) = \bigcup \{g_w(K, y) \cup g_r(K, y) : x = y, \text{ or } x \in f(y)\}.$$

Since $g_w(K, x)$ and $g_r(K, x)$ are subsets of $x \setminus \text{sup}(K)$, it suffices to show that if $x \in f(y)$, then $g_w(K, y)$ and $g_r(K, y)$ are subsets of $x \setminus \text{sup}(K)$. Since w and r are conditions, $g_w(K, y)$ and $g_r(K, y)$ are subsets of $y \setminus \text{sup}(K)$. Hence, it suffices to show that $g_w(K, y)$ and $g_r(K, y)$ are subsets of x .

Claim 1: The result holds if K , x , and y are all in N .

If K , x , and y are all in N , then $K \in f(x)$ and $x \in f(y)$ imply by Lemma 7.13(1,2) that $K \in f_w(x)$ and $x \in f_w(y)$. Since w is a condition, it follows that $g_w(K, y) \subseteq g_w(K, x) \subseteq x$. To show that $g_r(K, y) \subseteq x$, assume that $(K, y) \in \text{dom}(g_r)$, which means that $K \in f_r(y)$. Then by Lemma 7.8(3), $g_r(K, y) \subseteq g_w(K, y)$. But we just showed that $g_w(K, y) \subseteq x$, so we are done.

Claim 2: The result holds if K , x , and y are all in $\text{dom}(f_r)$.

If K , x , and y are all in $\text{dom}(f_r)$, then $K \in f(x)$ and $x \in f(y)$ imply by Lemma 7.13(3) that $K \in f_r(x)$ and $x \in f_r(y)$. Since r is a condition,

$$g_r(K, y) \subseteq g_r(K, x) \subseteq x.$$

To show that $g_w(K, y) \subseteq x$, assume that $(K, y) \in \text{dom}(g_w)$, which means that $K \in f_w(y)$. In particular, K and y are in N . So if x is also in N , then we are done by Claim 1. Assume that $x \notin N$.

Note that $y \in S$. For if $y \notin S$, then y is countable. So $y \in N$ implies that $Sk(y) \subseteq N$, and hence $x \in N$, which is false. Therefore, $y = \alpha$ for some $\alpha \in \text{dom}(f_r) \cap S$. Since $N \in A_r$ and $\alpha \in N$, $N \cap \alpha \in f_r(\alpha)$ by Definition 4.2(6). As $x \in f_r(\alpha)$, x and $N \cap \alpha$ are membership comparable. But $x \notin N$. Therefore, either $x = N \cap \alpha$ or $N \cap \alpha \in Sk(x)$. In either case, $N \cap \alpha \subseteq x$. As $w \in N$, we have that

$$g_w(K, y) = g_w(K, \alpha) \subseteq N \cap \alpha \subseteq x.$$

The rest of the proof splits up into the cases 1–7 of Definition 7.11 in the definition of $f(y)$.

⁵As in the previous section, we let $g_w(K, y)$ denote the empty set in the case that $(K, y) \notin \text{dom}(g_w)$, and similarly with g_r .

Case 1: $y \in \text{dom}(f_w) \setminus S$ and $f(y) = f_w(y)$. Then K , x , and y are in N . So we are done by Claim 1.

Case 2: $y = \alpha \in \text{dom}(f_w) \cap S \cap \text{dom}(f_r)$ and $f(y) = f_w(\alpha) \cup f_r(\alpha)$. Then $y \in N$. By Definition 4.2(6), $N \cap \alpha \in f_r(\alpha)$.

Assume that $(K, \alpha) \in \text{dom}(g_w)$, which means that $K \in f_w(\alpha)$. We will show that $g_w(K, \alpha) \subseteq x$. In particular, $K \in N$. If x is also in N , then we are done by Claim 1. So assume that $x \notin N$.

Since $x \in f(y) = f_w(\alpha) \cup f_r(\alpha)$ and $x \notin N$, we have that $x \in f_r(\alpha)$. So both $N \cap \alpha$ and x are in $f_r(\alpha)$. As $x \notin N$, either $x = N \cap \alpha$ or $N \cap \alpha \in Sk(x)$. In either case, $N \cap \alpha \subseteq x$. But $g_w(K, \alpha) \subseteq N \cap \alpha$, since $w \in N$. Therefore, $g_w(K, \alpha) \subseteq x$.

Assume that $(K, \alpha) \in \text{dom}(g_r)$, which means that $K \in f_r(\alpha)$. We will show that $g_r(K, \alpha) \subseteq x$. So K and α are both in $\text{dom}(f_r)$. If x is also in $\text{dom}(f_r)$, then we are done by Claim 2. So assume that $x \notin \text{dom}(f_r)$. But $x \in f(y) = f_w(\alpha) \cup f_r(\alpha)$, so $x \in f_w(\alpha)$. So $x \in N$. As $K \in Sk(x)$, also $K \in N$. So K , x , and α are all in N , and we are done by Claim 1.

Case 3: $y = \alpha \in (\text{dom}(f_w) \cap S) \setminus \text{dom}(f_r)$. Since $\alpha \notin \text{dom}(f_r)$, we have that $(K, \alpha) \notin \text{dom}(g_r)$. Therefore, $g_r(K, \alpha) = \emptyset \subseteq x$.

Assume that $(K, \alpha) \in \text{dom}(g_w)$, which means that $K \in f_w(\alpha)$. We will show that $g_w(K, \alpha) \subseteq x$. In particular, K and α are both in N . So if x is also in N , then we are done by Claim 1.

Suppose that $x \notin N$. Then by the definition of $f(\alpha)$, we have that $x = M \cap \alpha$, for some $M \in A_r$ with $N \leq M$ and $\alpha \in M$. Then $\alpha \in M \cap N \cap \kappa$, so $\alpha < \beta_{M,N}$ by Proposition 2.11. Since $N \leq M$, we have that $N \cap \alpha \subseteq M \cap \alpha = x$. As $w \in N$, it follows that $g_w(K, \alpha) \subseteq N \cap \alpha \subseteq x$.

Before handling cases 4–7, let us note that in each of these cases, $y \notin N$. This is immediate in cases 4, 5, and 6, and follows from Lemma 7.9 in case 7. Consequently, $(K, y) \notin \text{dom}(g_w)$, and hence $g_w(K, y) = \emptyset \subseteq x$. Thus, we only need to show that $g_r(K, y) \subseteq x$.

Assume that $(K, y) \in \text{dom}(g_r)$, which means that $K \in f_r(y)$. We will show that $g_r(K, y) \subseteq x$. If $x \in \text{dom}(f_r)$, then we are done by Claim 2. Hence, we may assume that $x \notin \text{dom}(f_r)$.

Case 4: For some $\alpha \in \text{dom}(f_r) \cap S \cap N$, $N \cap \alpha \in f_r(y) \cup \{y\}$, and $f(y) = f_w(\alpha) \cup f_r(y)$. By Definition 4.2(6), $N \cap \alpha \in f_r(\alpha)$. Since $x \notin \text{dom}(f_r)$, it follows that $x \in f_w(\alpha)$. In particular, $x \in N$, and therefore also $K \in N$.

By Lemma 7.12(2), since $\alpha \in \text{dom}(f_w)$, $f_w(\alpha) \subseteq f(\alpha)$. Hence, $x \in f(\alpha)$. We also know that $K \in f(x)$. Thus, by case 2 handled above,

$$g_r(K, \alpha) \subseteq x.$$

Since $K \in f_r(y)$, $N \cap \alpha \in f_r(y) \cup \{y\}$, and $K \in N$, we have that $K \in f_r(N \cap \alpha)$. Therefore, $K \in f_r(\alpha)$. By Definition 4.2(5), it follows that

$$g_r(K, y) \subseteq g_r(K, N \cap \alpha).$$

Since $r \in D_N$, by Definition 7.3(3) we have that

$$g_r(K, N \cap \alpha) \subseteq g_r(K, \alpha).$$

Putting it all together,

$$g_r(K, y) \subseteq g_r(K, N \cap \alpha) \subseteq g_r(K, \alpha) \subseteq x.$$

Case 5: $y \in \text{dom}(f_r) \setminus N$, $f_r(y) \cap N = \emptyset$, and $f(y) = f_r(y)$. Since $x \in f(y) = f_r(y)$, we have that $x \in \text{dom}(f_r)$. But this contradicts the fact that $x \notin \text{dom}(f_r)$.

Case 6: $y \in \text{dom}(f_r) \setminus N$, and $f(y) = f_r(y) \cup f_w(M)$, where M is the membership largest element of $f_r(y) \cap N$. So either $x \in f_r(y)$ or $x \in f_w(M)$. Since $x \notin \text{dom}(f_r)$, we have that $x \in f_w(M)$. Then $x \in Sk(M)$, and thus $K \in Sk(M)$.

Since K and M are in $f_r(y)$ and $K \in Sk(M)$, it follows that $K \in f_r(M)$. Since r is a condition, $g_r(K, y) \subseteq g_r(K, M)$. Now $x \in f_w(M)$, and $f(M) = f_w(M)$ by Definition 7.11(1). Thus, $K \in f(x)$ and $x \in f(M)$. By case 1 handled above, $g_r(K, M) \subseteq x$. Thus, $g_r(K, y) \subseteq g_r(K, M) \subseteq x$.

Case 7: $y = M \cap \alpha$, where $M \in A_r$, $N \leq M$, and $\alpha \in (M \cap \text{dom}(f_w) \cap S) \setminus \text{dom}(f_r)$. By Lemma 7.9, y is not in $\text{dom}(f_r)$. But we assumed that $K \in f_r(y)$, so we have a contradiction. Thus, this case does not occur. $\square$

We are ready to define the amalgam $w \oplus_N r$.

Definition 7.18. Let $N \in \mathcal{X}$ be simple and $r \in D_N$. Suppose that $w \in N \cap \mathbb{P}$ and $w \leq r \restriction N$. Define $w \oplus_N r$ as the object (f, g, A) satisfying:

- (1) $f := f_w \oplus_N f_r$;
- (2) $g := g_w \oplus_N g_r$;
- (3) $A := A_w \cup A_r$.

We will now show that $w \oplus_N r$ is a condition which is below w and r . We have completed most of the work for this proof in the preceding lemmas.

Proposition 7.19. Let $N \in \mathcal{X}$ be simple and $r \in D_N$. Then for all $w \leq r \restriction N$ in $N \cap \mathbb{P}$, w and r are compatible. In fact, $w \oplus_N r$ is in $\mathbb{P}$ and $w \oplus_N r \leq w, r$.

Proof. Let $w \oplus_N r = (f, g, A)$. We will prove that $w \oplus_N r$ is a condition which is below w and r . To show that $w \oplus_N r$ is a condition, we verify requirements (1)–(7) of Definition 4.2.

(1) We apply Proposition 2.25. Since $r \in D_N$, we have that for all $M \in A_r$, if $M < N$ then $M \cap N \in A_r$. Also, A_w is adequate, and by Lemma 7.8(1),

$$A_r \cap N \subseteq A_w \subseteq N.$$

By Proposition 2.25, $A_w \cup A_r = A$ is adequate.

(2,3) These statements are immediate from Lemmas 7.14 and 7.15.

(4) By Definition 7.16, g is a function whose domain is the set of pairs (K, x) such that $K \in f(x)$. And by Lemma 7.17, for all $(K, x) \in \text{dom}(g)$, $g(K, x) \subseteq x \setminus \text{sup}(K)$. Also, $g(K, x)$ is finite by Definition 7.16.

(5) Let $K \in f(L)$ and $L \in f(x)$. We will show that $g(K, x) \subseteq g(K, L)$. So let $\xi \in g(K, x)$. Then by Definition 7.16, either $\xi \in g_w(K, x) \cup g_r(K, x)$, or for some y with $x \in f(y)$, $\xi \in g_w(K, y) \cup g_r(K, y)$. In the second case, $L \in f(x)$ and $x \in f(y)$ imply by requirement (3) that $L \in f(y)$. So letting $z := x$ in the first case, and

$z := y$ in the second case, we have that $L \in f(z)$ and $\xi \in g_w(K, z) \cup g_r(K, z)$. By Definition 7.16, it follows that $\xi \in g(K, L)$.

(6) Let $\alpha \in \text{dom}(f) \cap S$ and $M \in A$ with $\alpha \in M$. We will show that $M \cap \alpha \in f(\alpha)$. By the definition of $\text{dom}(f)$ given in Definition 7.11, clearly $\alpha \in \text{dom}(f_w) \cup \text{dom}(f_r)$. Also, $A = A_w \cup A_r$, so either $M \in A_w$ or $M \in A_r$.

First, assume that $\alpha \in \text{dom}(f_r)$. If $M \in A_r$, then since r is a condition, $M \cap \alpha \in f_r(\alpha) \subseteq f(\alpha)$. Suppose that $M \in A_w$. Then $\alpha \in M \in N$, so $\alpha \in \text{dom}(f_r) \cap N \subseteq \text{dom}(f_w)$ by Lemma 7.8(2). Since w is a condition, $M \cap \alpha \in f_w(\alpha) \subseteq f(\alpha)$.

Secondly, assume that $\alpha \in \text{dom}(f_w) \setminus \text{dom}(f_r)$. If $M \in A_w$, then since w is a condition, $M \cap \alpha \in f_w(\alpha) \subseteq f(\alpha)$. Suppose that $M \in A_r$. If $N \leq M$, then $M \cap \alpha \in f(\alpha)$ by Definition 7.11(3).

Suppose that $M < N$. Then since $r \in D_N$, we have that $M \cap N \in A_r \cap N \subseteq A_w$ by Lemma 7.8(1). Since $\alpha \in \text{dom}(f_w)$ and $\alpha \in M \cap N$, it follows that $M \cap N \cap \alpha \in f_w(\alpha) \subseteq f(\alpha)$. Now $\alpha \in M \cap N \cap \kappa$ implies that $\alpha < \beta_{M,N}$ by Proposition 2.11. Since $M < N$, Lemma 2.15 implies that

$$M \cap \alpha = M \cap \beta_{M,N} \cap \alpha = M \cap N \cap \alpha.$$

Thus, $M \cap \alpha \in f(\alpha)$.

(7) As in (1) above, the assumptions of Proposition 3.5 hold for A_r and A_w . Therefore,

$$r^*(A) = r^*(A_w \cup A_r) = r^*(A_w) \cup r^*(A_r).$$

As w and r are conditions,

$$r^*(A_w) \cap S \subseteq \text{dom}(f_w), \quad r^*(A_r) \cap S \subseteq \text{dom}(f_r).$$

But $\text{dom}(f_w) \subseteq \text{dom}(f)$ and $\text{dom}(f_r) \subseteq \text{dom}(f)$. Hence,

$$r^*(A) \cap S = (r^*(A_w) \cap S) \cup (r^*(A_r) \cap S) \subseteq \text{dom}(f).$$

This completes the proof that $w \oplus_N r$ is a condition. Now we show that $w \oplus_N r \leq w, r$. First, we prove that $w \oplus_N r \leq w$ by verifying properties (a)–(d) of Definition 4.2 for w . (a) Since $A = A_w \cup A_r$, clearly $A_w \subseteq A$. (b) follows from Lemma 7.12(1,2). (c) is immediate from Definition 7.16, and (d) was proved in Lemma 7.13(2).

Secondly, we prove that $w \oplus_N r \leq r$ by verifying properties (a)–(d) of Definition 4.2 for r . (a) Since $A = A_w \cup A_r$, clearly $A_r \subseteq A$. (b) follows from Lemma 7.12(1,3). (c) is immediate from Definition 7.16, and (d) was proved in Lemma 7.13(3). $\square$

Corollary 7.20. *The forcing poset $\mathbb{P}$ is strongly proper on a stationary set. In particular, it preserves ω_1 .*

Proof. By Assumption 2.22, the set of $N \in \mathcal{X}$ such that N is simple is stationary in $P_{\omega_1}(H(\lambda))$. So it suffices to show that for all simple $N \in \mathcal{X}$, for all $p \in N \cap \mathbb{P}$, there is $q \leq p$ such that q is strongly N -generic.

Let $p \in N \cap \mathbb{P}$. By Lemma 7.1, fix $q \leq p$ with $N \in A_q$. We claim that q is strongly N -generic. So let D be a dense subset of $N \cap \mathbb{P}$, and we will show that D is predense below q . Let $r \leq q$, and we will find $w \in D$ which is compatible with r .

Since $N \in A_r$, we can apply Lemma 7.4 and fix $s \leq r$ such that $s \in D_N$. Then by Lemma 7.6, $s \restriction N$ is in $N \cap \mathbb{P}$. As D is dense in $N \cap \mathbb{P}$, fix $w \leq s \restriction N$ in D . By

Proposition 7.19, w and s are compatible. Since $s \leq r$, it follows that w and r are compatible. $\square$

It follows from Corollaries 6.16 and 7.20 that $\mathbb{P}$ preserves ω_1 and κ . By the next lemma, no cardinal in between ω_1 and κ survives.

Lemma 7.21. *If μ is a cardinal and $\omega_1 < \mu < \kappa$, then $\mathbb{P}$ forces that μ is not a cardinal.*

Proof. Let G be a generic filter on $\mathbb{P}$. Define

$$A := \{M : \exists p \in G \ M \in A_p, \ \mu \in M\}.$$

Consider M and N in A . Then there is $p \in G$ with M and N in A_p . Since $\mu \in M \cap N \cap \kappa$, $\mu < \beta_{M,N}$ by Proposition 2.11. If $M \cap \omega_1 = N \cap \omega_1$, then $M \sim N$ by Lemma 2.17, and therefore, $M \cap \beta_{M,N} = N \cap \beta_{M,N}$. Since $\mu < \beta_{M,N}$, it follows that $M \cap \mu = N \cap \mu$. This proves that the map which sends a member of the set

$$A^* := \{M \cap \mu : M \in A\}$$

to its intersection with ω_1 is an injective function from A^* into ω_1 . Hence, in $V[G]$, A^* has size less than or equal to ω_1 .

A density argument using Lemma 7.1 shows that for all $\xi < \mu$, there is $N \in A$ with $\xi \in N$. It follows that $\bigcup A^* = \mu$. So in $V[G]$, μ is the union of a collection of countable sets of size at most ω_1 . This implies that in $V[G]$, μ has size at most ω_1 . Since $\omega_1 < \mu$, μ is not a cardinal in $V[G]$. $\square$

Corollary 7.22. *The forcing poset $\mathbb{P}$ forces that $\kappa = \omega_2$.*

Proof. Immediate from Corollaries 6.16 and 7.20 and Lemma 7.21. $\square$

§8. Further analysis

The goal of the next two sections is to prove that certain quotients of the forcing poset $\mathbb{P}$ satisfy the ω_1 -approximation property. This fact will follow from the equation

$$(q \oplus_N p) \restriction Q = (q \restriction Q) \oplus_{N \cap Q} (p \restriction Q),$$

which is proved in Proposition 8.6. Lemmas 8.1 and 8.2 provide some additional information about $\mathbb{P}$ which we will need to prove the approximation property in Section 9. Then Lemmas 8.3, 8.4, and 8.5 prepare us for proving Proposition 8.6. The information which we provide here will be used again in Part III to analyze products of the partial square forcing poset.

Lemma 8.1. *Let $N \in \mathcal{X}$ be simple and $r \in D_N$. Suppose that v and w are in $N \cap \mathbb{P}$ and*

$$w \leq v \leq r \restriction N.$$

Then $w \oplus_N r \leq v \oplus_N r$.

The proof of this lemma is straightforward, but due to the multitude of cases in Definition 7.11, it is also somewhat lengthy.

Proof. Let $s := v \oplus_N r$ and $t := w \oplus_N r$. We will prove that $t \leq s$. We verify properties (a)–(d) of Definition 4.2 for s and t .

(a) By Definition 7.18, $A_s = A_v \cup A_r$ and $A_t = A_w \cup A_r$. Since $w \leq v$, $A_v \subseteq A_w$. Therefore, $A_s \subseteq A_t$.

(b,d) The domain of f_s is equal to the union of $\text{dom}(f_v)$, $\text{dom}(f_r)$, and the set $\{M \cap \alpha : M \in A_r, N \leq M, \alpha \in (M \cap \text{dom}(f_v) \cap S) \setminus \text{dom}(f_r)\}$.

The domain of f_t is equal to the union of $\text{dom}(f_w)$, $\text{dom}(f_r)$, and the set

$$\{M \cap \alpha : M \in A_r, N \leq M, \alpha \in (M \cap \text{dom}(f_w) \cap S) \setminus \text{dom}(f_r)\}.$$

But $w \leq v$ implies that $\text{dom}(f_v) \subseteq \text{dom}(f_w)$. It is easy to check from the above definitions and the fact that $\text{dom}(f_v) \subseteq \text{dom}(f_w)$ that $\text{dom}(f_s) \subseteq \text{dom}(f_t)$.

Let $x \in \text{dom}(f_s)$, and we will prove that $f_s(x) \subseteq f_t(x)$. At the same time, we will also show that if K is in $\text{dom}(f_s)$ and $K \in f_t(x)$, then $K \in f_s(x)$. Note that these assertions imply (b) and (d). The proof will split into the seven cases of Definition 7.11 for how $f_s(x)$ is defined. First, we prove a claim.

Claim 1: If $K \in \text{dom}(f_s)$ and $K \in f_w(x)$, then $K \in f_s(x)$.

Since $K \in f_w(x)$, K and x are in N . By Lemma 7.13(1), K and x are in $\text{dom}(f_s) \cap N = \text{dom}(f_v)$. So K and x are in $\text{dom}(f_v)$ and $K \in f_w(x)$. Since $w \leq v$, it follows that $K \in f_v(x)$. But $s \leq v$ implies that $f_v(x) \subseteq f_s(x)$. So $K \in f_s(x)$.

Case 1: $x \in \text{dom}(f_v) \setminus S$ and $f_s(x) = f_v(x)$. Since $w \leq v$, $x \in \text{dom}(f_w) \setminus S$, and so $f_t(x) = f_w(x)$. But $w \leq v$ implies that $f_v(x) \subseteq f_w(x)$, hence $f_s(x) \subseteq f_t(x)$.

Suppose that $K \in \text{dom}(f_s)$ and $K \in f_t(x)$. Then $K \in f_t(x) = f_w(x)$. So $K \in f_s(x)$ by Claim 1.

Case 2: $x = \alpha \in \text{dom}(f_v) \cap S \cap \text{dom}(f_r)$ and $f_s(\alpha) = f_v(\alpha) \cup f_r(\alpha)$. Since $\text{dom}(f_v) \subseteq \text{dom}(f_w)$, $\alpha \in \text{dom}(f_w) \cap S \cap \text{dom}(f_r)$, so $f_t(\alpha) = f_w(\alpha) \cup f_r(\alpha)$. Also, $w \leq v$ implies that $f_v(\alpha) \subseteq f_w(\alpha)$. Therefore, $f_s(\alpha) \subseteq f_t(\alpha)$.

Suppose that $K \in \text{dom}(f_s)$ and $K \in f_t(\alpha)$. Then $K \in f_t(\alpha) = f_w(\alpha) \cup f_r(\alpha)$. So either $K \in f_w(\alpha)$ or $K \in f_r(\alpha)$. If $K \in f_w(\alpha)$, then $K \in f_s(\alpha)$ by Claim 1. If $K \in f_r(\alpha)$, then $K \in f_s(\alpha)$ by definition.

Case 3: $x = \alpha \in (\text{dom}(f_v) \cap S) \setminus \text{dom}(f_r)$ and

$$f_s(\alpha) = f_v(\alpha) \cup \{M \cap \alpha : M \in A_r, N \leq M, \alpha \in M\}.$$

Since $\text{dom}(f_v) \subseteq \text{dom}(f_w)$, $\alpha \in (\text{dom}(f_w) \cap S) \setminus \text{dom}(f_r)$, and therefore

$$f_t(\alpha) = f_w(\alpha) \cup \{M \cap \alpha : M \in A_r, N \leq M, \alpha \in M\}.$$

Since $f_v(\alpha) \subseteq f_w(\alpha)$, it follows that $f_s(\alpha) \subseteq f_t(\alpha)$.

Suppose that $K \in \text{dom}(f_s)$ and $K \in f_t(\alpha)$. Then $K \in f_t(\alpha) = f_w(\alpha) \cup \{M \cap \alpha : M \in A_r, N \leq M, \alpha \in M\}$. If K is in the second set of this union, then $K \in f_s(\alpha)$ by definition. If $K \in f_w(\alpha)$, then $K \in f_s(\alpha)$ by Claim 1.

Case 4: $x \in \text{dom}(f_r) \setminus N$, and for some $\alpha \in \text{dom}(f_r) \cap S \cap N$, $N \cap \alpha \in f_r(x) \cup \{x\}$. Then $f_s(x) = f_v(\alpha) \cup f_r(x)$. Clearly we are also in case 4 in the definition of $f_t(x)$. So $f_t(x) = f_w(\alpha) \cup f_r(x)$. Since $w \leq v$, $f_v(\alpha) \subseteq f_w(\alpha)$. Therefore, $f_s(x) \subseteq f_t(x)$.

Suppose that $K \in \text{dom}(f_s)$ and $K \in f_t(x)$. Then $K \in f_w(\alpha) \cup f_r(x)$. If $K \in f_r(x)$, then $K \in f_s(x)$ by definition. Suppose that $K \in f_w(\alpha)$. Then K is in N . So K and α are in $\text{dom}(f_s) \cap N = \text{dom}(f_v)$ by Lemma 7.13(1). Since $K \in f_w(\alpha)$ and $w \leq v$, it follows that $K \in f_v(\alpha)$. So $K \in f_s(x)$ by definition.

Case 5: $x \in \text{dom}(f_r) \setminus N$, case 4 is false, and $f_r(x) \cap N = \emptyset$. Then clearly we are also in case 5 in the definition of $f_t(x)$. So by definition, $f_s(x)$ and $f_t(x)$ are both equal to $f_r(x)$. In particular, $f_s(x) \subseteq f_t(x)$. Also, if $K \in \text{dom}(f_s)$ and $K \in f_t(x)$, then $K \in f_t(x) = f_s(x)$.

Case 6: $x \in \text{dom}(f_r) \setminus N$, case 4 is false, and $f_s(x) = f_r(x) \cup f_v(M)$, where M is the membership largest element of $f_r(x) \cap N$. Then we are obviously also in case 6 in the definition of f_t . So $f_t(x) = f_r(x) \cup f_w(M)$. Since $w \leq v$, we have that $f_v(M) \subseteq f_w(M)$. Therefore, $f_s(x) \subseteq f_t(x)$.

Suppose that $K \in \text{dom}(f_s)$ and $K \in f_t(x)$. Then $K \in f_t(x) = f_r(x) \cup f_w(M)$. If $K \in f_r(x)$, then $K \in f_s(x)$ by definition. Otherwise, $K \in f_w(M)$. Since $M \in \text{dom}(f_r) \subseteq \text{dom}(f_s)$, Claim 1 implies that $K \in f_s(M)$. But $M \in f_r(x) \subseteq f_s(x)$. So $K \in f_s(M)$ and $M \in f_s(x)$. Since s is a condition, $K \in f_s(x)$.

Case 7: $x = M \cap \alpha$, where $M \in A_r$, $N \leq M$, and $\alpha \in (M \cap \text{dom}(f_v) \cap S) \setminus \text{dom}(f_r)$. Then

$$f_s(M \cap \alpha) = f_s(\alpha) \cap Sk(M \cap \alpha).$$

Since $w \leq v$, $\alpha \in (M \cap \text{dom}(f_w) \cap S) \setminus \text{dom}(f_r)$. So we are also in case 7 in the definition of $f_t(x)$. Therefore,

$$f_t(M \cap \alpha) = f_t(\alpha) \cap Sk(M \cap \alpha).$$

Since $f_s(\alpha) \subseteq f_t(\alpha)$ by case 3 handled above, it follows that $f_s(M \cap \alpha) \subseteq f_t(M \cap \alpha)$.

Suppose that $K \in \text{dom}(f_s)$ and $K \in f_t(M \cap \alpha)$. Then by the definition of $f_t(M \cap \alpha)$, $K \in f_t(\alpha)$ and $K \in Sk(M \cap \alpha)$. Since $\alpha \in \text{dom}(f_v)$ and $s \leq v$, $\alpha \in \text{dom}(f_s)$. By case 3 handled above, it follows that $K \in f_s(\alpha)$. So $K \in f_s(\alpha) \cap Sk(M \cap \alpha) = f_s(M \cap \alpha)$.

This completes the proof of (b) and (d).

(c) Let $(K, x) \in \text{dom}(g_s)$, and we will show that $g_s(K, x) \subseteq g_t(K, x)$. By Definition 7.16, $K \in f_s(x)$ and

$$g_s(K, x) = \bigcup \{g_v(K, y) \cup g_r(K, y) : x = y, \text{ or } x \in f_s(y)\}.$$

Since $f_s(x) \subseteq f_t(x)$ as just shown, $K \in f_t(x)$, and by Definition 7.16,

$$g_t(K, x) = \bigcup \{g_w(K, y) \cup g_r(K, y) : x = y, \text{ or } x \in f_t(y)\}.$$

Let $\xi \in g_s(K, x)$. To show that $\xi \in g_t(K, x)$, we will consider the different possibilities for why ξ is in $g_s(K, x)$.

Case 1: $\xi \in g_v(K, x)$. Then $(K, x) \in \text{dom}(g_v)$, which means that $K \in f_v(x)$. Since $w \leq v$, $K \in f_w(x)$ and $g_v(K, x) \subseteq g_w(K, x)$. So $\xi \in g_w(K, x) \subseteq g_t(K, x)$.

Case 2: $\xi \in g_v(K, y)$, where $x \in f_s(y)$. Then $(K, y) \in \text{dom}(g_v)$, which means that $K \in f_v(y)$. Since $w \leq v$, $K \in f_w(y)$ and $g_v(K, y) \subseteq g_w(K, y)$. So $\xi \in g_w(K, y)$. But $x \in f_s(y) \subseteq f_t(y)$. So by definition, $g_w(K, y) \subseteq g_t(K, x)$. So $\xi \in g_t(K, x)$.

Case 3: $\xi \in g_r(K, x)$. Since $t \leq r$, it follows that $g_r(K, x) \subseteq g_t(K, x)$, so $\xi \in g_t(K, x)$.

Case 4: $\xi \in g_r(K, y)$, where $x \in f_s(y)$. By (b), $f_s(y) \subseteq f_t(y)$. So $x \in f_t(y)$. Since $\xi \in g_r(K, y)$ and $x \in f_t(y)$, it follows by definition that $\xi \in g_t(K, x)$. $\square$

Lemma 8.2. *Let $N \in \mathcal{X}$ and $Q \in \mathcal{Y}$. Let $p \in \mathbb{P}$, and suppose that $N \in A_p$. Then there is $s \leq p$ such that $s \in D_N \cap D_Q$.*

Proof. We construct s in several steps. We begin by applying Lemma 7.2 to find $q \leq p$ such that for all $M \in A_q$, if $M < N$ then $M \cap N \in A_q$. Next, we apply Lemma 6.1 to fix $r \leq q$ such that for all $M \in A_r$, $M \cap Q \in A_r$, and moreover, $A_r = A_q \cup \{M \cap Q : M \in A_q\}$.

We claim that for all $M \in A_r$, if $M < N$ then $M \cap N \in A_r$. This is certainly true if $M \in A_q$, so assume that $M = M_1 \cap Q$, where $M_1 \in A_q$. By Lemma 2.29, $M_1 \sim M_1 \cap Q = M$. Since $M < N$, it follows that $M_1 < N$ by Lemma 2.18. As $M_1 \in A_q$, we have that $M_1 \cap N \in A_q$ by the choice of q . Hence,

$$M \cap N = (M_1 \cap Q) \cap N = (M_1 \cap N) \cap Q.$$

But $M_1 \cap N \in A_q$ implies that $M \cap N = (M_1 \cap N) \cap Q \in A_r$ by the definition of A_r .

Now apply Lemma 4.9 to find $s \leq r$ such that $A_s = A_r$ and whenever $K \in f_s(x)$ and $x \in f_s(y)$, then $g_s(K, x) \subseteq g_s(K, y)$. Then $s \leq p$ and $s \in D_N \cap D_Q$. $\square$

The next three lemmas will prepare us for proving Proposition 8.6.

Lemma 8.3. *Let $N \in \mathcal{X}$ be simple, $Q \in \mathcal{Y} \cap N$ be simple, and $p \in D_N \cap D_Q$. Then $p \restriction N \in D_Q$ and $p \restriction Q \in D_{N \cap Q}$.*

Recall that by Lemma 2.30, $N \cap Q$ is simple.

Proof. We prove first that $p \restriction N \in D_Q$, which by Definition 6.2 means that for all $M \in A_{p \restriction N}$, $M \cap Q \in A_{p \restriction N}$. So let $M \in A_{p \restriction N}$. Then by Definition 7.5, $M \in A_{p \restriction N} = A_p \cap N$. As $p \in D_Q$, we have that $M \cap Q \in A_p$. And since M and Q are in N , $M \cap Q \in N$. Thus, $M \cap Q \in A_p \cap N = A_{p \restriction N}$.

Next, we prove that $p \restriction Q \in D_{N \cap Q}$. First, we show that $N \cap Q \in A_{p \restriction Q}$. Since $A_{p \restriction Q} = A_p \cap Q$ by Definition 6.4, we need to show that $N \cap Q \in A_p \cap Q$. Since Q is simple, $N \cap Q \in Q$. As $p \in D_N$, $N \in A_p$, and since $p \in D_Q$, it follows that $N \cap Q \in A_p$. So $N \cap Q \in A_p \cap Q$.

Secondly, let $M \in A_{p \restriction Q} = A_p \cap Q$ be such that $M < N \cap Q$, and we will show that

$$M \cap (N \cap Q) \in A_{p \restriction Q} = A_p \cap Q.$$

Since $M \cap N \in \mathcal{X}$ and Q is simple, we have that $M \cap N \cap Q \in Q$. By Lemma 2.29, $N \sim N \cap Q$. Since $M < N \cap Q$, Lemma 2.18 implies that $M < N$. As $p \in D_N$ and $M \in A_p$, it follows that $M \cap N \in A_p$. Since $p \in D_Q$, $M \cap N \cap Q \in A_p$. So $M \cap N \cap Q \in A_p \cap Q$, as required.

Thirdly, let $K \in f_{p \restriction Q}(x)$ and $x \in f_{p \restriction Q}(y)$, and we will show that $g_{p \restriction Q}(K, x) \subseteq g_{p \restriction Q}(K, y)$. By the definition of $p \restriction Q$, we have that $K \in f_p(x)$ and $x \in f_p(y)$. Since $p \in D_N$, $g_p(K, x) \subseteq g_p(K, y)$. As $g_p(K, x) = g_{p \restriction Q}(K, x)$ and $g_p(K, y) = g_{p \restriction Q}(K, y)$ by the definition of $p \restriction Q$, we are done. $\square$

Lemma 8.4. *Let $N \in \mathcal{X}$ be simple, $Q \in \mathcal{Y} \cap N$ be simple, and $p \in D_N \cap D_Q$. Then*

$$(p \restriction N) \restriction Q = (p \restriction Q) \restriction (N \cap Q).$$

Note that we needed Lemma 8.3 to see that $(p \restriction N) \restriction Q$ and $(p \restriction Q) \restriction (N \cap Q)$ are defined.

Proof. By Definitions 6.4 and 7.5, we have that

$$\begin{aligned} \text{dom}(f_{(p \restriction N) \restriction Q}) &= \text{dom}(f_{p \restriction N}) \cap Q = \text{dom}(f_p) \cap N \cap Q = \\ &= (\text{dom}(f_p) \cap Q) \cap (N \cap Q) = \text{dom}(f_{p \restriction Q}) \cap (N \cap Q) = \text{dom}(f_{(p \restriction Q) \restriction (N \cap Q)}). \end{aligned}$$

And for each $x \in \text{dom}(f_{(p \restriction N) \restriction Q})$,

$$f_{(p \restriction N) \restriction Q}(x) = f_{p \restriction N}(x) = f_p(x) \cap N = f_{p \restriction Q}(x) \cap N,$$

which, since $f_{p \restriction Q}(x) \subseteq Q$, is equal to

$$f_{p \restriction Q}(x) \cap (N \cap Q) = f_{(p \restriction Q) \restriction (N \cap Q)}(x).$$

Thus, $f_{(p \restriction N) \restriction Q} = f_{(p \restriction Q) \restriction (N \cap Q)}$.

Again by Definitions 6.4 and 7.5, we have that

$$\begin{aligned} \text{dom}(g_{(p \restriction N) \restriction Q}) &= \text{dom}(g_{p \restriction N}) \cap Q = \text{dom}(g_p) \cap N \cap Q = \\ &= (\text{dom}(g_p) \cap Q) \cap (N \cap Q) = \text{dom}(g_{p \restriction Q}) \cap (N \cap Q) = \text{dom}(g_{(p \restriction Q) \restriction (N \cap Q)}). \end{aligned}$$

And for each $(K, x) \in \text{dom}(g_{(p \restriction N) \restriction Q})$,

$$g_{(p \restriction N) \restriction Q}(K, x) = g_{p \restriction N}(K, x) = g_p(K, x) = g_{p \restriction Q}(K, x) = g_{(p \restriction Q) \restriction (N \cap Q)}(K, x).$$

This proves that $g_{(p \restriction N) \restriction Q} = g_{(p \restriction Q) \restriction (N \cap Q)}$.

Finally, by Definitions 6.4 and 7.5,

$$\begin{aligned} A_{(p \restriction N) \restriction Q} &= A_{p \restriction N} \cap Q = A_p \cap N \cap Q = \\ &= (A_p \cap Q) \cap (N \cap Q) = A_{p \restriction Q} \cap (N \cap Q) = A_{(p \restriction Q) \restriction (N \cap Q)}. \end{aligned}$$

□

Lemma 8.5. *Let $N \in \mathcal{X}$ be simple, $Q \in \mathcal{Y} \cap N$ be simple, and $p \in D_N \cap D_Q$. Suppose that $q \in N \cap D_Q$ and $q \leq p \restriction N$. Then:*

- (1) $q \oplus_N p$ is in D_Q ;
- (2) $q \restriction Q \in N \cap Q$ and

$$q \restriction Q \leq (p \restriction Q) \restriction (N \cap Q).$$

Proof. (1) Since $q \leq p \restriction N$, $q \oplus_N p$ is a condition which is below q and p . We claim that $q \oplus_N p$ is in D_Q , which means that for all $M \in A_{q \oplus_N p}$, $M \cap Q \in A_{q \oplus_N p}$. Now $A_{q \oplus_N p} = A_q \cup A_p$. So if $M \in A_{q \oplus_N p}$, then either $M \in A_q$ or $M \in A_p$. But q and p are both in D_Q , so in the first case, $M \cap Q \in A_q$, and in the second case, $M \cap Q \in A_p$. In either case, $M \cap Q \in A_q \cup A_p = A_{q \oplus_N p}$.

(2) Since q and Q are in N , $q \restriction Q \in N$. Also, $q \restriction Q \in Q$, so $q \restriction Q \in N \cap Q$. By Lemmas 8.3 and 8.4, $p \restriction N \in D_Q$ and

$$(p \restriction N) \restriction Q = (p \restriction Q) \restriction (N \cap Q).$$

Now $q \leq p \restriction N$, so by Lemma 6.6(3), we have that

$$q \restriction Q \leq (p \restriction N) \restriction Q = (p \restriction Q) \restriction (N \cap Q).$$

□

Proposition 8.6. *Let $N \in \mathcal{X}$ be simple, $Q \in \mathcal{Y} \cap N$ be simple, and suppose that $Q \cap \kappa \notin S$. Let $p \in D_N \cap D_Q$. Assume that $q \in N \cap D_Q$ and $q \leq p \restriction N$. Then*

$$(q \oplus_N p) \restriction Q = (q \restriction Q) \oplus_{N \cap Q} (p \restriction Q).$$

Note that Lemma 8.5 implies that $(q \oplus_N p) \restriction Q$ and $(q \restriction Q) \oplus_{N \cap Q} (p \restriction Q)$ are defined.

Proof. Let

$$s := (q \oplus_N p) \restriction Q$$

and

$$t := (q \restriction Q) \oplus_{N \cap Q} (p \restriction Q).$$

Our goal is to prove that $s = t$. The proof will be split into three steps, namely, showing that $A_s = A_t$, $f_s = f_t$, and $g_s = g_t$.

By Definitions 6.4 and 7.18, we have that

$$\begin{aligned} A_s &= A_{(q \oplus_N p) \restriction Q} = A_{q \oplus_N p} \cap Q = (A_q \cup A_p) \cap Q = \\ &= (A_q \cap Q) \cup (A_p \cap Q) = A_{q \restriction Q} \cup A_{p \restriction Q} = A_{(q \restriction Q) \oplus_{N \cap Q} (p \restriction Q)} = A_t. \end{aligned}$$

So $A_s = A_t$.

We begin the proof of $f_s = f_t$ by showing that $\text{dom}(f_t) \subseteq \text{dom}(f_s)$. So let $x \in \text{dom}(f_t)$, and we will show that $x \in \text{dom}(f_s)$. By Definition 7.11, either

- (a) $x \in \text{dom}(f_{q \restriction Q})$, or
- (b) $x \in \text{dom}(f_{p \restriction Q})$, or
- (c) $x = M \cap \alpha$, where $M \in A_{p \restriction Q}$, $N \cap Q \leq M$, and $\alpha \in (M \cap \text{dom}(f_{q \restriction Q}) \cap S) \setminus \text{dom}(f_{p \restriction Q})$.

By Definition 6.4, the domain of f_s is equal to $\text{dom}(f_{q \oplus_N p}) \cap Q$. So it suffices to show that $x \in \text{dom}(f_{q \oplus_N p})$ and $x \in Q$. We consider cases a, b, and c separately.

(a) $x \in \text{dom}(f_{q \restriction Q}) = \text{dom}(f_q) \cap Q$. So $x \in Q$. Also, $\text{dom}(f_q) \subseteq \text{dom}(f_{q \oplus_N p})$, so $x \in \text{dom}(f_{q \oplus_N p})$.

(b) $x \in \text{dom}(f_{p \restriction Q}) = \text{dom}(f_p) \cap Q$. Then $x \in Q$. Also, $\text{dom}(f_p) \subseteq \text{dom}(f_{q \oplus_N p})$, so $x \in \text{dom}(f_{q \oplus_N p})$.

(c) $x = M \cap \alpha$, where $M \in A_{p \restriction Q}$, $N \cap Q \leq M$, and $\alpha \in (M \cap \text{dom}(f_{q \restriction Q}) \cap S) \setminus \text{dom}(f_{p \restriction Q})$. Then $M \in A_{p \restriction Q} = A_p \cap Q$ and $\alpha \in \text{dom}(f_{q \restriction Q}) = \text{dom}(f_q) \cap Q$. In particular, M and α are in Q , so $M \cap \alpha = x$ is in Q .

It remains to show that $M \cap \alpha \in \text{dom}(f_{q \oplus_N p})$. By the definition of the domain of $f_{q \oplus_N p}$ in Definition 7.11, it suffices to show that $M \in A_p$, $N \leq M$, and $\alpha \in (M \cap \text{dom}(f_q) \cap S) \setminus \text{dom}(f_p)$. We know that $M \in A_p$ from the last paragraph. By Lemma 2.29, $N \sim N \cap Q$, and since $N \cap Q \leq M$, it follows by Lemma 2.18 that $N \leq M$. In the previous paragraph, we observed that $\alpha \in \text{dom}(f_q) \cap Q$. Also, by the choice of α , $\alpha \in M \cap S$ and $\alpha \notin \text{dom}(f_{p \restriction Q}) = \text{dom}(f_p) \cap Q$. Since $\alpha \in Q$, the latter statement implies that $\alpha \notin \text{dom}(f_p)$. To summarize, $\alpha \in (M \cap \text{dom}(f_q) \cap S) \setminus \text{dom}(f_p)$, as required.

This completes the proof that $\text{dom}(f_t) \subseteq \text{dom}(f_s)$. To finish the proof that $f_s = f_t$, we show that if $x \in \text{dom}(f_s)$, then $x \in \text{dom}(f_t)$ and $f_s(x) = f_t(x)$.

Let $x \in \text{dom}(f_s)$. Then

$$x \in \text{dom}(f_s) = \text{dom}(f_{q \oplus_N p}) \cap Q$$

and

$$f_s(x) = f_{(q \oplus_N p) \upharpoonright Q}(x) = f_{q \oplus_N p}(x).$$

Thus, it suffices to prove that $x \in \text{dom}(f_t)$ and

$$f_{q \oplus_N p}(x) = f_{(q \upharpoonright Q) \oplus_{N \cap Q} (p \upharpoonright Q)}(x).$$

The proof splits into the seven cases of Definition 7.11 for how $f_{q \oplus_N p}(x)$ is defined. We remind the reader that $x \in Q$, as just noted.

(1) $x \in \text{dom}(f_q) \setminus S$ and $f_s(x) = f_q(x)$. Then

$$x \in \text{dom}(f_q) \cap Q = \text{dom}(f_{q \upharpoonright Q}) \subseteq \text{dom}(f_t).$$

Also, $x \in \text{dom}(f_{q \upharpoonright Q}) \setminus S$ implies that $f_t(x)$ is defined as in case 1 of Definition 7.11. So $f_t(x) = f_{q \upharpoonright Q}(x) = f_q(x) = f_s(x)$.

(2) $x = \alpha \in \text{dom}(f_q) \cap S \cap \text{dom}(f_p)$ and $f_s(\alpha) = f_q(\alpha) \cup f_p(\alpha)$. Then $\alpha \in \text{dom}(f_q) \cap Q = \text{dom}(f_{q \upharpoonright Q}) \subseteq \text{dom}(f_t)$ and $\alpha \in \text{dom}(f_p) \cap Q = \text{dom}(f_{p \upharpoonright Q})$. Hence,

$$\alpha \in \text{dom}(f_{q \upharpoonright Q}) \cap S \cap \text{dom}(f_{p \upharpoonright Q}),$$

and therefore $f_t(\alpha)$ is defined as in case 2 of Definition 7.11. So

$$f_t(\alpha) = f_{q \upharpoonright Q}(\alpha) \cup f_{p \upharpoonright Q}(\alpha) = f_q(\alpha) \cup f_p(\alpha) = f_s(\alpha).$$

(3) $x = \alpha \in (\text{dom}(f_q) \cap S) \setminus \text{dom}(f_p)$ and

$$f_s(x) = f_q(\alpha) \cup \{M \cap \alpha : M \in A_p, N \leq M, \alpha \in M\}.$$

Then $\alpha \in \text{dom}(f_q) \cap Q = \text{dom}(f_{q \upharpoonright Q}) \subseteq \text{dom}(f_t)$. Also, $\alpha \notin \text{dom}(f_p)$ implies that $\alpha \notin \text{dom}(f_p) \cap Q = \text{dom}(f_{p \upharpoonright Q})$.

To summarize, we have that $\alpha \in (\text{dom}(f_{q \upharpoonright Q}) \cap S) \setminus \text{dom}(f_{p \upharpoonright Q})$, which means that we are in case 3 in the definition of $f_t(\alpha)$. So

$$f_t(\alpha) = f_{q \upharpoonright Q}(\alpha) \cup \{K \cap \alpha : K \in A_{p \upharpoonright Q}, N \cap Q \leq K, \alpha \in K\}.$$

Since $f_{q \upharpoonright Q}(\alpha) = f_q(\alpha)$, in order to show that $f_s(x) = f_t(x)$, the above equations imply that it suffices to show that

$$\begin{aligned} \{M \cap \alpha : M \in A_p, N \leq M, \alpha \in M\} &= \\ &= \{K \cap \alpha : K \in A_{p \upharpoonright Q}, N \cap Q \leq K, \alpha \in K\}. \end{aligned}$$

Let $K \cap \alpha$ be in the set on the right, where $K \in A_{p \upharpoonright Q}$, $N \cap Q \leq K$, and $\alpha \in K$. Then $K \in A_{p \upharpoonright Q} = A_p \cap Q$. By Lemma 2.29, $N \sim N \cap Q$. Since $N \cap Q \leq K$, Lemma 2.18 implies that $N \leq K$. So $K \in A_p$, $N \leq K$, and $\alpha \in K$. Thus, $K \cap \alpha$ is in the set on the left.

Conversely, let $M \cap \alpha$ be a member of the set on the left, where $M \in A_p$, $N \leq M$, and $\alpha \in M$. We will show that $M \cap \alpha$ is a member of the set on the right. It suffices to show that $M \cap \alpha$ is equal to $K \cap \alpha$, for some $K \in A_{p \upharpoonright Q}$ such that $N \cap Q \leq K$ and $\alpha \in K$.

Let $K := M \cap Q$. Since $p \in D_Q$ and $M \in A_p$, $M \cap Q \in A_p$. As Q is simple, $M \cap Q \in Q$. So $K = M \cap Q \in A_p \cap Q = A_{p \upharpoonright Q}$. By Lemma 2.29, $M \sim M \cap Q$ and $N \sim N \cap Q$. Since $N \leq M$, Lemma 2.18 implies that $N \cap Q \leq M \cap Q = K$. And $\alpha \in Q$, so $\alpha \in M \cap Q = K$. It remains to show that $M \cap \alpha = K \cap \alpha$. But $\alpha \in Q$ implies that $K \cap \alpha = M \cap Q \cap \alpha = M \cap \alpha$.

(4) $x \in \text{dom}(f_p) \setminus N$, and for some $\alpha \in \text{dom}(f_p) \cap S \cap N$, $N \cap \alpha \in f_p(x) \cup \{x\}$. Then $f_s(x) = f_q(\alpha) \cup f_p(x)$.

We claim that $\alpha \in Q$. Suppose for a contradiction that $\alpha \notin Q$. Since $\alpha \in S$ and $Q \cap \kappa \notin S$, it follows that $Q \cap \kappa < \alpha$. As $N \cap \alpha \in f_p(x) \cup \{x\}$, $N \cap \alpha$ is either equal to x or is in $f_p(x)$. In particular, since $x \in Q$, we have that

$$\sup(N \cap \alpha) \leq \sup(x) < Q \cap \kappa < \alpha.$$

So $\sup(N \cap \alpha) < Q \cap \kappa$. But $Q \in N$, so $Q \cap \kappa \in N \cap \alpha$, which contradicts that $\sup(N \cap \alpha) < Q \cap \kappa$. This completes the proof that $\alpha \in Q$.

Now we argue that we are in case 4 of Definition 7.11 for $f_t(x)$. We know that α and x are in $\text{dom}(f_p) \cap Q = \text{dom}(f_{p \upharpoonright Q})$. So $x \in \text{dom}(f_{p \upharpoonright Q}) \setminus (N \cap Q)$, $\alpha \in \text{dom}(f_{p \upharpoonright Q}) \cap S \cap (N \cap Q)$, and $N \cap \alpha \in f_p(x) \cup \{x\}$. Since $\alpha \in Q$, $N \cap \alpha = N \cap Q \cap \alpha$, so $N \cap Q \cap \alpha \in f_p(x) \cup \{x\} = f_{p \upharpoonright Q}(x) \cup \{x\}$. This proves that we are in case 4 in the definition of $f_t(x)$. So

$$f_t(x) = f_{q \upharpoonright Q}(\alpha) \cup f_{p \upharpoonright Q}(x) = f_q(\alpha) \cup f_p(x) = f_s(x).$$

(5,6) $x \in \text{dom}(f_p) \setminus N$ and case 4 fails. Then $x \in \text{dom}(f_p) \cap Q = \text{dom}(f_{p \upharpoonright Q})$, so $x \in \text{dom}(f_{p \upharpoonright Q}) \setminus (N \cap Q)$. In particular, $x \in \text{dom}(f_t)$.

We claim that case 4 of Definition 7.11 fails for $f_t(x)$. Suppose for a contradiction that for some $\alpha \in \text{dom}(f_{p \upharpoonright Q}) \cap S \cap (N \cap Q)$, we have that $(N \cap Q) \cap \alpha \in f_{p \upharpoonright Q}(x) \cup \{x\}$. Then $\alpha \in \text{dom}(f_{p \upharpoonright Q}) = \text{dom}(f_p) \cap Q$. Since $\alpha \in Q$, $N \cap Q \cap \alpha = N \cap \alpha$. So $\alpha \in \text{dom}(f_p) \cap S \cap N$ and $N \cap \alpha \in f_{p \upharpoonright Q}(x) \cup \{x\} = f_p(x) \cup \{x\}$. Hence, we are in case 4 of Definition 7.11 in the definition of $f_{q \oplus_{NP}}(x)$, which is a contradiction.

It follows that we are in either case 5 or 6 in the definition of $f_t(x)$. Assume that we are in case 5 in the definition of $f_{q \oplus_{NP}}(x)$. Then $f_p(x) \cap N = \emptyset$ and $f_s(x) = f_p(x)$. It follows that

$$f_{p \upharpoonright Q}(x) \cap (N \cap Q) = f_p(x) \cap (N \cap Q) \subseteq f_p(x) \cap N = \emptyset.$$

So we are also in case 5 in the definition of $f_t(x)$. Hence,

$$f_t(x) = f_{p \upharpoonright Q}(x) = f_p(x) = f_s(x).$$

Now assume that we are in case 6 in the definition of $f_{q \oplus_{NP}}(x)$. Then $f_s(x) = f_p(x) \cup f_q(M)$, where M is the membership largest element of $f_p(x) \cap N$. Since $M \in f_p(x)$, it follows that $M \in Sk(x)$. And since $x \in Q$, $M \in Q$. Thus, $M \in f_p(x) \cap Q = f_{p \upharpoonright Q}(x) \cap Q$. Since $M \in N$, M is in $f_{p \upharpoonright Q}(x) \cap (N \cap Q)$. In particular, $f_{p \upharpoonright Q}(x) \cap (N \cap Q)$ is nonempty, so we are in case 6 in the definition of $f_t(x)$.

We claim that M is the membership largest element of $f_{p \upharpoonright Q}(x) \cap (N \cap Q)$. Otherwise there is $M_1 \in f_{p \upharpoonright Q}(x) \cap (N \cap Q)$ such that $M \in Sk(M_1)$. But then

$$M_1 \in f_{p \upharpoonright Q}(x) \cap (N \cap Q) = f_p(x) \cap N \cap Q \subseteq f_p(x) \cap N.$$

Since M is the membership largest element of $f_p(x) \cap N$, this is a contradiction. By the definition of $f_t(x)$, we have that

$$f_t(x) = f_{p \upharpoonright Q}(x) \cup f_{q \upharpoonright Q}(M) = f_p(x) \cup f_q(M) = f_s(x).$$

(7) $x = M \cap \alpha$, where $M \in A_p$, $N \leq M$, and $\alpha \in (M \cap \text{dom}(f_q) \cap S) \setminus \text{dom}(f_p)$. Then $f_s(x) = f_s(\alpha) \cap Sk(M \cap \alpha)$.

We claim that $M \cap \alpha$ is equal to $K \cap \alpha$, for some $K \in A_{p \upharpoonright Q}$ such that $N \cap Q \leq K$ and $\alpha \in (K \cap \text{dom}(f_{q \upharpoonright Q}) \cap S) \setminus \text{dom}(f_{p \upharpoonright Q})$. Let $K := M \cap Q$. Since $M \in A_p$ and $p \in D_Q$, $M \cap Q \in A_p$. As Q is simple, $M \cap Q \in Q$. So $K = M \cap Q \in A_p \cap Q = A_{p \upharpoonright Q}$.

By Lemma 2.29, $M \sim M \cap Q$ and $N \sim N \cap Q$. Since $N \leq M$, it follows by Lemma 2.18 that $N \cap Q \leq M \cap Q = K$.

Suppose for a moment that $\alpha \in Q$. Then $\alpha \in M \cap Q = K$ and $\alpha \in \text{dom}(f_q) \cap Q = \text{dom}(f_q \upharpoonright Q)$. Also, since $\alpha \notin \text{dom}(f_p)$ and $\text{dom}(f_p \upharpoonright Q) = \text{dom}(f_p) \cap Q$, it follows that $\alpha \notin \text{dom}(f_p \upharpoonright Q)$. So $\alpha \in (K \cap \text{dom}(f_q \upharpoonright Q) \cap S) \setminus \text{dom}(f_p \upharpoonright Q)$. Finally, assuming that $\alpha \in Q$, we have that

$$K \cap \alpha = M \cap Q \cap \alpha = M \cap \alpha.$$

Thus, assuming that $\alpha \in Q$, we have shown that $K \cap \alpha = M \cap \alpha$ is in the domain of f_t and $f_t(K \cap \alpha)$ is defined as in case 7 of Definition 7.11. So $f_t(K \cap \alpha) = f_t(\alpha) \cap Sk(K \cap \alpha)$. By case 3 handled above, $f_s(\alpha) = f_t(\alpha)$. So

$$\begin{aligned} f_t(M \cap \alpha) &= f_t(K \cap \alpha) = f_t(\alpha) \cap Sk(K \cap \alpha) = \\ &= f_s(\alpha) \cap Sk(M \cap \alpha) = f_s(M \cap \alpha). \end{aligned}$$

It remains to prove that $\alpha \in Q$. Suppose for a contradiction that $\alpha \notin Q$. Since $p \in D_Q$ and $M \in A_p$, we have that $M \cap Q \in A_p$. By Lemma 2.29, $M \sim M \cap Q$. As $M \cap \alpha = x \in Q$ and $\alpha \notin Q$, clearly $\alpha = \min((M \cap \kappa) \setminus (Q \cap \kappa))$. By Lemma 3.6, we have that

$$\alpha = \min((M \cap \kappa) \setminus \beta_{M \cap Q, M}).$$

Now M and $M \cap Q$ are both in A_p and $M \sim M \cap Q$. Therefore, $\alpha \in r^*(A_p) \cap S$. Since p is a condition, $\alpha \in \text{dom}(f_p)$. But this contradicts our original choice of α .

This completes the proof that $f_s = f_t$.

Now we show that $g_s = g_t$. Since the domain of g_s is equal to the set of pairs (K, x) such that $K \in f_s(x)$, and the domain of g_t is equal to the set of pairs (K, x) such that $K \in f_t(x)$, the fact that $f_s = f_t$ implies that $\text{dom}(g_s) = \text{dom}(g_t)$.

Let $(K, x) \in \text{dom}(g_s)$, and we will show that $g_s(K, x) = g_t(K, x)$. So $K \in f_s(x) = f_{q \oplus_{Np}}(x)$. By Definition 6.4, we have that

$$g_s(K, x) = g_{(q \oplus_{Np}) \upharpoonright Q}(K, x) = g_{q \oplus_{Np}}(K, x).$$

Hence, by Definition 7.16,

$$g_s(K, x) = \bigcup \{g_q(K, y) \cup g_p(K, y) : x = y, \text{ or } x \in f_{q \oplus_{Np}}(y)\}.$$

Also, since $g_q \upharpoonright Q = g_q \upharpoonright Q$ and $g_p \upharpoonright Q = g_p \upharpoonright Q$, we have that

$$\begin{aligned} g_t(K, x) &= g_{(q \upharpoonright Q) \oplus_{N \cap Q} (p \upharpoonright Q)}(K, x) = \\ &= \bigcup \{g_q(K, y) \cup g_p(K, y) : x = y, \text{ or } x \in f_t(y)\}. \end{aligned}$$

Therefore, to show that $g_s(K, x) = g_t(K, x)$, it suffices to show that for any ordinal ξ , the following are equivalent:

- (1) $\xi \in g_q(K, y) \cup g_p(K, y)$, for some y such that either $x = y$ or $x \in f_{q \oplus_{Np}}(y)$;
- (2) $\xi \in g_q(K, y) \cup g_p(K, y)$, for some y such that either $x = y$ or $x \in f_t(y)$.

Obviously (1) and (2) are equivalent in the special case that $x = y$.

(2) implies (1): Suppose that $\xi \in g_q(K, y) \cup g_p(K, y)$, where $x \in f_t(y)$. Then

$$x \in f_t(y) = f_s(y) = f_{(q \oplus_{Np}) \upharpoonright Q}(y) = f_{q \oplus_{Np}}(y).$$

So $\xi \in g_q(K, y) \cup g_p(K, y)$, where $x \in f_{q \oplus_{Np}}(y)$. Hence, (1) holds.

(1) implies (2): Suppose that $\xi \in g_q(K, y) \cup g_p(K, y)$, where $x \in f_{q \oplus_N p}(y)$. If $y \in Q$, then

$$y \in \text{dom}(f_{q \oplus_N p}) \cap Q = \text{dom}(f_{(q \oplus_N p) \upharpoonright Q}) = \text{dom}(f_s)$$

and

$$x \in f_{q \oplus_N p}(y) = f_{(q \oplus_N p) \upharpoonright Q}(y) = f_s(y) = f_t(y).$$

Therefore, (2) holds.

The more difficult case is when y is not in Q . We split the proof into the two cases of whether ξ is in $g_q(K, y)$ or in $g_p(K, y)$.

Case 1: $\xi \in g_q(K, y)$. Then in particular, $g_q(K, y)$ is nonempty, which implies that $K \in f_q(y)$. So K and y are in $\text{dom}(f_q)$. Since $q \in N$, K and y are in N .

We claim that $x \in \text{dom}(f_q)$. Assume for a moment that this claim is true, and we finish the proof. We have that K , x , and y are in $\text{dom}(f_q)$, and also, $K \in f_s(x) = f_{q \oplus_N p}(x)$ and $x \in f_{q \oplus_N p}(y)$. Since $q \oplus_N p \leq q$, it follows that $K \in f_q(x)$ and $x \in f_q(y)$. As q is a condition, we have that $g_q(K, y) \subseteq g_q(K, x)$. Thus, $\xi \in g_q(K, x)$, which implies that (2) holds, as required.

Suppose for a contradiction that x is not in $\text{dom}(f_q)$. If $x \in N$, then $x \in \text{dom}(f_{q \oplus_N p}) \cap N = \text{dom}(f_q)$ by Lemma 7.13(1). So $x \notin N$. Since $x \in f_{q \oplus_N p}(y)$ and $y \in N$, the only way that x would not be in N is if $y \in S$.

So $y = \alpha$, for some $\alpha \in \text{dom}(f_q) \cap S$. Since $N \in A_p \subseteq A_{q \oplus_N p}$, we have that $N \cap \alpha \in f_{q \oplus_N p}(\alpha)$. And since $x \in f_{q \oplus_N p}(\alpha)$ and $x \notin N$, $N \cap \alpha$ is either equal to x or is in $f_{q \oplus_N p}(x)$. In particular, $\sup(N \cap \alpha) \leq \sup(x)$.

Since $s \in Q$ and $x \in \text{dom}(f_s)$, $x \in Q$. Hence, $\sup(x) < Q \cap \kappa$. Since $y = \alpha$ is not in Q , $Q \cap \kappa \leq \alpha$. But $\alpha \in S$ and $Q \cap \kappa \notin S$, so $Q \cap \kappa < \alpha$. Therefore,

$$\sup(N \cap \alpha) \leq \sup(x) < Q \cap \kappa < \alpha.$$

But $Q \in N$ implies that $Q \cap \kappa \in N \cap \alpha$, which contradicts that $\sup(N \cap \alpha) < Q \cap \kappa$.

Case 2: $\xi \in g_p(K, y)$. Then in particular, $g_p(K, y)$ is nonempty, which implies that $K \in f_p(y)$. So K and y are in $\text{dom}(f_p)$.

The easier case is when $x \in \text{dom}(f_p)$. Then since $K \in f_s(x) = f_{q \oplus_N p}(x)$, $x \in f_{q \oplus_N p}(y)$, and $q \oplus_N p \leq p$, it follows that $K \in f_p(x)$ and $x \in f_p(y)$. Since p is a condition, it follows that $g_p(K, y) \subseteq g_p(K, x)$. So $\xi \in g_p(K, x)$, and (2) holds.

Assume that $x \notin \text{dom}(f_p)$. Since $x \in \text{dom}(f_{q \oplus_N p})$, Definition 7.11 implies that either $x \in \text{dom}(f_q)$, or $x = M \cap \alpha$ for some $M \in A_p$ with $N \leq M$ and some $\alpha \in (M \cap \text{dom}(f_q) \cap S) \setminus \text{dom}(f_p)$.

Case 2a: $x \in \text{dom}(f_q)$. Then $x \in N$. Since $K \in f_s(x)$ and $x \notin S$, $K \in N$ as well. By Lemma 7.8(2), $K \in \text{dom}(f_p) \cap N \subseteq \text{dom}(f_q)$. So $K \in \text{dom}(f_q)$. Since K and x are in $\text{dom}(f_q)$, $K \in f_{q \oplus_N p}(x)$, and $q \oplus_N p \leq q$, it follows that $K \in f_q(x)$.

First, assume that $y \in N$. By Lemma 7.8(2), $y \in \text{dom}(f_p) \cap N \subseteq \text{dom}(f_q)$. So x and y are in $\text{dom}(f_q)$. Since $x \in f_{q \oplus_N p}(y)$ and $q \oplus_N p \leq q$, we have that $x \in f_q(y)$. So $K \in f_q(x)$ and $x \in f_q(y)$. Since q is a condition, $g_q(K, y) \subseteq g_q(K, x)$. On the other hand, by Lemma 7.8(3), $(K, y) \in \text{dom}(g_p) \cap N \subseteq \text{dom}(g_q)$ and $g_p(K, y) \subseteq g_q(K, y)$. Thus,

$$\xi \in g_p(K, y) \subseteq g_q(K, y) \subseteq g_q(K, x).$$

So (2) holds.

Secondly, assume that $y \notin N$. Let us consider cases 1–7 of Definition 7.11 in the definition of $f_{q \oplus_{Np}}(y)$. Since $y \notin N$, cases 1, 2, and 3 are false. As $K \in f_p(y) \cap N$, case 5 is false. Case 7 is false by Lemma 7.9, since $y \in \text{dom}(f_p)$. So we are left with cases 4 and 6.

In case 4, $y \in \text{dom}(f_p) \setminus N$, and $N \cap \alpha \in f_p(y) \cup \{y\}$ for some $\alpha \in \text{dom}(f_p) \cap S \cap N$. Then $f_{q \oplus_{Np}}(y) = f_q(\alpha) \cup f_p(y)$. Since $x \in f_{q \oplus_{Np}}(y)$ and $x \notin \text{dom}(f_p)$, it follows that $x \in f_q(\alpha)$.

Since $K \in f_p(y) \cap N$ and $N \cap \alpha \in f_p(y) \cup \{y\}$, we have that $K \in f_p(N \cap \alpha)$. Since p is a condition, $g_p(K, y) \subseteq g_p(K, N \cap \alpha)$. As $N \cap \alpha \in f_p(\alpha)$ and $p \in D_N$, it follows that $g_p(K, N \cap \alpha) \subseteq g_p(K, \alpha)$. So $g_p(K, y) \subseteq g_p(K, \alpha)$. Since K and α are in N , $g_p(K, \alpha) \subseteq g_q(K, \alpha)$ by Lemma 7.8(3). Now $K \in f_q(x)$ and $x \in f_q(\alpha)$, so $g_q(K, \alpha) \subseteq g_q(K, x)$ since q is a condition. Putting it all together,

$$\xi \in g_p(K, y) \subseteq g_p(K, \alpha) \subseteq g_q(K, \alpha) \subseteq g_q(K, x).$$

So (2) holds.

In case 6, we have that case 4 fails, and $f_{q \oplus_{Np}}(y) = f_p(y) \cup f_q(M)$, where M is the membership largest element of $f_p(y) \cap N$. Since $x \in f_{q \oplus_{Np}}(y)$ and $x \notin \text{dom}(f_p)$, we have that $x \in f_q(M)$. Also, $K \in f_p(y) \cap N$, so $K \in f_p(M) \cup \{M\}$. But $K \in Sk(x)$ and $x \in f_q(M)$, so $K \neq M$. Hence, $K \in f_p(M)$. As p is a condition, $g_p(K, y) \subseteq g_p(K, M)$. Also, since $K \in f_p(M)$ and K and M are in N , $K \in f_q(M)$ by Lemma 7.8(2). As $x \in f_q(M)$ and $K \in Sk(x)$, $K \in f_q(x)$. Since q is a condition, $g_q(K, M) \subseteq g_q(K, x)$. And by Lemma 7.8(3), $g_p(K, M) \subseteq g_q(K, M)$. Therefore,

$$\xi \in g_p(K, y) \subseteq g_p(K, M) \subseteq g_q(K, M) \subseteq g_q(K, x).$$

So (2) holds.

Case 2b: $x = M \cap \alpha$ for some $M \in A_p$ with $N \leq M$ and some $\alpha \in (M \cap \text{dom}(f_q) \cap S) \setminus \text{dom}(f_p)$. Note that $\alpha \in \text{dom}(f_q)$ implies that $\alpha \in N$. We will show, in fact, that this case is impossible.

We claim that $\alpha < Q \cap \kappa$. If not, then since $\alpha \in S$ and $Q \cap \kappa \notin S$, we have that $Q \cap \kappa < \alpha$. Now $M \cap \alpha = x \in Q$, so $\sup(x) < Q \cap \kappa$. And $\alpha \in M \cap N \cap \kappa$, so $\alpha < \beta_{M,N}$ by Proposition 2.11. Therefore, $Q \cap \kappa < \alpha < \beta_{M,N}$. Since $N \leq M$, we have that $N \cap \beta_{M,N} \subseteq M$. So $N \cap \alpha \subseteq M \cap \alpha$. Therefore,

$$\sup(N \cap \alpha) \leq \sup(M \cap \alpha) = \sup(x) < Q \cap \kappa < \alpha.$$

So $\sup(N \cap \alpha) < Q \cap \kappa$. Since $Q \in N$, we have that $Q \cap \kappa \in N \cap \alpha$, which contradicts that $\sup(N \cap \alpha) < Q \cap \kappa$.

So indeed, $\alpha < Q \cap \kappa$. Since $y \notin Q$, Lemma 4.3(4) implies that $Q \cap \kappa \leq \sup(y)$. Hence, $\alpha < \sup(y)$. But we know that $M \cap \alpha = x \in f_{q \oplus_{Np}}(y)$. And by Definition 7.11 and Lemma 4.6, the only value of y for which $M \cap \alpha$ is in $f_{q \oplus_{Np}}(y)$ is either $y = \alpha$ or $y = L \cap \alpha$ for some L . Neither of these cases is possible, since $\alpha < \sup(y)$. $\square$

§9. The approximation property

In this section we will prove that certain quotients of the forcing poset $\mathbb{P}$ have the ω_1 -approximation property.

Lemma 9.1. *Suppose that $Q \in \mathcal{Y}$ is simple and $Q \prec (H(\lambda), \in, \mathbb{P})$. Then $Q \cap \mathbb{P}$ is a regular suborder of $\mathbb{P}$.*

Proof. If p and q are in $Q \cap \mathbb{P}$ and are compatible in $\mathbb{P}$, then by the elementarity of Q , there is $r \in Q \cap \mathbb{P}$ such that $r \leq p, q$. So p and q are compatible in $Q \cap \mathbb{P}$.

Let B be a maximal antichain of $Q \cap \mathbb{P}$, and we will show that B is predense in $\mathbb{P}$. So let $p \in \mathbb{P}$, and we will find $s \in B$ which is compatible with p . Since the set D_Q is dense by Lemma 6.3, fix $q \leq p$ in D_Q . Then $q \restriction Q$ exists and is in $Q \cap \mathbb{P}$.

Since B is maximal, we can find $s \in B$ such that s and $q \restriction Q$ are compatible in $Q \cap \mathbb{P}$. Fix $w \leq s, q \restriction Q$ in $Q \cap \mathbb{P}$. By Proposition 6.15, w and q are compatible in $\mathbb{P}$. Fix $t \leq w, q$. Then $t \leq q \leq p$ and $t \leq w \leq s$. So p and s are compatible. $\square$

Theorem 9.2. *Suppose that $Q \in \mathcal{Y}$ is simple, $Q \prec (H(\lambda), \in, \mathbb{P})$, and $Q \cap \kappa \notin S$. Then $Q \cap \mathbb{P}$ forces that $\mathbb{P}/\dot{G}_{Q \cap \mathbb{P}}$ has the ω_1 -approximation property.⁶*

Proof. By Lemma 1.4, it suffices to show that $\mathbb{P}$ forces that the pair

$$(V[\dot{G}_{\mathbb{P}} \cap Q], V[\dot{G}_{\mathbb{P}}])$$

has the ω_1 -approximation property. So let p, μ , and $\dot{k}$ be given such that μ is an ordinal, and p forces in $\mathbb{P}$ that $\dot{k} : \mu \rightarrow On$ is a function satisfying that for any countable set a in $V[\dot{G}_{\mathbb{P}} \cap Q]$, $\dot{k} \restriction a \in V[\dot{G}_{\mathbb{P}} \cap Q]$. We will find an extension of p which forces that $\dot{k}$ is in $V[\dot{G}_{\mathbb{P}} \cap Q]$.

Fix a regular cardinal θ large enough so that $\mathbb{P}, \mu$, and $\dot{k}$ are members of $H(\theta)$. By the stationarity of the simple models in $\mathcal{X}$ as described in Assumption 2.22, we can fix a countable set $M^* \prec H(\theta)$ which contains the parameters $\mathbb{P}, Q, p, \mu, \dot{k}$, and satisfies that the set $M^* \cap H(\lambda)$ is in $\mathcal{X}$ and is simple.

Let $M := M^* \cap H(\lambda)$. Note that since $\mathbb{P} \subseteq H(\lambda)$, $M^* \cap \mathbb{P} = M \cap \mathbb{P}$. In particular, $p \in M \cap \mathbb{P}$. Also, observe that $Q \in M^* \cap H(\lambda) = M$.

By Lemma 7.1, we can fix $p_0 \leq p$ such that $M \in A_{p_0}$. Since $M^* \cap \mu$ is in V , by the choice of p and $\dot{k}$ we can fix $p_1 \leq p_0$ and a $(Q \cap \mathbb{P})$ -name $\dot{s}$ such that

$$p_1 \Vdash_{\mathbb{P}} \dot{k} \restriction (M^* \cap \mu) = \dot{s}^{\dot{G}_{\mathbb{P}} \cap Q}.$$

Since $M \in A_{p_1}$, apply Lemma 8.2 to fix $p_2 \leq p_1$ such that $p_2 \in D_M \cap D_Q$.

Note that since $p_2 \leq p$ and $p \in M$, it follows that $p_2 \restriction M \leq p$ by Lemma 7.7. So it will suffice to prove that $p_2 \restriction M$ forces that $\dot{k}$ is in $V[\dot{G}_{\mathbb{P}} \cap Q]$.

Claim 1: If $t \leq p_2$ is in D_Q , $\nu \in M^* \cap \mu$, and $t \Vdash_{\mathbb{P}} \dot{k}(\nu) = x$ (or $t \Vdash_{\mathbb{P}} \dot{k}(\nu) \neq x$, respectively) then $t \restriction Q \Vdash_{Q \cap \mathbb{P}} \dot{s}(\nu) = x$ (or $t \restriction Q \Vdash_{Q \cap \mathbb{P}} \dot{s}(\nu) \neq x$, respectively).

We will prove only the main part of Claim 1, since the parenthetical part has the essentially the same proof. So assume that $t \Vdash_{\mathbb{P}} \dot{k}(\nu) = x$. If $t \restriction Q \nVdash_{Q \cap \mathbb{P}} \dot{s}(\nu) = x$, then there is $u \leq t \restriction Q$ in $Q \cap \mathbb{P}$ such that $u \Vdash_{Q \cap \mathbb{P}} \dot{s}(\nu) \neq x$. By Proposition 6.15, u and t are compatible in $\mathbb{P}$. Fix $z \in \mathbb{P}$ such that $z \leq u, t$.

Fix a generic filter G on $\mathbb{P}$ with $z \in G$. Then $t \in G$, which implies that $\dot{k}^G(\nu) = x$. Also $p_1 \in G$, which implies that

$$\dot{k}^G \restriction (M^* \cap \mu) = \dot{s}^{G \cap Q}.$$

It follows that $\dot{s}^{G \cap Q}(\nu) = x$. But $z \leq u$, so $u \in G \cap Q$. By the choice of u , $\dot{s}^{G \cap Q}(\nu) \neq x$, which is a contradiction. This completes the proof of Claim 1.

⁶As discussed in the introduction of the paper, the proof of this result is similar to the proof of [8, Lemma 2.22], except that we replace the tidy condition property of [8, Definition 2.20] with Proposition 8.6.

Claim 2: For all $q \leq p_2 \restriction M$ in D_Q , $\nu < \mu$, and x ,

$$q \Vdash_{\mathbb{P}} \dot{k}(\nu) = x \implies \forall r \in \mathbb{P}((r \leq p_2 \restriction M \wedge r \leq q \restriction Q) \implies (r \Vdash_{\mathbb{P}} \dot{k}(\nu) = x)).$$

Note that $p_2 \restriction M$, Q , D_Q , μ , $\dot{k}$, and $\mathbb{P}$ are in M^* . So by the elementarity of M^* , it suffices to show that the statement holds in M^* .

Suppose for a contradiction that there exists $q \leq p_2 \restriction M$ in $M^* \cap D_Q$, $\nu \in M^* \cap \mu$, and $x \in M^*$ such that

$$q \Vdash_{\mathbb{P}} \dot{k}(\nu) = x,$$

but there is $r_0 \in M^* \cap \mathbb{P}$ with $r_0 \leq p_2 \restriction M$ and $r_0 \leq q \restriction Q$ such that

$$r_0 \nVdash_{\mathbb{P}} \dot{k}(\nu) = x.$$

By the elementarity of M^* , we can fix $r \leq r_0$ in $M^* \cap D_Q$ such that

$$r \Vdash_{\mathbb{P}} \dot{k}(\nu) \neq x.$$

Then $r \leq p_2 \restriction M$ and $r \leq q \restriction Q$. Since $r \leq q \restriction Q$, it follows that $r \restriction Q \leq q \restriction Q$ by Lemma 6.6(2).

We have that $q \leq p_2 \restriction M$ and $q \in M^* \cap \mathbb{P} = M \cap \mathbb{P}$. By Proposition 7.19, $q \oplus_M p_2$ is a condition which is below q and p_2 . Similarly, $r \leq p_2 \restriction M$ and $r \in M \cap \mathbb{P}$. By Proposition 7.19, $r \oplus_M p_2$ is a condition which is below r and p_2 .

By Proposition 8.6, we have that

$$(q \oplus_M p_2) \restriction Q = (q \restriction Q) \oplus_{M \cap Q} (p_2 \restriction Q),$$

and

$$(r \oplus_M p_2) \restriction Q = (r \restriction Q) \oplus_{M \cap Q} (p_2 \restriction Q).$$

We would like to apply Lemma 8.1 to $M \cap Q$, $p_2 \restriction Q$, $q \restriction Q$, and $r \restriction Q$. Let us check that the assumptions of Lemma 8.1 hold for these objects. By Lemma 2.30, $M \cap Q$ is simple. Since $p_2 \in D_M \cap D_Q$, it follows that $p_2 \restriction Q \in D_{M \cap Q}$ by Lemma 8.3. As q , r , and Q are in M^* , we have that $q \restriction Q$ and $r \restriction Q$ are in $M^* \cap Q = M \cap Q$. We observed above that $r \restriction Q \leq q \restriction Q$. Finally, $q \leq p_2 \restriction M$ implies that $q \restriction Q \leq (p_2 \restriction Q) \restriction (M \cap Q)$ by Lemma 8.5.

Thus, all of the assumptions of Lemma 8.1 hold. So by Lemma 8.1,

$$(r \restriction Q) \oplus_{M \cap Q} (p_2 \restriction Q) \leq (q \restriction Q) \oplus_{M \cap Q} (p_2 \restriction Q).$$

Combining this with the equalities above, we have that

$$(r \oplus_M p_2) \restriction Q \leq (q \oplus_M p_2) \restriction Q.$$

We claim that this last inequality is impossible. In fact, we will show that $(r \oplus_M p_2) \restriction Q$ and $(q \oplus_M p_2) \restriction Q$ are incompatible. This contradiction will complete the proof of Claim 2.

We know that $r \Vdash_{\mathbb{P}} \dot{k}(\nu) \neq x$, and therefore, since $r \oplus_M p_2 \leq r$, $r \oplus_M p_2 \nVdash_{\mathbb{P}} \dot{k}(\nu) \neq x$. By Lemma 8.5(1), $r \oplus_M p_2$ is in D_Q . So by Claim 1,

$$(r \oplus_M p_2) \restriction Q \Vdash_{Q \cap \mathbb{P}} \dot{s}(\nu) \neq x.$$

Similarly, $q \Vdash_{\mathbb{P}} \dot{k}(\nu) = x$, and therefore, since $q \oplus_M p_2 \leq q$, $q \oplus_M p_2 \Vdash_{\mathbb{P}} \dot{k}(\nu) = x$. By Lemma 8.5(1), $q \oplus_M p_2$ is in D_Q . So by Claim 1,

$$(q \oplus_M p_2) \restriction Q \Vdash_{Q \cap \mathbb{P}} \dot{s}(\nu) = x.$$

So indeed, $(r \oplus_M p_2) \restriction Q$ and $(q \oplus_M p_2) \restriction Q$ are incompatible, since they force contradictory information.

Now we finish the proof that $p_2 \restriction M$ forces that $\dot{k} \in V[\dot{G}_{\mathbb{P}} \cap Q]$. Let G be a generic filter on $\mathbb{P}$ with $p_2 \restriction M \in G$, and we will show that $\dot{k}^G \in V[G \cap Q]$.

In the model $V[G \cap Q]$, define a partial function $h : \mu \rightarrow V$ by letting, for every $\nu < \mu$, $h(\nu) = x$ iff there exists $t \in G \cap Q$ such that for every $r \in \mathbb{P}$, if r is below both $p_2 \restriction M$ and t , then

$$r \Vdash_{\mathbb{P}}^V \dot{k}(\nu) = x.$$

We claim that $h = \dot{k}^G$.

First, let us show that h is well-defined. So assume that t_0 and t_1 are in $G \cap Q$ and witness respectively that $h(\nu) = x_0$ and $h(\nu) = x_1$. We will show that $x_0 = x_1$. Fix $u \leq t_0, t_1$ in $G \cap Q$. Since $p_2 \restriction M$ is in G , we can fix $r \leq u, p_2 \restriction M$ in G . Now r is below both $p_2 \restriction M$ and t_0 , so by the choice of t_0 , $r \Vdash_{\mathbb{P}}^V \dot{k}(\nu) = x_0$. Similarly, r is below both $p_2 \restriction M$ and t_1 , so by the choice of t_1 , $r \Vdash_{\mathbb{P}}^V \dot{k}(\nu) = x_1$. Thus, $r \Vdash_{\mathbb{P}}^V x_0 = x_1$, which implies that $x_0 = x_1$.

Secondly, we prove that $h = \dot{k}^G$. As just shown, h is a well-defined function whose domain is a subset of μ . So it suffices to show that for all $\nu < \mu$, $\nu \in \text{dom}(h)$ and $h(\nu) = \dot{k}^G(\nu)$. Fix $\nu < \mu$, and let $x := \dot{k}^G(\nu)$. Fix $q \leq p_2 \restriction M$ in $G \cap D_Q$ such that $q \Vdash_{\mathbb{P}} \dot{k}(\nu) = x$. Then by Claim 2, for all $r \in \mathbb{P}$, if r is an extension of both $p_2 \restriction M$ and $q \restriction Q$, then $r \Vdash_{\mathbb{P}} \dot{k}(\nu) = x$.

Let $t := q \restriction Q$. By Lemma 6.5, $q \leq t$. Since $q \in G$, it follows that $t \in G \cap Q$. So by the definition of h , in order to show that $h(\nu) = x$, it suffices to show that for all $r \in \mathbb{P}$, if r is an extension of both $p_2 \restriction M$ and $t = q \restriction Q$, then $r \Vdash_{\mathbb{P}}^V \dot{k}(\nu) = x$. But this statement is exactly what we observed to be true at the end of the previous paragraph. $\square$

Recall that in Theorem 9.2 we assumed that $Q \cap \kappa \notin S$. Suppose, on the other hand, that $Q \in \mathcal{Y}$ is simple, $Q \prec (H(\lambda), \in, \mathbb{P})$, and $Q \cap \kappa \in S$. Let G be a generic filter on $\mathbb{P}$. Let $\langle c_\alpha : \alpha \in S \rangle$ be the partial square sequence in $V[G]$ as defined in Section 5. Since $Q \cap \kappa \in S$, we have that $c_{Q \cap \kappa}$ is defined. Using the coherence property of the partial square sequence, one can show that every proper initial segment of $c_{Q \cap \kappa}$ is in $V[G \cap Q]$. But by a density argument, $c_{Q \cap \kappa}$ is not in $V[G \cap Q]$. Thus, the quotient $\mathbb{P}/(G \cap Q)$ does not have the ω_1 -approximation property in $V[G \cap Q]$.

Part 3. Combining Forcings

§10. A product forcing

We now develop a forcing poset which simultaneously adds partial square sequences on multiple stationary subsets of κ . This forcing poset will be a kind of side condition product forcing. Before we get started, we need to make some additional assumptions.

Assumption 10.1. *The cardinal λ introduced in Notation 2.1 is at least κ^+ .*

Notation 10.2. *Fix an ordinal $\lambda^* \leq \lambda$. Fix a sequence $\langle S_i : i < \lambda^* \rangle$ such that for all $i < \lambda^*$, $S_i \subseteq \Lambda$ is stationary in κ and for all $\alpha \in S_i$, $Sk(\alpha) \cap \kappa = \alpha$; moreover, for all $i < j < \lambda^*$, $S_i \cap S_j$ is nonstationary.*

Assumption 10.3. For all $i < j < \lambda^*$, there is a club set $C_{i,j} \subseteq \kappa$ satisfying that $S_i \cap S_j \cap C_{i,j} = \emptyset$, and moreover, $C_{i,j}$ is definable in the structure $\mathcal{A}$ of Notation 2.3 from i and j .

Notation 10.4. For each $i < \lambda^*$, let $\mathbb{P}_i$ denote the forcing poset defined in Definition 4.2 which adds a partial square sequence on S_i .

We introduce a side condition product forcing which combines the forcing posets $\mathbb{P}_i$, for all $i < \lambda^*$. Before giving the definition, we need the following technical lemma to make sure that the definition makes sense.

Lemma 10.5. Suppose that M and N are in $\mathcal{X}$, $M \leq N$, and $\gamma = \min((M \cap \kappa) \setminus \beta_{M,N})$. Then there is at most one ordinal $i < \lambda^*$ such that $i \in M \cap N$ and $\gamma \in S_i$.

Proof. Suppose for a contradiction that there are $i < j$ in $M \cap N \cap \lambda^*$ such that $\gamma \in S_i \cap S_j$. By Assumption 10.3, $S_i \cap S_j \cap C_{i,j} = \emptyset$, and by the elementarity of M and N , $C_{i,j} \in M \cap N$. By the elementarity of $M \cap N$, it is easy to show that $C_{i,j}$ is cofinal in $M \cap N \cap \kappa$. Since $M \leq N$, by Lemma 2.15 and the minimality of γ ,

$$M \cap N \cap \kappa = M \cap \beta_{M,N} = M \cap \gamma.$$

Therefore, $C_{i,j}$ is cofinal in $M \cap \gamma$. Since $C_{i,j} \in M$, by the elementarity of M it follows that $C_{i,j}$ is cofinal in γ . As $C_{i,j}$ is a club, $\gamma \in C_{i,j}$. But then $\gamma \in S_i \cap S_j \cap C_{i,j}$, contradicting the choice of $C_{i,j}$. $\square$

Definition 10.6. Let $\mathbb{Q}$ be the forcing poset consisting of pairs $p = (F_p, A_p)$ satisfying:

- (1) A_p is an adequate set;
- (2) F_p is a function whose domain is a finite subset of λ^* ;
- (3) for all $i \in \text{dom}(F_p)$, $F_p(i) \in \mathbb{P}_i$ and

$$\{M \in A_p : i \in M\} \subseteq A_{F_p(i)};^7$$

- (4) if M and N are in A_p , $M \sim N$, $i \in M \cap N \cap \lambda^*$, and the ordinal $\min((M \cap \kappa) \setminus \beta_{M,N})$ exists and is in S_i , then $i \in \text{dom}(F_p)$.

Let $q \leq p$ if $A_p \subseteq A_q$, $\text{dom}(F_p) \subseteq \text{dom}(F_q)$, and for all $i \in \text{dom}(F_p)$, $F_q(i) \leq F_p(i)$ in $\mathbb{P}_i$.

Note that since A_p is finite, Lemma 10.5 implies that there are only finitely many ordinals i as described in (4).

Let us see that we can add any ordinal in λ^* to the domain of F_p .

Definition 10.7. Let $p \in \mathbb{Q}$, and let x be a finite subset of $\lambda^* \setminus \text{dom}(F_p)$. Define

$$p \uplus x$$

as the pair (F, A) satisfying:

- (1) $A := A_p$;
- (2) $\text{dom}(F) := \text{dom}(F_p) \cup x$;
- (3) for all $j \in \text{dom}(F_p)$, $F(j) := F_p(j)$, and for all $i \in x$,

$$F(i) := (\emptyset, \emptyset, B_i),$$

where

$$B_i := \{M \in A_p : i \in M\}.$$

⁷Recall from Definition 4.2 that $F_p(i) = (f_{F_p(i)}, g_{F_p(i)}, A_{F_p(i)})$.

Lemma 10.8. *Let $p \in \mathbb{Q}$, and let x be a finite subset of $\lambda^* \setminus \text{dom}(F_p)$. Then:*

- (1) $p \uplus x \in \mathbb{Q}$;
- (2) $p \uplus x \leq p$;
- (3) $x \subseteq \text{dom}(F_{p \uplus x})$;
- (4) *whenever $q \leq p$ and $x \subseteq \text{dom}(F_q)$, then $q \leq p \uplus x$.*

Proof. (3) is immediate.

(1) To see that $p \uplus x$ is a condition, requirements (1), (2), and (4) of Definition 10.6 are immediate. For requirement (3), for all $j \in \text{dom}(F_p)$, $F_{p \uplus x}(j) = F_p(j) \in \mathbb{P}_j$ and $\{M \in A_p : j \in M\} \subseteq A_{F_p(j)} = A_{F_{p \uplus x}(j)}$, since p is a condition. Consider $i \in x$. Then by definition, $F_{p \uplus x}(i) = (\emptyset, \emptyset, B_i)$, where $B_i = \{M \in A_p : i \in M\}$. Thus, it suffices to show that $(\emptyset, \emptyset, B_i) \in \mathbb{P}_i$.

We check that $(\emptyset, \emptyset, B_i)$ satisfies properties (1)–(7) of Definition 4.2. (1) B_i is adequate, because it is a subset of A_p . (2)–(6) are vacuously true. We claim that (7) is vacuously true as well.

Suppose that $\gamma \in r^*(B_i) \cap S_i$. Then for some M and N in B_i , $M \sim N$ and $\gamma = \min((M \cap \kappa) \setminus \beta_{M,N})$. By the definition of B_i , M and N are in A_p and $i \in M \cap N \cap \lambda^*$. So M and N are in A_p , $M \sim N$, $i \in M \cap N \cap \lambda^*$, and $\min((M \cap \kappa) \setminus \beta_{M,N})$ exists and is in S_i . Since p is a condition, Definition 10.6(4) implies that $i \in \text{dom}(F_p)$. But $i \in x$ and $x \cap \text{dom}(F_p) = \emptyset$, which is a contradiction.

(2) It is trivial to check that $p \uplus x \leq p$.

(4) Assume that $q \leq p$ and $x \subseteq \text{dom}(F_q)$, and we will show that $q \leq p \uplus x$. Since $q \leq p$, $A_{p \uplus x} = A_p \subseteq A_q$ and $\text{dom}(F_p) \subseteq \text{dom}(F_q)$. Since $x \subseteq \text{dom}(F_q)$, $\text{dom}(F_{p \uplus x}) = \text{dom}(F_p) \cup x \subseteq \text{dom}(F_q)$.

Let $j \in \text{dom}(F_{p \uplus x})$, and we will show that $F_q(j) \leq F_{p \uplus x}(j)$. If $j \in \text{dom}(F_p)$, then by definition, $F_{p \uplus x}(j) = F_p(j)$. And since $q \leq p$, $F_q(j) \leq F_p(j)$. So $F_q(j) \leq F_{p \uplus x}(j)$.

Suppose that $i \in x$. We claim that $F_q(i) \leq F_{p \uplus x}(i)$, that is, $F_q(i) \leq (\emptyset, \emptyset, B_i)$, where $B_i = \{M \in A_p : i \in M\}$. We verify properties (a)–(d) of Definition 4.2. Note that (b), (c), and (d) are vacuously true. For (a), since $i \in \text{dom}(F_q)$, by Definition 10.6(3) we have that $\{M \in A_q : i \in M\} \subseteq A_{F_q(i)}$. But $q \leq p$ implies that $A_p \subseteq A_q$. Hence, $B_i = \{M \in A_p : i \in M\} \subseteq \{M \in A_q : i \in M\} \subseteq A_{F_q(i)}$. $\square$

The next two easy lemmas will be useful in what follows.

Lemma 10.9. *Let $p \in \mathbb{Q}$. For each $i \in \text{dom}(F_p)$, suppose that r_i is a condition in $\mathbb{P}_i$ such that $r_i \leq F_p(i)$ in $\mathbb{P}_i$. Define r as follows:*

- (1) $A_r := A_p$;
- (2) $\text{dom}(F_r) := \text{dom}(F_p)$;
- (3) *for all $i \in \text{dom}(F_r)$, $F_r(i) := r_i$.*

Then $r \in \mathbb{Q}$ and $r \leq p$.

Proof. Straightforward. $\square$

Lemma 10.10. *Let x be a finite subset of λ^* , and assume that for each $i \in x$, D_i is a dense subset of $\mathbb{P}_i$. Then for any $p \in \mathbb{Q}$, there is $r \leq p$ satisfying:*

- (1) $A_r = A_p$;
- (2) $\text{dom}(F_r) = \text{dom}(F_p) \cup x$;

(3) for each $i \in x$, $F_r(i) \in D_i$.

Proof. Let $q := p \uplus (x \setminus \text{dom}(F_p))$. Then q is a condition, $A_q = A_p$, $\text{dom}(F_q) = \text{dom}(F_p) \cup x$, and $q \leq p$.

For each $i \in x$, fix $r_i \leq F_q(i)$ in D_i . By Lemma 10.9, there is $r \leq q$ such that $A_r = A_q = A_p$, $\text{dom}(F_r) = \text{dom}(F_q) = \text{dom}(F_p) \cup x$, and for all $i \in \text{dom}(F_q)$, if $i \in x$ then $F_r(i) = r_i$, and if $i \notin x$ then $F_r(i) = F_q(i)$. Then r is as required. $\square$

The next result justifies our informal use of the word “product” to describe $\mathbb{Q}$.

Proposition 10.11. *For each $i < \lambda^*$, there is a projection mapping from a dense subset of $\mathbb{Q}$ into $\mathbb{P}_i$.*

Proof. Let D be the set of conditions $q \in \mathbb{Q}$ such that $i \in \text{dom}(F_q)$, together with the maximum condition $(\emptyset, \emptyset)$. By Lemma 10.8, if $p \in \mathbb{Q}$, then there is $q \leq p$ such that $i \in \text{dom}(F_q)$. Thus, D is dense in $\mathbb{Q}$.

Define $\pi_i : D \rightarrow \mathbb{P}_i$ as follows. Let $\pi_i(\emptyset, \emptyset)$ be the maximum condition of $\mathbb{P}_i$, namely, $(\emptyset, \emptyset, \emptyset)$. If $q \in D$ and q is not the maximum condition, then $i \in \text{dom}(F_q)$. In that case, let $\pi_i(q) := F_q(i)$.

We claim that π_i is a projection mapping. Obviously, π_i maps the maximum condition of $\mathbb{Q}$ to the maximum condition of $\mathbb{P}_i$.

Suppose that $q \leq p$ in D , and we will show that $\pi_i(q) \leq \pi_i(p)$ in $\mathbb{P}_i$. This is immediate if p is the maximum condition of $\mathbb{Q}$, so assume that $i \in \text{dom}(F_p)$. Then since $q \leq p$, we have that $\pi_i(q) = F_q(i) \leq F_p(i) = \pi_i(p)$.

Suppose that $v \leq \pi_i(p)$ in $\mathbb{P}_i$, and we will find $r \leq p$ in D such that $\pi_i(r) = v$. First, assume that p is not the maximum condition of $\mathbb{Q}$. Then $v \leq \pi_i(p) = F_p(i)$. By Lemma 10.9, there exists $r \leq p$ satisfying that $F_r(i) = v$ and $F_r(j) = F_p(j)$ for all $j \in \text{dom}(F_p) \setminus \{i\}$. Then $\pi_i(r) = v$, as required.

Secondly, assume that p is the maximum condition of $\mathbb{Q}$. We construct a condition r as follows. Let $A_r := \emptyset$, and let F_r be the function with domain equal to $\{i\}$ such that $F_r(i) = v$. Since $A_r = \emptyset$, it is easy to check that r is a condition, with most properties of Definition 10.6 being vacuously true. Also, $r \leq p$, since p is the maximum condition, and $\pi_i(r) = F_r(i) = v$. $\square$

It follows that if G is a generic filter on $\mathbb{Q}$, then $\pi_i[G \cap D]$ generates a generic filter on $\mathbb{P}_i$, where D is the dense subset of $\mathbb{Q}$ which is the domain of π_i . We will prove in Section 11 that $\mathbb{Q}$ preserves ω_1 and is κ -c.c. It follows by Corollary 7.22 that $\mathbb{Q}$ collapses κ to become ω_2 . And by Proposition 5.4, $\mathbb{Q}$ adds a partial square sequence on S_i , for all $i < \lambda^*$. Since $\mathbb{Q}$ is κ -c.c., it also preserves the stationarity of S_i . Hence, for all $i < \lambda^*$, $\mathbb{Q}$ forces that S_i is a stationary subset of $\omega_2 \cap \text{cof}(\omega_1)$ in the approachability ideal $I[\omega_2]$. See Corollary 11.22 below for more details.

We conclude this section by introducing a set s^* which is analogous to the set r^* from Parts I and II.

Definition 10.12. *Let A be an adequate set. Define $s^*(A)$ as the set of $i < \lambda^*$ such that for some M and N in A , $M \sim N$, $i \in M \cap N$, and the ordinal $\min((M \cap \kappa) \setminus \beta_{M,N})$ exists and is in S_i .*

Note that requirement (4) of Definition 10.6 is equivalent to the statement that $s^*(A_p) \subseteq \text{dom}(F_p)$.

Observe that if $A \subseteq B$, then $s^*(A) \subseteq s^*(B)$.

The following is an analogue of Propositions 3.5 and 3.8 for s^* .

Proposition 10.13. *Let A be an adequate set, $N \in \mathcal{X} \cup \mathcal{Y}$ be simple, and suppose that:*

- (1) *if $N \in \mathcal{X}$, then $N \in A$ and for all $M \in A$, if $M < N$ then $M \cap N \in A$;*
- (2) *if $N \in \mathcal{Y}$, then for all $M \in A$, $M \cap N \in A$.*

Let B be an adequate set such that

$$A \cap N \subseteq B \subseteq N.$$

Then

$$s^*(A \cup B) = s^*(A) \cup s^*(B).$$

Proof. By Propositions 2.25 and 2.28, $A \cup B$ is adequate. The reverse inclusion is immediate. For the forward inclusion, let $K \in A$ and $M \in B$, and assume that $K \sim M$, $i \in K \cap M \cap \lambda^*$, γ is equal to either $\min((K \cap \kappa) \setminus \beta_{K,M})$ or $\min((M \cap \kappa) \setminus \beta_{K,M})$, and $\gamma \in S_i$. We will show that $i \in s^*(A) \cup s^*(B)$.

We claim that $K \cap N \in B$. First, assume that $N \in \mathcal{X}$. Since $M \in N$, $M \cap \omega_1 < N \cap \omega_1$. As $K \sim M$, $K \cap \omega_1 = M \cap \omega_1$ by Lemma 2.17(2). So $K \cap \omega_1 < N \cap \omega_1$, and hence $K < N$ by Lemma 2.17(1). By (1), $K \cap N \in A$. As N is simple, $K \cap N \in N$. So $K \cap N \in A \cap N \subseteq B$. Secondly, assume that $N \in \mathcal{Y}$. Then since $K \in A$, $K \cap N \in A$ by (2). As N is simple, $K \cap N \in N$. So $K \cap N \in A \cap N \subseteq B$.

Since $M \in N$, $i \in N$. So $i \in (K \cap N) \cap M$. Also note that by Lemmas 2.18, 2.26, and 2.29, $K \sim K \cap N \sim M$.

We consider the two possibilities for γ . Suppose that $\gamma = \min((K \cap \kappa) \setminus \beta_{K,M})$. Then by Lemma 3.4(3) in the case that $N \in \mathcal{X}$ and Lemma 3.7(3) in the case that $N \in \mathcal{Y}$, either $\gamma = \min((K \cap N \cap \kappa) \setminus \beta_{K \cap N, M})$, or $\gamma = \min((K \cap \kappa) \setminus \beta_{K, K \cap N})$. In the first case, since $\gamma \in S_i$, $i \in (K \cap N) \cap M$, and $K \cap N$ and M are in B , it follows that $i \in s^*(B)$. In second case, since $\gamma \in S_i$, $i \in (K \cap N) \cap K$, and $K \cap N$ and K are in A , we have that $i \in s^*(A)$.

Now suppose that $\gamma = \min((M \cap \kappa) \setminus \beta_{K,M})$. Then by Lemma 3.4(2) in the case that $N \in \mathcal{X}$ and Lemma 3.7(2) in the case that $N \in \mathcal{Y}$, $\gamma = \min((M \cap \kappa) \setminus \beta_{M, K \cap N})$. Since $\gamma \in S_i$, $i \in (K \cap N) \cap M$, and $K \cap N$ and M are in B , it follows that $i \in s^*(B)$. $\square$

§11. Amalgamation

In this section we will prove cardinal preservation results for $\mathbb{Q}$, namely, that $\mathbb{Q}$ is strongly proper on a stationary set and is κ -c.c. The arguments are simpler than those from Sections 6 and 7, and as a result we are able to handle the amalgamation arguments for countable and uncountable models at the same time. The order of topics and results is similar to that of Sections 6 and 7.

Lemma 11.1. *Let $p \in \mathbb{Q}$, $N \in \mathcal{X}$, and suppose that $p \in N$. Then there is $r \leq p$ such that $N \in A_r$.*

Proof. Since $p \in N$, for all $i \in \text{dom}(F_p)$, $F_p(i) \in N$. So we can apply Lemma 7.1 and fix, for each $i \in \text{dom}(F_p)$, a condition $q_i \leq F_p(i)$ in $\mathbb{P}_i$ such that $N \in A_{q_i}$. Now apply Lemma 10.9 and fix $q \leq p$ such that $A_q = A_p$, $\text{dom}(F_q) = \text{dom}(F_p)$, and for each $i \in \text{dom}(F_p)$, $F_q(i) = q_i$.

Define r by letting $F_r := F_q$ and $A_r := A_q \cup \{N\}$. It is easy to see that if r is a condition, then $r \leq q$ and $N \in A_r$. So we will be done if we can prove that r is a condition. We verify requirements (1)–(4) of Definition 10.6.

(1) Since $A_p = A_q$ and $p \in N$, A_r is adequate by Lemma 2.16. (2) is immediate. (4) Since $M \in N$ for all $M \in A_q$, easily $s^*(A_r) = s^*(A_q \cup \{N\}) = s^*(A_q)$. Since q is a condition, $s^*(A_q) \subseteq \text{dom}(F_q) = \text{dom}(F_r)$.

(3) Let $i \in \text{dom}(F_r)$. Then $F_r(i) = F_q(i) = q_i$, which is in $\mathbb{P}_i$. Let $M \in A_r = A_q \cup \{N\}$ and suppose that $i \in M$. We will show that $M \in A_{q_i}$. Since $M \in A_r$, either $M \in A_q$ or $M = N$.

First, assume that $M = N$. Then by the choice of q_i , $M = N \in A_{q_i}$, and we are done. Secondly, assume that $M \in A_q$. Now $A_q = A_p$ and $\text{dom}(F_r) = \text{dom}(F_q) = \text{dom}(F_p)$. So $M \in A_p$ and $i \in \text{dom}(F_p)$. Since p is a condition, $M \in A_{F_p(i)}$. But $q_i \leq F_p(i)$, so $A_{F_p(i)} \subseteq A_{q_i}$. Hence, $M \in A_{q_i}$. $\square$

Lemma 11.2. *Let $p \in \mathbb{Q}$ and $N \in \mathcal{X} \cup \mathcal{Y}$. Suppose that if $N \in \mathcal{X}$, then $N \in A_p$. Then there is $s \leq p$ satisfying:*

- (1) *if $N \in \mathcal{X}$, then for all $M \in A_s$, if $M < N$ then $M \cap N \in A_s$, and moreover, $A_s = A_p \cup \{M \cap N : M \in A_p, M < N\}$;*
- (2) *if $N \in \mathcal{Y}$, then for all $M \in A_s$, $M \cap N \in A_s$, and moreover, $A_s = A_p \cup \{M \cap N : M \in A_p\}$.*

Proof. Define

$$B := A_p \cup \{M \cap N : M \in A_p, M < N\}$$

in the case that $N \in \mathcal{X}$, and

$$B := A_p \cup \{M \cap N : M \in A_p\}$$

in the case that $N \in \mathcal{Y}$. By Propositions 2.24 and 2.27, B is adequate. Define

$$q := p \uplus (s^*(B) \setminus \text{dom}(F_p)).$$

By Definition 10.7 and Lemma 10.8, $q \in \mathbb{Q}$, $q \leq p$, $A_q = A_p$, and $\text{dom}(F_q) = \text{dom}(F_p) \cup s^*(B)$.

If $N \in \mathcal{X}$, then since $N \in A_q$, Definition 10.6(3) implies that for all $i \in \text{dom}(F_q) \cap N$, $N \in A_{F_q(i)}$. Applying Lemma 7.2 in the case that $N \in \mathcal{X}$ and Lemma 6.1 in the case that $N \in \mathcal{Y}$, we can fix, for each $i \in \text{dom}(F_q) \cap N$, a condition $r_i \leq F_q(i)$ in $\mathbb{P}_i$ satisfying:

- (1) if $N \in \mathcal{X}$, then for all $M \in A_{r_i}$, if $M < N$ then $M \cap N \in A_{r_i}$;
- (2) if $N \in \mathcal{Y}$, then for all $M \in A_{r_i}$, $M \cap N \in A_{r_i}$.

Now apply Lemma 10.9 and fix $r \leq q$ such that $A_r = A_q$, $\text{dom}(F_r) = \text{dom}(F_q)$, for each $i \in \text{dom}(F_q) \cap N$, $F_r(i) = r_i$, and for each $i \in \text{dom}(F_q) \setminus N$, $F_r(i) = F_q(i)$.

Finally, define

$$s := (F_r, B).$$

We claim that s is as required. Note that if s is a condition, then clearly $s \leq r$. Also, by Propositions 2.24 and 2.27, since $A_p = A_q = A_r$, $A_s = B$ satisfies (1) and (2) of the lemma.

It remains to show that s is a condition. We verify requirements (1)–(4) of Definition 10.6. (1) We already observed that B is adequate. (2) is immediate. (4) We have that

$$s^*(A_s) = s^*(B) \subseteq \text{dom}(F_q) = \text{dom}(F_r) = \text{dom}(F_s).$$

(3) Let $i \in \text{dom}(F_s)$. Then $F_s(i) = F_r(i) \in \mathbb{P}_i$, since r is a condition. Suppose that $M \in A_s = B$ and $i \in M$. We will show that $M \in A_{F_s(i)}$. If $M \in A_p = A_r$,

then since $i \in \text{dom}(F_s) = \text{dom}(F_r)$, it follows that $M \in A_{F_r(i)} = A_{F_s(i)}$ since r is a condition.

Suppose that $M \in B \setminus A_p$. Then $M = M_1 \cap N$ for some $M_1 \in A_p$, where $M_1 < N$ in the case that $N \in \mathcal{X}$. Then $i \in M = M_1 \cap N$, so $i \in M_1$ and $i \in N$. Now M_1 is in $A_p = A_r$. Since i is in $M_1 \cap \text{dom}(F_r)$ and r is a condition, it follows that M_1 is in $A_{F_r(i)}$. Also, $i \in \text{dom}(F_r) \cap N = \text{dom}(F_q) \cap N$, so by the choice of r , $F_r(i) = r_i$. And by the choice of r_i , $M_1 \cap N \in A_{r_i}$. So $M = M_1 \cap N \in A_{r_i} = A_{F_r(i)} = A_{F_s(i)}$. $\square$

Notation 11.3. Let $N \in \mathcal{X}$. For each $i < \lambda^*$, let $D_{i,N}$ denote the set of conditions in $\mathbb{P}_i$ defined as D_N in Definition 7.3.

We introduce an analogue $D(N)$ of $D_{i,N}$ for $\mathbb{Q}$.

Definition 11.4. Let $N \in \mathcal{X}$. Define $D(N)$ as the set of conditions $p \in \mathbb{Q}$ satisfying:

- (1) $N \in A_p$;
- (2) for all $M \in A_p$, if $M < N$ then $M \cap N \in A_p$;
- (3) for all $i \in \text{dom}(F_p) \cap N$, $F_p(i) \in D_{i,N}$.

Lemma 11.5. Let $N \in \mathcal{X}$. Then for any condition $p \in \mathbb{Q}$, if $N \in A_p$, then there is $r \leq p$ such that $r \in D(N)$.

Proof. Let $p \in \mathbb{P}$. By Lemma 11.2(1), fix $q \leq p$ such that for all $M \in A_q$, if $M < N$ then $M \cap N \in A_q$. Since $N \in A_q$, for each $i \in \text{dom}(F_q) \cap N$, $N \in A_{F_q(i)}$ by Definition 10.6(3). So by Lemma 7.4, for each $i \in \text{dom}(F_q) \cap N$, we can fix $r_i \leq F_q(i)$ such that $r_i \in D_{i,N}$. By Lemma 10.9, there is $r \leq q$ such that $A_r = A_q$, $\text{dom}(F_r) = \text{dom}(F_q)$, for all $i \in \text{dom}(F_r) \cap N$, $F_r(i) = r_i$, and for all $i \in \text{dom}(F_r) \setminus N$, $F_r(i) = F_q(i)$. Then $r \leq p$ and $r \in D(N)$. $\square$

Lemma 11.6. Let $N \in \mathcal{X}$ and $q \in D(N)$. Let x be a finite subset of $\lambda^* \setminus \text{dom}(F_q)$. Then $q \uplus x \in D(N)$.

Proof. Let $r := q \uplus x$. Since $q \in D(N)$ and $A_q = A_r$, we have that $N \in A_r$, and for all $M \in A_r$, if $M < N$ then $M \cap N \in A_r$. Also, since $q \in D(N)$, for all $i \in \text{dom}(F_q) \cap N$, $F_r(i) = F_q(i) \in D_{i,N}$. It remains to show that for all $i \in x \cap N$, $F_r(i) \in D_{i,N}$.

Let $i \in x \cap N$. By Definition 10.7, $F_r(i) = (\emptyset, \emptyset, B_i)$, where $B_i = \{M \in A_q : i \in M\}$. We will verify that $(\emptyset, \emptyset, B_i)$ satisfies requirements (1), (2), and (3) of Definition 7.3.

(1) Since $q \in D(N)$, $N \in A_q$. As $i \in N$, $N \in B_i$ by definition. (3) is immediate, since $f_{F_r(i)} = \emptyset$. (2) Suppose that $M \in B_i$ and $M < N$, and we will show that $M \cap N \in B_i$. By the definition of B_i , $M \in A_q$ and $i \in M$. So $i \in M \cap N$. Since $q \in D(N)$ and $M < N$, $M \cap N \in A_q$. So $M \cap N \in A_q$ and $i \in M \cap N$, which by definition implies that $M \cap N \in B_i$. $\square$

Notation 11.7. Let $P \in \mathcal{Y}$. For each $i < \lambda^*$, let $D_{i,P}$ denote the set of conditions in $\mathbb{P}_i$ defined as D_P in Definition 6.2.

We introduce an analogue $D(P)$ of $D_{i,P}$ for $\mathbb{Q}$.

Definition 11.8. Let $P \in \mathcal{Y}$. Define $D(P)$ as the set of conditions $p \in \mathbb{Q}$ satisfying:

- (1) for all $M \in A_p$, $M \cap P \in A_p$;
- (2) for all $i \in \text{dom}(F_p) \cap P$, $F_p(i) \in D_{i,P}$.

Lemma 11.9. *Let $P \in \mathcal{Y}$. Then $D(P)$ is dense in $\mathbb{Q}$.*

Proof. Let $p \in \mathbb{Q}$. By Lemma 11.2(2), we can find $q \leq p$ such that for all $M \in A_q$, $M \cap P \in A_q$. Let $x := \text{dom}(F_q) \cap P$. For each $i \in x$, the set $D_{i,P}$ is dense in $\mathbb{P}_i$ by Lemma 6.3. By Lemma 10.10, fix $r \leq q$ such that $A_r = A_q$, $\text{dom}(F_r) = \text{dom}(F_q) \cup x = \text{dom}(F_q)$, and for each $i \in x$, $F_r(i) \in D_{i,P}$. Then $r \leq p$ and $r \in D(P)$. $\square$

Lemma 11.10. *Let $P \in \mathcal{Y}$ and $q \in D(P)$. Let x be a finite subset of $\lambda^* \setminus \text{dom}(F_q)$. Then $q \uplus x \in D(P)$.*

Proof. Let $r := q \uplus x$. Then by Definition 10.7, $A_r = A_q$. Since $q \in D(P)$, it follows that for all $M \in A_r$, $M \cap P \in A_r$. It remains to show that for all $i \in \text{dom}(F_r) \cap P$, $F_r(i) \in D_{i,P}$.

By Definition 10.7, $\text{dom}(F_r) = \text{dom}(F_q) \cup x$, for all $i \in \text{dom}(F_q)$, $F_r(i) = F_q(i)$, and for all $i \in x$, $F_r(i) = (\emptyset, \emptyset, B_i)$, where $B_i = \{M \in A_q : i \in M\}$. Since $q \in D(P)$, for all $i \in \text{dom}(F_q) \cap P$, $F_r(i) = F_q(i) \in D_{i,P}$.

It remains to show that for all $i \in x \cap P$, $F_r(i) = (\emptyset, \emptyset, B_i)$ is in $D_{i,P}$. By Definition 6.2, we need to show that for all $M \in B_i$, $M \cap P \in B_i$. So let $M \in B_i$. Then $M \in A_q$ and $i \in M$. So $i \in M \cap P$. Since $q \in D(P)$, $M \cap P \in A_q$. Hence, $M \cap P \in A_q$ and $i \in M \cap P$, which means that $M \cap P \in B_i$. $\square$

Definition 11.11. *Suppose that $N \in \mathcal{X} \cup \mathcal{Y}$ is simple and $q \in D(N)$. Define $q \upharpoonright N$ as the pair (F, A) satisfying:*

- (1) $A := A_q \cap N$;
- (2) $\text{dom}(F) := \text{dom}(F_q) \cap N$;
- (3) *for all $i \in \text{dom}(F)$, $F(i) := F_q(i) \upharpoonright N$, as defined in Definition 7.5 if $N \in \mathcal{X}$, and as defined in Definition 6.4 if $N \in \mathcal{Y}$.*

Note that (3) makes sense because $F_q(i) \in D_{i,N}$, for all $i \in \text{dom}(F_q) \cap N$.

Lemma 11.12. *Suppose that $N \in \mathcal{X} \cup \mathcal{Y}$ is simple and $q \in D(N)$. Then $q \upharpoonright N$ is in $N \cap \mathbb{Q}$ and $q \leq q \upharpoonright N$.*

Proof. Let $q \upharpoonright N = (F, A)$. Then $A = A_q \cap N$ and $\text{dom}(F) = \text{dom}(F_q) \cap N$ are finite subsets of N , and hence are in N . For each $i \in \text{dom}(F)$, $F(i) = F_q(i) \upharpoonright N$ is in N by Lemmas 6.5 and 7.6. So F is in N . Since A and F are in N , so is $q \upharpoonright N$.

To prove that $q \upharpoonright N$ is in $\mathbb{Q}$, we verify requirements (1)–(4) of Definition 10.6. (1) and (2) are immediate. For (3), let $i \in \text{dom}(F)$. Then $F(i) = F_q(i) \upharpoonright N$ is in $\mathbb{P}_i$ by Lemmas 6.5 and 7.6. Suppose that $M \in A$ and $i \in M$, and we will show that $M \in A_{F(i)}$. Then $M \in A = A_q \cap N$, so $M \in A_q$ and $M \in N$. Since q is a condition, the fact that $M \in A_q$ and $i \in M \cap \text{dom}(F_q)$ implies that $M \in A_{F_q(i)}$. Since $F(i) = F_q(i) \upharpoonright N$, Definitions 6.4 and 7.5 imply that

$$M \in A_{F_q(i)} \cap N = A_{F_q(i) \upharpoonright N} = A_{F(i)}.$$

For (4), suppose that K and L are in A , $K \sim L$, $i \in K \cap L \cap \lambda^*$, and the ordinal $\min((K \cap \kappa) \setminus \beta_{K,L})$ exists and is in S_i . We will show that $i \in \text{dom}(F)$. Since $A = A_q \cap N$, K and L are in A_q and in N . As q is a condition, i must be in $\text{dom}(F_q)$. Since $K \in N$ and $i \in K$, we have that $i \in N$. So $i \in \text{dom}(F_q) \cap N = \text{dom}(F)$.

This completes the proof that $q \upharpoonright N$ is a condition. Now we show that $q \leq q \upharpoonright N$. We have that $A = A_q \cap N \subseteq A_q$ and $\text{dom}(F) = \text{dom}(F_q) \cap N \subseteq \text{dom}(F_q)$. Let $i \in \text{dom}(F)$, and we will show that $F_q(i) \leq F(i)$ in $\mathbb{P}_i$. But $F(i) = F_q(i) \upharpoonright N$ and $F_q(i) \leq F_q(i) \upharpoonright N$ in $\mathbb{P}_i$ by Lemmas 6.5 and 7.6. $\square$

The next lemma will be needed in Section 12.

Lemma 11.13. *Let $N \in \mathcal{X} \cup \mathcal{Y}$ be simple and $q \in D(N)$.*

- (1) *Suppose that $p \in N \cap \mathbb{Q}$ and $q \leq p$. Then $q \restriction N \leq p$.*
- (2) *Suppose that p is in $D(N)$ and $q \leq p$. Then $q \restriction N \leq p \restriction N$.*

Proof. (1) Since $p \in N$ and $q \leq p$, we have that $A_p \subseteq A_q \cap N = A_{q \restriction N}$ and $\text{dom}(F_p) \subseteq \text{dom}(F_q) \cap N = \text{dom}(F_{q \restriction N})$. Let $i \in \text{dom}(F_p)$, and we will show that $F_{q \restriction N}(i) \leq F_p(i)$ in $\mathbb{P}_i$. Since $q \leq p$, we have that $F_q(i) \leq F_p(i)$. As $F_p(i) \in N \cap \mathbb{P}_i$, it follows that $F_{q \restriction N}(i) = F_q(i) \restriction N \leq F_p(i)$ by Lemmas 6.6(1) and 7.7.

(2) By Lemma 11.12, $p \leq p \restriction N$. So $q \leq p \restriction N$. Now $p \restriction N \in N$, so by (1), $q \restriction N \leq p \restriction N$. $\square$

We will now begin analyzing the situation where $q \in D(N)$ and $w \leq q \restriction N$ is in $N \cap \mathbb{Q}$.

Lemma 11.14. *Let $N \in \mathcal{X} \cup \mathcal{Y}$ be simple and $q \in D(N)$. Suppose that $w \in N \cap \mathbb{Q}$ and $w \leq q \restriction N$. Then:*

- (1) $A_q \cap N \subseteq A_w$;
- (2) $\text{dom}(F_q) \cap N \subseteq \text{dom}(F_w)$, and for all $i \in \text{dom}(F_q) \cap N$, $F_w(i) \leq F_q(i) \restriction N$ in $\mathbb{P}_i$.

Proof. Immediate from the definition of $q \restriction N$ and the fact that $w \leq q \restriction N$. $\square$

Note that in (2) above, if $i \in \text{dom}(F_q) \cap N$, then since $w \in N$, $F_w(i)$ is a condition in $N \cap \mathbb{P}_i$ which is below $F_q(i) \restriction N$ in $\mathbb{P}_i$. As $F_q(i) \in D_{i,N}$, it follows by Propositions 6.15 and 7.19 that $F_w(i) \oplus_N F_q(i)$ is a condition in $\mathbb{P}_i$ which is below $F_w(i)$ and $F_q(i)$.

As in Sections 6 and 7, we are going to show that whenever $w \leq q \restriction N$, where $q \in D(N)$ and $w \in N \cap \mathbb{Q}$, then w and q are compatible. We will define a specific lower bound of w and q , namely, $w \oplus^N q$. However, unlike the situation in Sections 6 and 7, the condition $w \oplus^N q$ will exist only under the assumption that $\text{dom}(F_w) \subseteq \text{dom}(F_q)$.

Definition 11.15. *Let $N \in \mathcal{X} \cup \mathcal{Y}$ be simple and $q \in D(N)$. Suppose that $w \in N \cap \mathbb{Q}$ and $w \leq q \restriction N$. Assume, moreover, that $\text{dom}(F_w) \subseteq \text{dom}(F_q)$. Define $w \oplus^N q$ as the pair (F, A) satisfying:*

- (1) $A := A_w \cup A_q$;
- (2) $\text{dom}(F) := \text{dom}(F_q)$;
- (3) *for all $i \in \text{dom}(F_q) \setminus \text{dom}(F_w)$, $F(i) := F_q(i)$, and for all $i \in \text{dom}(F_w) \cap \text{dom}(F_q)$, $F(i) := F_w(i) \oplus_N F_q(i)$, as defined in Definition 7.18 if $N \in \mathcal{X}$, and as defined in Definition 6.14 if $N \in \mathcal{Y}$.*

Proposition 11.16. *Let $N \in \mathcal{X} \cup \mathcal{Y}$ be simple and $q \in D(N)$. Suppose that $w \in N \cap \mathbb{Q}$ and $w \leq q \restriction N$. Assume, moreover, that $\text{dom}(F_w) \subseteq \text{dom}(F_q)$. Then w and q are compatible. In fact, $w \oplus^N q$ is in $\mathbb{Q}$ and $w \oplus^N q \leq w, q$.*

Proof. Let $w \oplus^N q = (F, A)$. To prove that $w \oplus^N q$ is a condition, we verify requirements (1)–(4) of Definition 10.6. For (1), the set $A = A_w \cup A_q$ is adequate by Propositions 2.25 and 2.28. For (2), obviously F is a function whose domain is a finite subset of λ^* .

For (4), by Proposition 10.13 we have that

$$\begin{aligned} s^*(A) &= s^*(A_w \cup A_q) = s^*(A_w) \cup s^*(A_q) \subseteq \\ &\subseteq \text{dom}(F_w) \cup \text{dom}(F_q) = \text{dom}(F_q) = \text{dom}(F). \end{aligned}$$

It remains to prove (3). Let $i \in \text{dom}(F)$. If $i \in \text{dom}(F_q) \setminus \text{dom}(F_w)$, then $F(i) = F_q(i)$, which is in $\mathbb{P}_i$ since q is a condition. If $i \in \text{dom}(F_q) \cap \text{dom}(F_w)$, then $F(i) = F_w(i) \oplus_N F_q(i)$, which is in $\mathbb{P}_i$ by Propositions 6.15 and 7.19.

Assume that $i \in \text{dom}(F)$, and we will show that

$$\{M \in A_{w \oplus^N q} : i \in M\} \subseteq A_{F(i)},$$

that is,

$$\{M \in A_w \cup A_q : i \in M\} \subseteq A_{F(i)}.$$

First, assume that $i \in \text{dom}(F_q) \setminus \text{dom}(F_w)$, so $F(i) = F_q(i)$. Since $\text{dom}(F_q) \cap N \subseteq \text{dom}(F_w)$ by Lemma 11.14(2), it follows that $i \notin N$. In particular, if $M \in A_w$ then $M \in N$, so i cannot be in M since otherwise it would be in N . It follows that

$$\{M \in A_w \cup A_q : i \in M\} = \{M \in A_q : i \in M\}.$$

Since $F(i) = F_q(i)$, q being a condition implies that

$$\{M \in A_q : i \in M\} \subseteq A_{F_q(i)} = A_{F(i)}.$$

Secondly, assume that $i \in \text{dom}(F_q) \cap \text{dom}(F_w)$. Then $F(i) = F_w(i) \oplus_N F_q(i)$. By Definitions 6.14 and 7.18,

$$A_{F(i)} = A_{F_w(i)} \cup A_{F_q(i)}.$$

Since w and q are conditions,

$$\{M \in A_w : i \in M\} \subseteq A_{F_w(i)}, \quad \{M \in A_q : i \in M\} \subseteq A_{F_q(i)}.$$

Therefore,

$$\{M \in A_w \cup A_q : i \in M\} \subseteq A_{F_w(i)} \cup A_{F_q(i)} = A_{F(i)}.$$

This completes the proof that $w \oplus^N q$ is in $\mathbb{Q}$.

It remains to show that $w \oplus^N q \leq q, w$. First, we prove that $w \oplus^N q \leq w$. We have that $A_w \subseteq A_w \cup A_q = A$. Since $\text{dom}(F_w) \subseteq \text{dom}(F_q)$ by assumption, it follows that $\text{dom}(F_w) \subseteq \text{dom}(F_q) = \text{dom}(F)$. Let $i \in \text{dom}(F_w)$, and we will show that $F(i) \leq F_w(i)$ in $\mathbb{P}_i$. But $F(i) = F_w(i) \oplus_N F_q(i)$, which is less than or equal to $F_w(i)$ in $\mathbb{P}_i$ by Propositions 6.15 and 7.19.

Secondly, we prove that $w \oplus^N q \leq q$. We have that $A_q \subseteq A_w \cup A_q = A$, and $\text{dom}(F) = \text{dom}(F_q)$. Let $i \in \text{dom}(F_q)$, and we will show that $F(i) \leq F_q(i)$ in $\mathbb{P}_i$. If $i \notin \text{dom}(F_w)$, then $F(i) = F_q(i)$, and we are done. If $i \in \text{dom}(F_w)$, then $F(i) = F_w(i) \oplus_N F_q(i)$, which is less than or equal to $F_q(i)$ in $\mathbb{P}_i$ by Propositions 6.15 and 7.19. $\square$

In the above amalgamation result, we assumed that $\text{dom}(F_w) \subseteq \text{dom}(F_q)$. To prove the amalgamation result in general, we need a lemma.

Lemma 11.17. *Suppose that $N \in \mathcal{X} \cup \mathcal{Y}$ is simple and $q \in D(N)$. Let x be a finite subset of $\lambda^* \setminus \text{dom}(F_q)$. Then*

$$(q \uplus x) \upharpoonright N = (q \upharpoonright N) \uplus (x \cap N).$$

Recall that by Lemmas 11.6 and 11.10, $q \uplus x$ is in $D(N)$.

Proof. By Definitions 10.7 and 11.11,

$$A_{(q \uplus x) \upharpoonright N} = A_{q \uplus x} \cap N = A_q \cap N = A_{q \upharpoonright N} = A_{(q \upharpoonright N) \uplus (x \cap N)}.$$

Also,

$$\begin{aligned} \text{dom}(F_{(q \uplus x) \upharpoonright N}) &= \text{dom}(F_{q \uplus x}) \cap N = (\text{dom}(F_q) \cup x) \cap N = \\ &= (\text{dom}(F_q) \cap N) \cup (x \cap N) = \text{dom}(F_{q \upharpoonright N}) \cup (x \cap N) = \text{dom}(F_{(q \upharpoonright N) \uplus (x \cap N)}). \end{aligned}$$

Let $i \in \text{dom}(F_{(q \uplus x) \upharpoonright N})$, and we will show that

$$F_{(q \uplus x) \upharpoonright N}(i) = F_{(q \upharpoonright N) \uplus (x \cap N)}(i).$$

By the above equalities, we have that either $i \in \text{dom}(F_{q \upharpoonright N})$ or $i \in x \cap N$.

First, assume that $i \in \text{dom}(F_{q \upharpoonright N})$. Then by Definitions 10.7 and 11.11, we have that $F_{(q \upharpoonright N) \uplus (x \cap N)}(i) = F_{q \upharpoonright N}(i) = F_q(i) \upharpoonright N$. On the other hand, since $\text{dom}(F_{q \upharpoonright N}) = \text{dom}(F_q) \cap N$, it follows that $i \in \text{dom}(F_q)$, and hence $F_{(q \uplus x) \upharpoonright N}(i) = F_{q \uplus x}(i) \upharpoonright N = F_q(i) \upharpoonright N$.

Now assume that $i \in x \cap N$. Then by definition,

$$F_{(q \upharpoonright N) \uplus (x \cap N)}(i) = (\emptyset, \emptyset, B),$$

where

$$B = \{M \in A_{q \upharpoonright N} : i \in M\}.$$

Also,

$$F_{(q \uplus x) \upharpoonright N}(i) = F_{q \uplus x}(i) \upharpoonright N = (\emptyset, \emptyset, C) \upharpoonright N,$$

where

$$C = \{M \in A_q : i \in M\}.$$

Hence, it suffices to show that

$$(\emptyset, \emptyset, B) = (\emptyset, \emptyset, C) \upharpoonright N.$$

By Definitions 6.4 and 7.5,

$$(\emptyset, \emptyset, C) \upharpoonright N = (\emptyset, \emptyset, C \cap N).$$

So it suffices to show that

$$B = C \cap N.$$

But $M \in B$ iff $(M \in A_{q \upharpoonright N} = A_q \cap N \text{ and } i \in M)$ iff $M \in C \cap N$. $\square$

Proposition 11.18. *Let $N \in \mathcal{X} \cup \mathcal{Y}$ be simple and $q \in D(N)$. Then for all $w \leq q \upharpoonright N$ in $N \cap \mathbb{Q}$, w and q are compatible. In fact, let $x := \text{dom}(F_w) \setminus \text{dom}(F_q)$. Then $w \leq (q \uplus x) \upharpoonright N$, $\text{dom}(F_w) \subseteq \text{dom}(F_{q \uplus x})$, and $w \oplus^N (q \uplus x)$ is less than or equal to w , $q \uplus x$, and q .*

Proof. Note that $x \subseteq N$. By Lemma 11.17,

$$(q \uplus x) \upharpoonright N = (q \upharpoonright N) \uplus (x \cap N).$$

By Lemma 10.8(4), since $w \leq q \upharpoonright N$ and $x \cap N = x \subseteq \text{dom}(F_w)$, it follows that

$$w \leq (q \upharpoonright N) \uplus (x \cap N).$$

Hence, $w \leq (q \uplus x) \upharpoonright N$. As $x = \text{dom}(F_w) \setminus \text{dom}(F_q)$, clearly $\text{dom}(F_w) \subseteq \text{dom}(F_q) \cup x = \text{dom}(F_{q \uplus x})$. By Proposition 11.16, it follows that $w \oplus^N (q \uplus x)$ is a condition which is less than or equal to w and $q \uplus x$. Since $q \uplus x \leq q$, also $w \oplus^N (q \uplus x) \leq q$. $\square$

Corollary 11.19. *The forcing poset $\mathbb{Q}$ is strongly proper on a stationary set. In particular, it preserves ω_1 .*

Proof. By Assumption 2.22, the set of $N \in \mathcal{X}$ such that N is simple is stationary. So it suffices to show that for all simple $N \in \mathcal{X}$, for all $p \in N \cap \mathbb{Q}$, there is $q \leq p$ such that q is strongly N -generic.

Let $p \in N \cap \mathbb{Q}$. By Lemma 11.1, fix $q \leq p$ with $N \in A_q$. We claim that q is strongly N -generic. So let D be a dense subset of $N \cap \mathbb{Q}$, and we will show that D is predense below q . Let $r \leq q$, and we will find $w \in D$ which is compatible with r .

Since $N \in A_r$, we can apply Lemma 11.5 to fix $s \leq r$ such that $s \in D(N)$. Then by Lemma 11.12, $s \restriction N$ is in $N \cap \mathbb{Q}$. As D is dense in $N \cap \mathbb{Q}$, fix $w \leq s \restriction N$ in D . By Proposition 11.18, w and s are compatible. Since $s \leq r$, it follows that w and r are compatible. $\square$

Corollary 11.20. *Suppose that $P \in \mathcal{Y}$ is simple and $P \prec (H(\lambda), \in, \mathbb{Q})$. Then the maximum condition of $\mathbb{Q}$ is strongly P -generic. Moreover, $P \cap \mathbb{Q}$ is a regular suborder of $\mathbb{Q}$.*

Proof. Let D be a dense subset of $P \cap \mathbb{Q}$, and we will show that D is predense in $\mathbb{Q}$. So let $p \in \mathbb{Q}$, and we will find w in D which is compatible with p . By Lemma 11.9, fix $q \leq p$ in $D(P)$. Then $q \restriction P$ is in $P \cap \mathbb{Q}$ by Lemma 11.12. Since D is dense in $P \cap \mathbb{Q}$, fix $w \leq q \restriction P$ in D . By Proposition 11.18, w and q are compatible. Since $q \leq p$, it follows that w and p are compatible. This completes the proof that the maximum condition in $\mathbb{Q}$ is strongly P -generic.

Now we show that $P \cap \mathbb{Q}$ is a regular suborder of $\mathbb{Q}$. If p and q are in $P \cap \mathbb{Q}$ and are compatible in $\mathbb{Q}$, then by the elementarity of P , there is $r \in P \cap \mathbb{Q}$ with $r \leq p, q$. So p and q are compatible in $P \cap \mathbb{Q}$.

Let $B \subseteq P \cap \mathbb{Q}$ be a maximal antichain of $P \cap \mathbb{Q}$, and we will prove that B is predense in $\mathbb{Q}$. Let D be the set of conditions in $P \cap \mathbb{Q}$ which are below some member of B . Then D is dense in $P \cap \mathbb{Q}$. Since the maximum condition is strongly P -generic, D is predense in $\mathbb{Q}$. It easily follows that B is predense in $\mathbb{Q}$. $\square$

Corollary 11.21. *The forcing poset $\mathbb{Q}$ is κ -c.c.*

Proof. Let A be an antichain of $\mathbb{Q}$, and suppose for a contradiction that A has size at least κ . Without loss of generality, assume that A is maximal. By Assumption 2.23, there are stationarily many simple models in $\mathcal{Y}$. So we can fix a simple model $P \in \mathcal{Y}$ such that $P \prec (H(\lambda), \in, \mathbb{Q}, A)$. As A has size at least κ and $|P| < \kappa$, fix $s \in A \setminus P$.

By Lemma 11.9, fix $q \leq s$ such that $q \in D(P)$. By Lemma 11.12, $q \restriction P$ is a condition in $P \cap \mathbb{Q}$. By the elementarity of P and the maximality of A , there is $t \in A \cap P$ which is compatible with $q \restriction P$. By elementarity, fix $w \in P \cap \mathbb{Q}$ such that $w \leq q \restriction P, t$.

By Proposition 11.18, w and q are compatible. Fix $v \leq w, q$. Then $v \leq w \leq t$ and $v \leq q \leq s$. Hence, s and t are compatible. But s and t are in A and A is an antichain. Therefore, $s = t$. This is impossible, since $t \in P$ and $s \notin P$. $\square$

The next result summarizes the main properties which we have proven about the forcing poset $\mathbb{Q}$.

Corollary 11.22. *The forcing poset $\mathbb{Q}$ preserves ω_1 , is κ -c.c., forces that κ is equal to ω_2 , and forces that for all $i < \lambda^*$, $S_i \in I[\omega_2]$.*

Proof. By Corollaries 11.19 and 11.21, $\mathbb{Q}$ preserves ω_1 and is κ -c.c. Let $i < \lambda^*$, and consider a generic filter G on $\mathbb{Q}$. Then by Proposition 10.11 and the comments

which follow, there is a generic filter H on $\mathbb{P}_i$ such that $V[H] \subseteq V[G]$. By Corollary 7.22, κ is equal to ω_2 in $V[H]$. Since $V[H] \subseteq V[G]$, it follows that any cardinal μ such that $\omega_1 < \mu < \kappa$ has size ω_1 in $V[G]$. Therefore, $\kappa = \omega_2$ in $V[G]$.

By Proposition 5.4, there is a partial square sequence on S_i in $V[H]$. But being a partial square sequence is upwards absolute between $V[H]$ and $V[G]$, since they have the same ω_1 and ω_2 . So there is a partial square sequence on S_i in $V[G]$. Therefore, $S_i \in I[\omega_2]$ in $V[G]$. $\square$

§12. Approximation

In this section we will prove that if $P \in \mathcal{Y}$ is simple, $P \prec (H(\lambda), \in, \mathbb{Q})$, and for all $i \in P \cap \lambda^*$, $P \cap \kappa \notin S_i$, then $P \cap \mathbb{Q}$ forces that $\mathbb{Q}/\dot{G}_{P \cap \mathbb{Q}}$ has the ω_1 -approximation property. The proof is similar to the analogous result given in Sections 8 and 9 for the forcing poset $\mathbb{P}$, albeit somewhat easier. The order of topics and results follows along the same lines as in those previous sections.

Lemma 12.1. *Let $N \in \mathcal{X}$ be simple and $q \in D(N)$. Suppose that v and w are in $N \cap \mathbb{Q}$ and*

$$w \leq v \leq q \restriction N.$$

Assume, moreover, that $\text{dom}(F_w) \subseteq \text{dom}(F_q)$. Then $w \oplus^N q \leq v \oplus^N q$.

Note that since $w \leq v$, $\text{dom}(F_v) \subseteq \text{dom}(F_w) \subseteq \text{dom}(F_q)$. So $v \oplus^N q$ is defined.

Proof. Let $s := v \oplus^N q$ and $t := w \oplus^N q$. We will prove that $t \leq s$. Since $w \leq v$, $A_v \subseteq A_w$. By Definition 11.15, we have that

$$A_s = A_v \cup A_q \subseteq A_w \cup A_q = A_t.$$

So $A_s \subseteq A_t$. Also, by Definition 11.15, $\text{dom}(F_s) = \text{dom}(F_q) = \text{dom}(F_t)$.

Let $i \in \text{dom}(F_s)$, and we will show that $F_t(i) \leq F_s(i)$ in $\mathbb{P}_i$. First, assume that $i \notin N$. Then by Definition 11.15, $F_s(i) = F_q(i)$ and $F_t(i) = F_q(i)$, and we are done.

Secondly, assume that $i \in N$. Then by Lemma 11.14, $i \in \text{dom}(F_q) \cap N \subseteq \text{dom}(F_v) \subseteq \text{dom}(F_w)$. So by Definition 11.15, $F_s(i) = F_v(i) \oplus_N F_q(i)$ and $F_t(i) = F_w(i) \oplus_N F_q(i)$. Also, since v , w , and i are in N , so are $F_v(i)$ and $F_w(i)$.

As $q \in D(N)$ and $i \in \text{dom}(F_q) \cap N$, $F_q(i) \in D_{i,N}$. Also, since $w \leq v \leq q \restriction N$, we have that in $\mathbb{P}_i$,

$$F_w(i) \leq F_v(i) \leq F_q \restriction N(i) = F_q(i) \restriction N.$$

By Lemma 8.1, it follows that

$$F_t(i) = F_w(i) \oplus_N F_q(i) \leq F_v(i) \oplus_N F_q(i) = F_s(i).$$

$\square$

Lemma 12.2. *Let $N \in \mathcal{X}$ and $P \in \mathcal{Y}$. Let $p \in \mathbb{Q}$, and suppose that $N \in A_p$. Then there is $s \leq p$ such that $s \in D(N) \cap D(P)$.*

Proof. By Lemma 11.2(1), there is $q \leq p$ such that for all $M \in A_q$, if $M < N$ then $M \cap N \in A_q$. By Lemma 11.2(2), there is $r \leq q$ such that for all $M \in A_r$, $M \cap P \in A_r$, and moreover,

$$A_r = A_q \cup \{M \cap P : M \in A_q\}.$$

We claim that for all $M \in A_r$, if $M < N$ then $M \cap N \in A_r$. This is certainly true if $M \in A_q$, so assume that $M = M_1 \cap P$, where $M_1 \in A_q$. By Lemma 2.29,

$M_1 \sim M_1 \cap P = M$. Since $M < N$, it follows that $M_1 < N$ by Lemma 2.18. As $M_1 \in A_q$, we have that $M_1 \cap N \in A_q$ by the choice of q . Now $M \cap N = (M_1 \cap P) \cap N = (M_1 \cap N) \cap P$. But $M_1 \cap N \in A_q$ implies that $M \cap N = (M_1 \cap N) \cap P \in A_r$, by the definition of A_r .

Since r is a condition and $N \in A_r$, we have that for all $i \in \text{dom}(F_r) \cap N$, $N \in A_{F_r(i)}$. Let $x_1 := \text{dom}(F_r) \cap N$ and $x_2 := \text{dom}(F_r) \cap P$. Then:

- (1) For each $i \in x_1 \setminus x_2$, since $N \in A_{F_r(i)}$ we can fix, by Lemma 7.4, a condition $s_i \leq F_r(i)$ in $D_{i,N}$.
- (2) For each $i \in x_1 \cap x_2$, since $N \in A_{F_r(i)}$ we can fix, by Lemma 8.2, a condition $s_i \leq F_r(i)$ in $D_{i,N} \cap D_{i,P}$.
- (3) For each $i \in x_2 \setminus x_1$, we can fix, by Lemma 6.3, a condition $s_i \leq F_r(i)$ in $D_{i,P}$.

Now apply Lemma 10.9 to fix $s \leq r$ satisfying that $A_s = A_r$, $\text{dom}(F_s) = \text{dom}(F_r)$, for all $i \in x_1 \cup x_2$, $F_s(i) = s_i$, and for all $i \in \text{dom}(F_r) \setminus (x_1 \cup x_2)$, $F_s(i) = F_r(i)$. Then $s \leq p$ and $s \in D(N) \cap D(P)$. $\square$

The next three lemmas will be used in the proof of Proposition 12.6.

Lemma 12.3. *Let $N \in \mathcal{X}$ be simple, $P \in \mathcal{Y} \cap N$ be simple, and $p \in D(N) \cap D(P)$. Then $p \restriction N \in D(P)$ and $p \restriction P \in D(N \cap P)$.*

Proof. We prove first that $p \restriction N \in D(P)$, which means that for all $M \in A_{p \restriction N}$, $M \cap P \in A_{p \restriction N}$, and for all $i \in \text{dom}(F_{p \restriction N}) \cap P$, $F_{p \restriction N}(i) \in D_{i,P}$. Let $M \in A_{p \restriction N}$. Then $M \in A_{p \restriction N} = A_p \cap N$, so $M \in A_p \cap N$. Since $p \in D(P)$, we have that $M \cap P \in A_p$. And as M and P are in N , $M \cap P \in N$. Therefore, $M \cap P \in A_p \cap N = A_{p \restriction N}$.

Now let $i \in \text{dom}(F_{p \restriction N}) \cap P = \text{dom}(F_p) \cap N \cap P$, and we will show that $F_{p \restriction N}(i) \in D_{i,P}$. Since $i \in N \cap P$ and $p \in D(N) \cap D(P)$, $F_p(i) \in D_{i,N} \cap D_{i,P}$. By Lemma 8.3, it follows that $F_p(i) \restriction N \in D_{i,P}$. But $F_p(i) \restriction N = F_{p \restriction N}(i)$. This completes the proof that $p \restriction N \in D(P)$.

Next, we prove that $p \restriction P \in D(N \cap P)$. First, we show that $N \cap P \in A_{p \restriction P}$. Since $p \in D(N)$, $N \in A_p$. As $p \in D(P)$, $N \cap P \in A_p$. Since P is simple, $N \cap P \in P$. So $N \cap P \in A_p \cap P = A_{p \restriction P}$.

Secondly, we prove that if $M \in A_{p \restriction P}$ and $M < N \cap P$, then $M \cap N \cap P \in A_{p \restriction P}$. Let $M \in A_{p \restriction P}$, and assume that $M < N \cap P$. Then $M \in A_{p \restriction P} = A_p \cap P$. By Lemma 2.29, $N \sim N \cap P$. Since $M < N \cap P$, it follows by Lemma 2.18 that $M < N$. Since $p \in D(N)$ and $M < N$, it follows that $M \cap N \in A_p$. And as $p \in D(P)$, $M \cap N \cap P \in A_p$. Since P is simple, $M \cap N \cap P \in P$. So $M \cap N \cap P \in A_p \cap P = A_{p \restriction P}$.

Thirdly, we show that if $i \in \text{dom}(F_{p \restriction P}) \cap (N \cap P)$, then $F_{p \restriction P}(i) \in D_{i,N \cap P}$. Since $p \in D(P) \cap D(N)$ and $i \in N \cap P$, we have that $F_p(i) \in D_{i,N} \cap D_{i,P}$. By Lemma 8.3, $F_p(i) \restriction P$ is in $D_{i,N \cap P}$. But $F_p(i) \restriction P = F_{p \restriction P}(i)$. $\square$

Lemma 12.4. *Let $N \in \mathcal{X}$ be simple, $P \in \mathcal{Y} \cap N$ be simple, and $p \in D(N) \cap D(P)$. Then*

$$(p \restriction N) \restriction P = (p \restriction P) \restriction (N \cap P).$$

Note that we needed Lemma 12.3 to see that $(p \restriction N) \restriction P$ and $(p \restriction P) \restriction (N \cap P)$ are defined.

Proof. By Definition 11.11, we have that

$$\begin{aligned} A_{(p \upharpoonright N) \upharpoonright P} &= A_{p \upharpoonright N} \cap P = A_p \cap N \cap P = \\ &= (A_p \cap P) \cap (N \cap P) = A_{p \upharpoonright P} \cap (N \cap P) = A_{(p \upharpoonright P) \upharpoonright (N \cap P)}. \end{aligned}$$

And

$$\begin{aligned} \text{dom}(F_{(p \upharpoonright N) \upharpoonright P}) &= \text{dom}(F_{p \upharpoonright N}) \cap P = \text{dom}(F_p) \cap N \cap P = \\ &= (\text{dom}(F_p) \cap P) \cap (N \cap P) = \text{dom}(F_{p \upharpoonright P}) \cap (N \cap P) = \text{dom}(F_{(p \upharpoonright P) \upharpoonright (N \cap P)}). \end{aligned}$$

Let $i \in \text{dom}(F_{(p \upharpoonright N) \upharpoonright P})$, and we will show that

$$F_{(p \upharpoonright N) \upharpoonright P}(i) = F_{(p \upharpoonright P) \upharpoonright (N \cap P)}(i).$$

By the above equations, $i \in N \cap P$. Since $p \in D(N) \cap D(P)$, we have that $F_p(i) \in D_{i,N} \cap D_{i,P}$. So by Definition 11.11 and Lemma 8.4,

$$\begin{aligned} F_{(p \upharpoonright N) \upharpoonright P}(i) &= F_{p \upharpoonright N}(i) \upharpoonright P = (F_p(i) \upharpoonright N) \upharpoonright P = \\ &= (F_p(i) \upharpoonright P) \upharpoonright (N \cap P) = F_{p \upharpoonright P}(i) \upharpoonright (N \cap P) = F_{(p \upharpoonright P) \upharpoonright (N \cap P)}(i). \end{aligned}$$

□

Lemma 12.5. *Let $N \in \mathcal{X}$ be simple, $P \in \mathcal{Y} \cap N$ be simple, and $p \in D(N) \cap D(P)$. Suppose that $q \in N \cap D(P)$, $q \leq p \upharpoonright N$, and $\text{dom}(F_q) \subseteq \text{dom}(F_p)$. Then:*

- (1) $q \oplus^N p$ is in $D(P)$;
- (2) $q \upharpoonright P \in N \cap P$ and

$$q \upharpoonright P \leq (p \upharpoonright P) \upharpoonright (N \cap P).$$

Proof. (1) Let us prove that $q \oplus^N p$ is in $D(P)$, which means that for all $M \in A_{q \oplus^N p}$, $M \cap P \in A_{q \oplus^N p}$, and for all $i \in \text{dom}(F_{q \oplus^N p}) \cap P$, $F_{q \oplus^N p}(i) \in D_{i,P}$. Now $A_{q \oplus^N p} = A_q \cup A_p$. So if $M \in A_{q \oplus^N p}$, then either $M \in A_q$ or $M \in A_p$. But q and p are both in $D(P)$, so in the first case, $M \cap P \in A_q$, and in the second case, $M \cap P \in A_p$. In either case, $M \cap P \in A_q \cup A_p = A_{q \oplus^N p}$.

Now let $i \in \text{dom}(F_{q \oplus^N p}) \cap P$, and we will show that $F_{q \oplus^N p}(i) \in D_{i,P}$. By Definition 11.15, $\text{dom}(F_{q \oplus^N p}) = \text{dom}(F_p)$, for all $i \in \text{dom}(F_p) \setminus \text{dom}(F_q)$, $F_{q \oplus^N p}(i) = F_p(i)$, and for all $i \in \text{dom}(F_q) \cap \text{dom}(F_p)$, $F_{q \oplus^N p}(i) = F_q(i) \oplus_N F_p(i)$.

First, assume that $i \in \text{dom}(F_p) \setminus \text{dom}(F_q)$. Then $F_{q \oplus^N p}(i) = F_p(i)$. Since $p \in D(P)$ and $i \in \text{dom}(F_p) \cap P$, we have that $F_p(i) \in D_{i,P}$.

Secondly, assume that $i \in \text{dom}(F_p) \cap \text{dom}(F_q)$. Since $q \in N$, we have that $i \in N$. So $F_{q \oplus^N p}(i) = F_q(i) \oplus_N F_p(i)$. Thus, it suffices to show that $F_q(i) \oplus_N F_p(i) \in D_{i,P}$. This will follow from Lemma 8.5, provided that the assumptions of this lemma are true for $F_p(i)$ and $F_q(i)$.

Since $i \in N \cap P$ and $p \in D(N) \cap D(P)$, $F_p(i) \in D_{i,N} \cap D_{i,P}$. As q and i are in N , $F_q(i) \in N$, and since $q \in D(P)$ and $i \in P$, $F_q(i) \in N \cap D_{i,P}$. Finally, as $q \leq p \upharpoonright N$, $F_q(i) \leq F_{p \upharpoonright N}(i) = F_p(i) \upharpoonright N$. This completes the verification of the assumptions of Lemma 8.5. By Lemma 8.5(1), we have that $F_q(i) \oplus_N F_p(i)$ is in $D_{i,P}$.

(2) Since q and P are in N , $q \upharpoonright P \in N$. Also, $q \upharpoonright P \in P$, so $q \upharpoonright P \in N \cap P$. By Lemmas 12.3 and 12.4, we have that $p \upharpoonright N \in D(P)$ and

$$(p \upharpoonright N) \upharpoonright P = (p \upharpoonright P) \upharpoonright (N \cap P).$$

As $q \leq p \upharpoonright N$, it follows by Lemma 11.13(2) that

$$q \upharpoonright P \leq (p \upharpoonright N) \upharpoonright P = (p \upharpoonright P) \upharpoonright (N \cap P).$$

□

Proposition 12.6. *Let $N \in \mathcal{X}$ be simple, $P \in \mathcal{Y} \cap N$ be simple, and suppose that $P \cap \kappa \notin S_i$, for all $i \in P \cap \lambda^*$. Let $p \in D(N) \cap D(P)$, $q \in N \cap D(P)$, and $q \leq p \restriction N$. Assume, moreover, that $\text{dom}(F_q) \subseteq \text{dom}(F_p)$. Then*

$$(q \oplus^N p) \restriction P = (q \restriction P) \oplus^{N \cap P} (p \restriction P).$$

Note that since $\text{dom}(F_q) \subseteq \text{dom}(F_p)$, we also have that

$$\text{dom}(F_{q \restriction P}) = \text{dom}(F_q) \cap P \subseteq \text{dom}(F_p) \cap P = \text{dom}(F_{p \restriction P}).$$

By this fact and Lemma 12.5, it follows that $(q \oplus^N p) \restriction P$ and $(q \restriction P) \oplus^{N \cap P} (p \restriction P)$ are defined.

Proof. Let

$$s := (q \oplus^N p) \restriction P$$

and

$$t := (q \restriction P) \oplus^{N \cap P} (p \restriction P).$$

Our goal is to prove that $s = t$.

We have that

$$\begin{aligned} A_s &= A_{(q \oplus^N p) \restriction P} = A_{q \oplus^N p} \cap P = (A_q \cup A_p) \cap P = \\ &= (A_q \cap P) \cup (A_p \cap P) = A_{q \restriction P} \cup A_{p \restriction P} = A_{(q \restriction P) \oplus^{N \cap P} (p \restriction P)} = A_t. \end{aligned}$$

Thus, $A_s = A_t$.

Similarly,

$$\begin{aligned} \text{dom}(F_s) &= \text{dom}(F_{(q \oplus^N p) \restriction P}) = \text{dom}(F_{q \oplus^N p}) \cap P = \\ &= \text{dom}(F_p) \cap P = \text{dom}(F_{p \restriction P}) = \text{dom}(F_{(q \restriction P) \oplus^{N \cap P} (p \restriction P)}) = \text{dom}(F_t). \end{aligned}$$

So $\text{dom}(F_s) = \text{dom}(F_t)$.

Let $i \in \text{dom}(F_s)$, and we will show that $F_s(i) = F_t(i)$. Note that $\text{dom}(F_s) \subseteq P$, so $i \in P$. By definition, we have that

$$F_s(i) = F_{(q \oplus^N p) \restriction P}(i) = F_{q \oplus^N p}(i) \restriction P.$$

The definition of $F_{q \oplus^N p}(i)$ splits into two cases, depending on whether $i \in \text{dom}(F_p) \setminus \text{dom}(F_q)$, or $i \in \text{dom}(F_p) \cap \text{dom}(F_q)$.

First, assume that $i \in \text{dom}(F_p) \setminus \text{dom}(F_q)$. Then $F_{q \oplus^N p}(i) = F_p(i)$ by Definition 11.15. Thus, by the above,

$$F_s(i) = F_p(i) \restriction P.$$

Since $i \notin \text{dom}(F_q)$, also $i \notin \text{dom}(F_q) \cap P = \text{dom}(F_{q \restriction P})$. Thus, by Definition 11.15,

$$F_t(i) = F_{(q \restriction P) \oplus^{N \cap P} (p \restriction P)}(i) = F_{p \restriction P}(i) = F_p(i) \restriction P = F_s(i).$$

Secondly, assume that $i \in \text{dom}(F_p) \cap \text{dom}(F_q)$. Then $i \in N$. By Definition 11.15 and the above,

$$F_s(i) = F_{q \oplus^N p}(i) \restriction P = (F_q(i) \oplus_N F_p(i)) \restriction P.$$

Also, $i \in \text{dom}(F_p) \cap \text{dom}(F_q) \cap P = \text{dom}(F_{p \restriction P}) \cap \text{dom}(F_{q \restriction P})$. So by Definition 11.15,

$$\begin{aligned} F_t(i) &= F_{(q \restriction P) \oplus^{N \cap P} (p \restriction P)}(i) = F_{q \restriction P}(i) \oplus_{N \cap P} F_{p \restriction P}(i) = \\ &= (F_q(i) \restriction P) \oplus_{N \cap P} (F_p(i) \restriction P). \end{aligned}$$

Thus, to show that $F_s(i) = F_t(i)$, it suffices to show that

$$(F_q(i) \oplus_N F_p(i)) \restriction P = (F_q(i) \restriction P) \oplus_{N \cap P} (F_p(i) \restriction P).$$

This equation follows immediately from Proposition 8.6 for the conditions $F_q(i)$ and $F_p(i)$, so it is enough to verify that the assumptions of Proposition 8.6 hold.

Since $i \in P$, $P \cap \kappa \notin S_i$. As $i \in N \cap P$ and $p \in D(N) \cap D(P)$, $F_p(i) \in D_{i,N} \cap D_{i,P}$. Since $q \in N \cap D(P)$, $F_q(i) \in N \cap D_{i,P}$. And as $q \leq p \restriction N$, $F_q(i) \leq F_p \restriction N(i) = F_p(i) \restriction N$. Thus, all of the assumptions of Proposition 8.6 are true, and we are done. $\square$

Theorem 12.7. *Let $P \in \mathcal{Y}$ be simple, $P \prec (H(\lambda), \in, \mathbb{Q})$, and suppose that for all $i \in P \cap \lambda^*$, $P \cap \kappa \notin S_i$. Then $P \cap \mathbb{Q}$ forces that $\mathbb{Q}/\dot{G}_{P \cap \mathbb{Q}}$ has the ω_1 -approximation property.*

Recall that by Corollary 11.20, $P \cap \mathbb{Q}$ is a regular suborder of $\mathbb{Q}$.

The proof of this theorem is almost identical in several places to the proof of Theorem 9.2. In those places, we will ask the reader to refer to the proof of Theorem 9.2 for some of the details instead of repeating everything here.

Proof. By Lemma 1.4, it suffices to show that $\mathbb{Q}$ forces that the pair

$$(V[\dot{G}_{\mathbb{Q}} \cap P], V[\dot{G}_{\mathbb{Q}}])$$

has the ω_1 -approximation property. So let p , μ , and $\dot{k}$ be given such that μ is an ordinal, and p forces in $\mathbb{Q}$ that $\dot{k} : \mu \rightarrow On$ is a function satisfying that for any countable set a in $V[\dot{G}_{\mathbb{Q}} \cap P]$, $\dot{k} \restriction a \in V[\dot{G}_{\mathbb{Q}} \cap P]$. We will find an extension of p which forces that $\dot{k}$ is in $V[\dot{G}_{\mathbb{Q}} \cap P]$.

Fix a regular cardinal θ large enough so that $\mathbb{Q}$, μ , and $\dot{k}$ are members of $H(\theta)$. By the stationarity of the simple models in $\mathcal{X}$ as described in Assumption 2.22, fix a countable set $M^* \prec H(\theta)$ such that M^* contains the parameters $\mathbb{Q}$, P , p , μ , and $\dot{k}$, and satisfies that $M^* \cap H(\lambda)$ is in $\mathcal{X}$ and is simple.

Let $M := M^* \cap H(\lambda)$. Note that since $\mathbb{Q} \subseteq H(\lambda)$, $M \cap \mathbb{Q} = M^* \cap \mathbb{Q}$. In particular, $p \in M \cap \mathbb{Q}$. Also, note that since $P \in H(\lambda)$, we have that $P \in M$ and $M \cap P = M^* \cap P$.

By Lemma 11.1, fix $p_0 \leq p$ such that $M \in A_{p_0}$. By the choice of p and $\dot{k}$, and since $M^* \cap \mu$ is in V , we can fix $p_1 \leq p_0$ and a $(P \cap \mathbb{Q})$ -name $\dot{s}$ such that

$$p_1 \Vdash_{\mathbb{Q}} \dot{k} \restriction (M^* \cap \mu) = \dot{s}^{\dot{G}_{\mathbb{Q}} \cap P}.$$

Since $M \in A_{p_1}$, by Lemma 12.2 we can fix $p_2 \leq p_1$ such that $p_2 \in D(M) \cap D(P)$.

Since $p_2 \leq p$ and $p \in M$, it follows that $p_2 \restriction M \leq p$ by Lemma 11.13(1). So it suffices to prove that $p_2 \restriction M$ forces that $\dot{k}$ is in $V[\dot{G}_{\mathbb{Q}} \cap P]$.

Claim 1: If $t \leq p_2$ is in $D(P)$, $\nu \in M^* \cap \mu$, and $t \Vdash_{\mathbb{Q}} \dot{k}(\nu) = x$ (or $t \Vdash_{\mathbb{Q}} \dot{k}(\nu) \neq x$, respectively) then $t \restriction P \Vdash_{P \cap \mathbb{Q}} \dot{s}(\nu) = x$ (or $t \restriction P \Vdash_{P \cap \mathbb{Q}} \dot{s}(\nu) \neq x$, respectively).

The proof of Claim 1 is identical to the proof of Claim 1 of Theorem 9.2, except that the reference to Proposition 6.15 is replaced with a reference to Proposition 11.18.

Claim 2: For all $q \leq p_2 \restriction M$ in $D(P)$, $\nu < \mu$, and x ,

$$q \Vdash_{\mathbb{Q}} \dot{k}(\nu) = x \implies \forall r \in \mathbb{Q}((r \leq p_2 \restriction M \wedge r \leq q \restriction P) \implies (r \Vdash_{\mathbb{Q}} \dot{k}(\nu) = x)).$$

Note that $p_2 \upharpoonright M$, P , $D(P)$, μ , $\dot{k}$, and $\mathbb{Q}$ are in M^* . So by the elementarity of M^* , it suffices to show that the statement holds in M^* .

Suppose for a contradiction that there exists $q \leq p_2 \upharpoonright M$ in $M^* \cap D(P)$, $\nu \in M^* \cap \mu$, and $x \in M^*$ such that

$$q \Vdash_{\mathbb{Q}} \dot{k}(\nu) = x,$$

but there is $r_0 \in M^* \cap \mathbb{Q}$ with $r_0 \leq p_2 \upharpoonright M$ and $r_0 \leq q \upharpoonright P$ such that

$$r_0 \nVdash_{\mathbb{Q}} \dot{k}(\nu) = x.$$

By the elementarity of M^* , we can fix $r \leq r_0$ in $M^* \cap D(P)$ such that

$$r \Vdash_{\mathbb{Q}} \dot{k}(\nu) \neq x.$$

Then $r \leq p_2 \upharpoonright M$ and $r \leq q \upharpoonright P$. Since $r \leq q \upharpoonright P$ and $q \upharpoonright P \in P$, it follows that $r \upharpoonright P \leq q \upharpoonright P$ by Lemma 11.13(1).

Observe that if we let

$$q' := q \uplus (\text{dom}(F_r) \setminus \text{dom}(F_q))$$

and

$$r' := r \uplus (\text{dom}(F_q) \setminus \text{dom}(F_r)),$$

then q' and r' satisfy exactly the same properties which we stated that q and r satisfy, and moreover, $\text{dom}(F_{q'}) = \text{dom}(F_{r'})$. Let us check this observation carefully.

Since q and r are in M^* , so are q' and r' . And by Definition 10.7,

$$\text{dom}(F_{q'}) = \text{dom}(F_q) \cup \text{dom}(F_r) = \text{dom}(F_{r'}).$$

Since $q' \leq q \leq p_2 \upharpoonright M$, we have that $q' \leq p_2 \upharpoonright M$. As q and r are in $D(P)$, so are q' and r' by Lemma 11.10. And since $q' \leq q$ and $r' \leq r$, we have that $q' \Vdash_{\mathbb{Q}} \dot{k}(\nu) = x$ and $r' \Vdash_{\mathbb{Q}} \dot{k}(\nu) \neq x$. Finally, $r' \leq r \leq p_2 \upharpoonright M$ implies that $r' \leq p_2 \upharpoonright M$. Letting $y := \text{dom}(F_r) \setminus \text{dom}(F_q)$, the fact that $r \leq q \upharpoonright P$ implies by Lemmas 10.8(4) and 11.17 that

$$r' \leq r \leq (q \upharpoonright P) \uplus (y \cap P) = (q \uplus y) \upharpoonright P = q' \upharpoonright P,$$

so $r' \leq q' \upharpoonright P$. And by Lemma 11.13(1), this last inequality implies that $r' \upharpoonright P \leq q' \upharpoonright P$.

By replacing q and r with q' and r' respectively if necessary, we can assume without loss of generality that $\text{dom}(F_q) = \text{dom}(F_r)$. Let $x := \text{dom}(F_q) \setminus \text{dom}(F_{p_2})$. Define

$$p_3 := p_2 \uplus x.$$

Then by Proposition 11.18, q and r are below $p_3 \upharpoonright M$. Also, $q \oplus^M p_3$ is a condition below q and p_3 , and $r \oplus^M p_3$ is a condition below r and p_3 . Also, by Lemmas 11.6 and 11.10, $p_3 \in D(M) \cap D(P)$.

Since $\text{dom}(F_q)$ and $\text{dom}(F_r)$ are subsets of $\text{dom}(F_{p_3})$, by Proposition 12.6 we have that

$$(q \oplus^M p_3) \upharpoonright P = (q \upharpoonright P) \oplus^{M \cap P} (p_3 \upharpoonright P)$$

and

$$(r \oplus^M p_3) \upharpoonright P = (r \upharpoonright P) \oplus^{M \cap P} (p_3 \upharpoonright P).$$

We would like to apply Lemma 12.1 to $M \cap P$, $p_3 \upharpoonright P$, $q \upharpoonright P$, and $r \upharpoonright P$. Let us check that the assumptions of Lemma 12.1 hold for these objects. By Lemma 2.30, $M \cap P$ is simple. Since $p_3 \in D(M) \cap D(P)$, it follows that $p_3 \upharpoonright P \in D(M \cap P)$ by Lemma 12.3. As q , r , and P are in M^* , we have that $q \upharpoonright P$ and $r \upharpoonright P$ are in

$M^* \cap P = M \cap P$. Finally, we observed above that $r \restriction P \leq q \restriction P$, and $q \leq p_3 \restriction M$ implies that $q \restriction P \leq (p_3 \restriction P) \restriction (M \cap P)$ by Lemma 12.5.

Thus, all of the assumptions of Lemma 12.1 hold. Consequently,

$$(r \restriction P) \oplus^{M \cap P} (p_3 \restriction P) \leq (q \restriction P) \oplus^{M \cap P} (p_3 \restriction P).$$

Combining this with the equalities above, we have that

$$(r \oplus^M p_3) \restriction P \leq (q \oplus^M p_3) \restriction P.$$

We claim that this last inequality is impossible. In fact, we will show that $(r \oplus^M p_3) \restriction P$ and $(q \oplus^M p_3) \restriction P$ are incompatible. This contradiction will complete the proof of Claim 2.

We know that $r \Vdash_{\mathbb{Q}} \dot{k}(\nu) \neq x$, and therefore, since $r \oplus^M p_3 \leq r$, we have that $r \oplus^M p_3 \Vdash_{\mathbb{Q}} \dot{k}(\nu) \neq x$. By Claim 1,

$$(r \oplus^M p_3) \restriction P \Vdash_{P \cap \mathbb{Q}} \dot{s}(\nu) \neq x.$$

Similarly, $q \Vdash_{\mathbb{Q}} \dot{k}(\nu) = x$, and therefore, since $q \oplus^M p_3 \leq q$, we have that $q \oplus^M p_3 \Vdash_{\mathbb{Q}} \dot{k}(\nu) = x$. By Claim 1,

$$(q \oplus^M p_3) \restriction P \Vdash_{P \cap \mathbb{Q}} \dot{s}(\nu) = x.$$

Thus, indeed $(r \oplus^M p_3) \restriction P$ and $(q \oplus^M p_3) \restriction P$ are incompatible, since they force contradictory information. This completes the proof of Claim 2.

The proof that $p_2 \restriction M$ forces that $\dot{k}$ is in $V[\dot{G}_{\mathbb{Q}} \cap P]$ follows from Claim 2 in exactly the same way that the analogous conclusion in Theorem 9.2 followed from Claim 2 there. $\square$

§13. The consistency result

We now fulfill the mission of the paper and prove that it is consistent, relative to the consistency of a greatly Mahlo cardinal, that the approachability ideal $I[\omega_2]$ does not have a maximal set modulo clubs.

We work in a ground model V in which κ is a greatly Mahlo cardinal, $2^\kappa = \kappa^+$, and $\square_\kappa$ holds. The consistency of a greatly Mahlo cardinal easily implies the consistency of these assumptions.

It is a standard fact that κ being greatly Mahlo implies that there exists a sequence $\langle B_i : i < \kappa^+ \rangle$ of stationary subsets of κ satisfying the following properties:

- (1) for each $i < \kappa^+$, for all $\beta \in B_i$, β is strongly inaccessible;
- (2) for each $i < \kappa^+$, for all $\beta \in B_{i+1}$, $B_i \cap \beta$ is stationary in β ;
- (3) for all $i < j < \kappa^+$, there is a club set $C \subseteq \kappa$ such that $B_j \cap C \subseteq B_i$;
- (4) for each $i < \kappa^+$, $B_i \setminus B_{i+1}$ is stationary.

Such a sequence is obtained by iterating the Mahlo operation

$$M(A) := \{\alpha \in \kappa \cap \text{cof}(> \omega) : A \cap \alpha \text{ is stationary in } \alpha\},$$

starting with the set of inaccessibles in κ , and taking diagonal intersections of some form at limit stages. We refer the reader to [1, Section 4] for more information about greatly Mahlo cardinals.

The results of this paper up to now were made in the context of several fixed objects, together with some assumptions about these objects. Specifically, in Section 2 we fixed κ , λ , Λ , $\mathcal{A}$, $\mathcal{X}$, and $\mathcal{Y}$, satisfying the properties described in Notations

2.1, 2.2, 2.3, and 2.4, and Assumptions 2.5, 2.6, 2.19, 2.22, 2.23, and 10.1. In addition, in Section 10 we fixed an ordinal λ^* and a sequence $\langle S_i : i < \lambda^* \rangle$ satisfying Notation 10.2 and Assumption 10.3.

We now specify such objects explicitly and justify the properties which we have been assuming about them. We will refer to our previous paper [5] for some of the definitions and proofs.

The greatly Mahlo cardinal κ which we fixed at the beginning of this section is the cardinal described in Notation 2.1. The cardinal λ described in Notation 2.1 is equal to κ^+ .

We refer to [5, Notation 1.7] for the definition of Λ . In that paper, we have that $\Lambda = C^* \cap \text{cof}(> \omega)$, where C^* is a club subset of κ . The club set C^* , in turn, is defined in terms of a thin stationary set $T^* \subseteq P_{\omega_1}(\kappa)$. We must justify, therefore, the existence of a thin stationary set. But κ is strongly inaccessible, so we can let T^* be equal to the entire set $P_{\omega_1}(\kappa)$. The properties of κ , λ , and Λ described in Notations 2.1 and 2.2 and Assumption 10.1 are now immediate.

We refer to [5, Section 7] for the definitions of $\mathcal{A}$, $\mathcal{X}$, and $\mathcal{Y}$. At the beginning of that section, it is assumed that $2^\kappa = \kappa^+$ and $\square_\kappa$, which are exactly the same assumptions which we made above. Let $\mathcal{A}$ denote the structure which is obtained by expanding the structure on $H(\kappa^+)$ specified in [5, Notation 7.6] by adding the sequence $\langle B_i : i < \kappa^+ \rangle$ as a predicate. This structure has a well-ordering of $H(\kappa^+)$ as a predicate, and therefore has definable Skolem functions. It also has κ and Λ as constants. Thus, the description of $\mathcal{A}$ made in Notation 2.3 is satisfied.

We define $\mathcal{X}$ exactly as in [5, Notation 7.7]. Then by definition, for all $M \in \mathcal{X}$, M is a countable elementary substructure of $\mathcal{A}$. We define $\mathcal{Y}$ as the set of models P which are in the set defined in [5, Notation 7.8] and also satisfy that $\text{cf}(P \cap \kappa) > \omega$. Then by definition, for all $P \in \mathcal{Y}$, P is an elementary substructure of $\mathcal{A}$, $|P| < \kappa$, and $P \cap \kappa \in \kappa$. Thus, the properties of $\mathcal{X}$ and $\mathcal{Y}$ described in Notation 2.4 are satisfied.

The next lemma verifies Assumptions 2.5, 2.6, and 2.19.

Lemma 13.1. (1) *If P and Q are in $\mathcal{Y}$, then $P \cap Q \in \mathcal{Y}$;*
 (2) *if $M \in \mathcal{X}$ and $P \in \mathcal{Y}$, then $M \cap P \in \mathcal{X}$;*
 (3) *if M and N are in $\mathcal{X}$ and $\{M, N\}$ is adequate, then $M \cap N \in \mathcal{X}$;*
 (4) *if $M \in \mathcal{X}$, $\alpha \in \Lambda \cup \{\kappa\}$, and $Sk(\alpha) \cap \kappa = \alpha$ if $\alpha < \kappa$, then $M \cap \alpha \in Sk(\alpha)$.*

Proof. (1), (2), and (3) follow immediately from [5, Lemma 7.16]. (4) Note that $Sk(\kappa) \cap \kappa = \kappa$. Let $M \in \mathcal{X}$ and $\alpha \in \Lambda \cup \{\kappa\}$ be as in (4). Since κ is strongly inaccessible, for all $\beta < \alpha$, the cardinality of $P(\beta)$ is in $Sk(\alpha) \cap \kappa = \alpha$ by elementarity. So again by elementarity, $P(\beta) \subseteq Sk(\alpha)$. As M is countable and $\text{cf}(\alpha) > \omega$, $M \cap \alpha \in P(\beta)$ for some $\beta < \alpha$. Hence, $M \cap \alpha \in Sk(\alpha)$. $\square$

In [5, Definition 7.18], the notion of a simple model in $\mathcal{X} \cup \mathcal{Y}$ is defined. This notion is different from what we are calling simple in this paper, so let us momentarily refer to the property from [5, Definition 7.18] as strongly simple. By [5, Lemma 8.2], a set in $\mathcal{X} \cup \mathcal{Y}$ which is strongly simple is also simple in the sense that we are using in the present paper.

The next lemma verifies Assumption 2.22.

Lemma 13.2. *There are stationarily many sets $N \in P_{\omega_1}(H(\kappa^+))$ such that $N \in \mathcal{X}$ and N is simple.*

Proof. By [5, Proposition 7.20], there are stationarily many strongly simple models in $\mathcal{X}$. Since strongly simple implies simple, there are stationarily many simple models in $\mathcal{X}$. $\square$

The next lemma gives a sufficient criterion for a set P being a simple model in $\mathcal{Y}$.

Lemma 13.3. *Suppose that $P \in P_\kappa(H(\kappa^+))$ and P satisfies:*

- (1) $P \prec \mathcal{A}$;
- (2) $P \cap \kappa \in \kappa$;
- (3) $\text{cf}(\sup(P \cap \kappa^+)) = P \cap \kappa$.

Then $P \in \mathcal{Y}$ and P is simple.

Proof. By [5, Lemma 7.15], assumptions (1), (2), and (3) imply that $P \in \mathcal{Y}$. By [5, Lemma 8.3], assumption (3) implies that P is strongly simple, and hence simple. $\square$

The next lemma justifies Assumption 2.23.

Lemma 13.4. *The set of $P \in P_\kappa(H(\kappa^+))$ such that $P \in \mathcal{Y}$ and P is simple is stationary.*

Proof. Given a function $F : H(\kappa^+)^{<\omega} \rightarrow H(\kappa^+)$, build a membership increasing and continuous chain $\langle P_i : i < \kappa \rangle$ of elementary substructures of $\mathcal{A}$ which have size less than κ and are closed under F . Then there is a club of α such that $P_\alpha \cap \kappa = \alpha$. Fix a strongly inaccessible cardinal α in this club. Then P_α is closed under F and satisfies properties (1), (2), and (3) of Lemma 13.3. Hence, P_α is in $\mathcal{Y}$ and is simple. $\square$

Let the ordinal λ^* from Notation 10.2 be equal to κ^+ . Define, for each $i < \kappa^+$,

$$S_i := (B_i \setminus B_{i+1}) \cap C^* \cap C',$$

where C^* and C' are club subsets of κ such that $\Lambda = C^* \cap \text{cof}(> \omega)$, and for all $\alpha \in C'$, $Sk(\alpha) \cap \kappa = \alpha$. By the properties described at the beginning of the section for $\langle B_i : i < \kappa^+ \rangle$, each S_i is a stationary subset of $\kappa \cap \text{cof}(> \omega)$, and for all $\alpha \in S_i$, $\alpha \in \Lambda$ and $Sk(\alpha) \cap \kappa = \alpha$.

Consider $i < j < \kappa^+$. We claim that there is a club set $C_{i,j} \subseteq \kappa$, which is definable in $\mathcal{A}$ from i and j , such that $S_i \cap S_j \cap C_{i,j} = \emptyset$. Since $i + 1 \leq j$, we know that there exists a club C such that $B_j \cap C \subseteq B_{i+1}$. Let $C_{i,j}$ be the least such club in the well-ordering of $H(\kappa^+)$ which is a predicate of $\mathcal{A}$. Since $S_j \subseteq B_j$, $S_j \cap C_{i,j} \subseteq B_{i+1}$. But by definition, B_{i+1} is disjoint from S_i . Thus, $S_i \cap S_j \cap C_{i,j} = \emptyset$. This completes the verification of the properties described in Notation 10.2 and Assumption 10.3.

Finally, for each $i < \kappa^+$, let $\mathbb{P}_i$ denote the forcing poset defined in Definition 4.2 for adding a partial square sequence on S_i , and let $\mathbb{Q}$ be the product forcing defined in Definition 10.6.

This completes the choice of all of the background objects and the verification of all of the assumptions which we made about them. By Corollary 11.22, the forcing poset $\mathbb{Q}$ preserves ω_1 , is κ -c.c., forces that κ is equal to ω_2 , and forces that for all $i < \kappa^+$, $S_i \in I[\omega_2]$.

It remains to show that $\mathbb{Q}$ forces that $I[\omega_2]$ does not have a maximal set modulo clubs. We will prove a technical lemma about names and then finish the proof of the consistency result.

Lemma 13.5. *Suppose that $P \in \mathcal{Y}$ is simple and $P \prec (H(\kappa^+), \in, \mathbb{Q})$. Let $\dot{a} \in P$ be a nice $\mathbb{Q}$ -name for a countable subset of κ . Then for any generic filter G on $\mathbb{Q}$, $\dot{a}^G \in V[G \cap P]$.*

Recall that by Corollary 11.20, $P \cap \mathbb{Q}$ is a regular suborder of $\mathbb{Q}$. Therefore, $G \cap P$ is a generic filter on $P \cap \mathbb{Q}$ and $V[G \cap P] \subseteq V[G]$.

Proof. Let $\alpha := P \cap \kappa$. Since $\mathbb{Q}$ is κ -c.c., by elementarity we can fix a set $b \in P$ such that b is a bounded subset of κ and $\mathbb{Q}$ forces that $\dot{a} \subseteq b$. Note that $b \subseteq \alpha$.

Since $\dot{a}$ is a nice name, for each $\gamma < \kappa$ there is a unique antichain A_γ such that

$$(p, \gamma) \in \dot{a} \iff p \in A_\gamma.$$

Moreover, as $\mathbb{Q}$ forces that $\dot{a} \subseteq \alpha$, $A_\gamma = \emptyset$ for all $\gamma \in \kappa \setminus \alpha$. Let $\gamma < \alpha$. Since $\dot{a} \in P$, by elementarity A_γ is in P . As $\mathbb{Q}$ is κ -c.c., $|A_\gamma| < \kappa$. Since $P \cap \kappa \in \kappa$, we have that $A_\gamma \subseteq P \cap \mathbb{Q}$. It follows that $\dot{a}$ is actually a $(P \cap \mathbb{Q})$ -name. Since $P \cap \mathbb{Q}$ is a regular suborder of $\mathbb{Q}$, $\dot{a}^G = \dot{a}^{G \cap P}$. Thus, $\dot{a}^G \in V[G \cap P]$. $\square$

Theorem 13.6. *The forcing poset $\mathbb{Q}$ forces that the approachability ideal $I[\omega_2]$ does not have a maximal set modulo clubs.*

Proof. Suppose for a contradiction that there is a condition p and a sequence $\vec{a} = \langle \dot{a}_i : i < \kappa \rangle$ of $\mathbb{Q}$ -names for countable subsets of κ such that p forces that $S_{\vec{a}}$ is a maximal set in $I[\omega_2]$ modulo clubs. This means that p forces that whenever $S \in I[\omega_2]$, then there is a club $C \subseteq \omega_2$ such that $S \cap C \subseteq S_{\vec{a}}$.

Without loss of generality, we may assume that each $\dot{a}_i$ is a nice $\mathbb{Q}$ -name for a countable subset of κ . Since $\mathbb{Q}$ is κ -c.c., it follows that each name $\dot{a}_i$, and therefore the entire sequence of names $\vec{a}$, is a member of $H(\kappa^+)$.

In the ground model V , fix a set X satisfying:

- (1) $X \prec \mathcal{A}$ and $X \prec (H(\kappa^+), \in, \mathbb{Q}, \vec{a})$;
- (2) $|X| = \kappa$;
- (3) $\tau := X \cap \kappa^+$ is an ordinal in κ^+ ;
- (4) $X^{<\kappa} \subseteq X$.

This is possible since κ is strongly inaccessible. Note that by (3) and (4), $\text{cf}(\tau) = \kappa$.

Fix a membership increasing and continuous sequence $\langle P_i : i < \kappa \rangle$ of sets of size less than κ , whose union is equal to X , such that each P_i is an elementary substructure of $\mathcal{A}$ and $(H(\kappa^+), \in, \mathbb{Q}, \vec{a})$. This is possible since $X^{<\kappa} \subseteq X$ by (4). Note that by elementarity, $\sup(P_i \cap \kappa^+) \in P_{i+1}$ for all $i < \kappa^+$.

Using the properties of the sequence $\langle B_i : i < \kappa^+ \rangle$ described at the beginning of this section, we can fix, for each $i < \tau$, a club set $C_i \subseteq \kappa$ such that

$$B_\tau \cap C_i \subseteq B_i.$$

Define a function $F : \tau \times \kappa \rightarrow \kappa$ by letting

$$F(i, \gamma) := \min(C_i \setminus \gamma),$$

for all $(i, \gamma) \in \tau \times \kappa$. Note that $F \subseteq X$. Also, fix a club $C^\tau \subseteq \kappa$ such that $B_{\tau+1} \cap C^\tau \subseteq B_\tau$.

Fix a club $D \subseteq \kappa$ such that for all $\beta \in D$, $P_\beta \cap \kappa = \beta$, $\beta \in C^\tau$, and P_β is closed under the function F .

Claim 1: Suppose that $\beta \in D$ and β is strongly inaccessible. Then $P_\beta \in \mathcal{Y}$, P_β is simple, $P_\beta \prec (H(\kappa^+), \in, \mathbb{Q})$, $\beta \in C^\tau$, and for all $i \in P_\beta \cap \kappa^+$, $\beta \in C_i$.

The fact that P_β is closed under F easily implies that for all $i \in P_\beta \cap \kappa^+ = P_\beta \cap \tau$, $P_\beta \cap \kappa = \beta$ is a limit point of C_i , and therefore is in C_i . And $\beta \in C^\tau$ by the definition of D . The set P_β is an elementary substructure of $\mathcal{A}$ and $(H(\kappa^+), \in, \mathbb{Q})$ by the choice of the sequence $\langle P_i : i < \kappa \rangle$. Since P_β is the union of the sequence $\langle P_i : i < \beta \rangle$, and $\sup(P_i \cap \kappa^+) \in P_{i+1}$ for all $i < \beta$, it follows that

$$\text{cf}(\sup(P_\beta \cap \kappa^+)) = \text{cf}(\beta) = \beta = P_\beta \cap \kappa.$$

By Lemma 13.3, it follows that $P_\beta \in \mathcal{Y}$ and P_β is simple. This completes the proof of Claim 1.

Let G be a generic filter on $\mathbb{Q}$ with $p \in G$. For each $i < \kappa$, let $a_i := \dot{a}_i^G$, and let $\vec{a} := \langle a_i : i < \kappa \rangle$. Then by assumption, in $V[G]$ the set $S_{\vec{a}}$ is maximal in $I[\omega_2]$ modulo clubs. In $V[G]$ the set $S_{\tau+1}$ is in $I[\omega_2]$. Since $S_{\vec{a}}$ is maximal modulo clubs, fix a club $C \subseteq \kappa$ in $V[G]$ such that

$$S_{\tau+1} \cap C \subseteq S_{\vec{a}}.$$

As the set $S_{\tau+1}$ is stationary in $V[G]$, we can fix

$$\beta \in S_{\tau+1} \cap \lim(D) \cap C.$$

Then $\beta \in S_{\tau+1} \cap C \subseteq S_{\vec{a}}$. Let $P := P_\beta$.

Note that since $\beta \in S_{\tau+1}$, β is strongly inaccessible in V . Also, $\beta \in D$. So by Claim 1, $P \in \mathcal{Y}$, P is simple, $P \prec (H(\kappa^+), \in, \mathbb{Q})$, $\beta \in C^\tau$, and for all $i \in P \cap \kappa^+$, $\beta \in C_i$.

Since $\beta \in C^\tau$, $S_{\tau+1} \subseteq B_{\tau+1}$, and $B_{\tau+1} \cap C^\tau \subseteq B_\tau$, it follows that $\beta \in B_\tau$. Also, for all $i \in P \cap \kappa^+$, $\beta \in B_\tau \cap C_i \subseteq B_i$, so $\beta \in B_i$. Therefore, for all $i \in P \cap \kappa^+$, since $i+1 \in P \cap \kappa^+$ by elementarity, $\beta \in B_{i+1}$. But $S_i \subseteq B_i \setminus B_{i+1}$. So for all $i \in P \cap \kappa^+$, $P \cap \kappa = \beta \notin S_i$.

By Corollary 11.20 and Theorem 12.7, it follows that $P \cap \mathbb{Q}$ is a regular suborder of $\mathbb{Q}$ and $P \cap \mathbb{Q}$ forces that $\mathbb{Q}/\dot{G}_{P \cap \mathbb{Q}}$ has the ω_1 -approximation property.

Claim 2: The forcing poset $P \cap \mathbb{Q}$ is β -c.c.

Let A be an antichain of $P \cap \mathbb{Q}$, and we will prove that $|A| < \beta$. Without loss of generality, assume that A is maximal. Since β is strongly inaccessible and is a limit point of D , we have that $D \cap \beta$ is a club subset of β . As P is the union of the $\subseteq$ -increasing and continuous sequence $\langle P_i : i < \beta \rangle$, there is a club $E \subseteq \beta$ such that for all $\gamma \in E$:

- (1) $\gamma \in D \cap \beta$;
- (2) $P_\gamma \prec (P, \in, P \cap \mathbb{Q}, A)$.

Now $\beta \in B_{\tau+1}$, which implies that $B_\tau \cap \beta$ is stationary in β . Since E is a club subset of β , we can fix $\gamma \in E \cap B_\tau$. Then in particular, $\gamma \in D$ and γ is strongly inaccessible.

By Claim 1, it follows that $P_\gamma \in \mathcal{Y}$, P_γ is simple, and $P_\gamma \prec (H(\kappa^+), \in, \mathbb{Q})$. Therefore, by Corollary 11.20, $P_\gamma \cap \mathbb{Q}$ is a regular suborder of $\mathbb{Q}$. Since $P_\gamma \cap \mathbb{Q} \subseteq P \cap \mathbb{Q}$, it follows from Lemma 1.1 that $P_\gamma \cap \mathbb{Q}$ is a regular suborder of $P \cap \mathbb{Q}$.

We claim that $P_\gamma \cap A$ is a maximal antichain of $P_\gamma \cap \mathbb{Q}$. It is obviously an antichain. Let $v \in P_\gamma \cap \mathbb{Q}$. Then since A is a maximal antichain of $P \cap \mathbb{Q}$, there

is $s \in A$ such that s is compatible in $P \cap \mathbb{Q}$ with v . But P_γ is an elementary substructure of $(P, \in, P \cap \mathbb{Q}, A)$. So by elementarity, there is $s \in A \cap P_\gamma$ which is compatible in $P \cap \mathbb{Q}$ with v . Again by elementarity, s and v are compatible in $P_\gamma \cap \mathbb{Q}$. This completes the proof that $P_\gamma \cap A$ is a maximal antichain of $P_\gamma \cap \mathbb{Q}$.

As $P_\gamma \cap A$ is a maximal antichain of $P_\gamma \cap \mathbb{Q}$ and $P_\gamma \cap \mathbb{Q}$ is a regular suborder of $P \cap \mathbb{Q}$, it follows that $P_\gamma \cap A$ is predense in $P \cap \mathbb{Q}$. But A is a maximal antichain of $P \cap \mathbb{Q}$; therefore, it must be the case that $A = P_\gamma \cap A$. So $A \subseteq P_\gamma$. But as $P_\gamma \in P$, we have that $|P_\gamma| \in P \cap \kappa = \beta$ by elementarity. So $|A| < \beta$. This completes the proof of Claim 2.

Recall that $\beta \in S_{\bar{a}}$ in $V[G]$. Therefore, in $V[G]$ there is a set c which is cofinal in β with order type ω_1 , and for all $\xi < \beta$, $c \cap \xi \in \{a_i : i < \beta\}$. Now for all $i < \beta$, the name $\dot{a}_i$ is in P by elementarity. By Lemma 13.5, it follows that $\dot{a}_i^G = a_i$ is in $V[G \cap P]$.

So c is a cofinal subset of β with order type ω_1 , and every proper initial segment of c is in $V[G \cap P]$. It easily follows that whenever x is a countable set in $V[G \cap P]$, then $c \cap x \in V[G \cap P]$. Since $\mathbb{Q}/(G \cap P)$ has the ω_1 -approximation property in $V[G \cap P]$, we have that $c \in V[G \cap P]$. As c has order type ω_1 and β is a strongly inaccessible cardinal in V , it follows that β is no longer regular in $V[G \cap P]$. But by Claim 2, $P \cap \mathbb{Q}$ is β -c.c., and so $P \cap \mathbb{Q}$ preserves the regularity of β . Since $V[G \cap P]$ is a generic extension of V by the forcing poset $P \cap \mathbb{Q}$, we have a contradiction. $\square$

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