

CLUB ISOMORPHISMS ON HIGHER ARONSZAJN TREES

JOHN KRUEGER

ABSTRACT. We prove the consistency, assuming an ineffable cardinal, of the statement that **CH** holds and any two normal countably closed ω_2 -Aronszajn trees are club isomorphic. This work generalizes to higher cardinals the property of Abraham-Shelah [1] that any two normal ω_1 -Aronszajn trees are club isomorphic, which follows from **PFA**. The statement that any two normal countably closed ω_2 -Aronszajn trees are club isomorphic implies that there are no ω_2 -Suslin trees, so our proof also expands on the method of Laver-Shelah [5] for obtaining the ω_2 -Suslin hypothesis.

One of the earliest applications of iterated forcing was the theorem of Solovay and Tennenbaum [7] on the consistency of Suslin's hypothesis, which states that there does not exist an ω_1 -Suslin tree. Later Abraham and Shelah [1] formulated a strengthening of Suslin's hypothesis, namely, the statement that any two normal ω_1 -Aronszajn trees are club isomorphic, and proved its consistency. The former result on Suslin's hypothesis was proved using the method of finite support iterations of ω_1 -c.c. forcings, and the latter was established using Shelah's technique of countable support iterations of proper forcing. Suslin's hypothesis also follows from $\mathbf{MA}_{\omega_1}$, whereas the Abraham-Shelah property follows from **PFA** but not from $\mathbf{MA}_{\omega_1}$. Using a much more sophisticated argument than appeared in the original consistency proof on Suslin's hypothesis, Laver and Shelah [5] proved the consistency, assuming a weakly compact cardinal, of **CH** together with the ω_2 -Suslin hypothesis, which asserts the nonexistence of an ω_2 -Suslin tree.

In this paper, we generalize the Abraham-Shelah result on club isomorphisms of normal ω_1 -Aronszajn trees to normal countably closed ω_2 -Aronszajn trees, showing that it is consistent with **CH**, assuming an ineffable cardinal, that any two normal countably closed ω_2 -Aronszajn trees are club isomorphic. We note that it is not consistent with **CH** that any two normal ω_2 -Aronszajn are club isomorphic, since club isomorphisms preserve the closure of levels of the tree, and under **CH** there are normal ω_2 -Aronszajn trees which are countably closed and ones which are not closed at any level. Nevertheless, our generalization of Abraham-Shelah's property to ω_2 does indeed imply the nonexistence of an ω_2 -Suslin tree, and thus provides a natural strengthening of the ω_2 -Suslin hypothesis which is analogous to the situation on ω_1 .

The Laver-Shelah consistency proof involved a countable support iteration of specializing forcings, and it employed a weakly compact cardinal κ which is collapsed to ω_2 in order to obtain the κ -c.c. of the iteration. Similarly, we will use a

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countable support forcing iteration at an ineffable cardinal of forcings which add club isomorphisms between pairs of normal countably closed ω_2 -Aronszajn trees. The main complication, besides having to iterate a more complex forcing poset, is replacing the κ -c.c. property of the iteration, which appeared in the Laver-Shelah proof, with κ -properness. This complication is handled by forcing over a stronger large cardinal, namely an ineffable cardinal, as well as employing in the iteration a rudimentary kind of side condition. We propose that the ideas and methods used in our proof could be useful for generalizing other consequences of PFA from ω_1 to higher cardinals.

The first three sections are mostly preliminary, handling strong genericity, the ineffability ideal, and trees respectively. In Section 4 we present a generalization of the Abraham-Shelah forcing for adding club isomorphisms between normal ω_1 -Aronszajn trees to countably closed normal ω_2 -Aronszajn trees. In Section 5 we develop a kind of countable support forcing iteration of the forcing from Section 4. The main difficulty is proving that the forcing iteration preserves ω_2 , and this is handled in Sections 6 and 7. We note that Sections 6 and 7 contain as a proper subset adaptations of all of the arguments from the Laver-Shelah forcing construction [5].

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1. STRONG GENERICITY

In this section, we will develop the material on strongly generic conditions which we will need in the results from Sections 6 and 7, which form the heart of the consistency proof. Much of this material was originally developed by Mitchell [6].

Fix a forcing poset $\mathbb{P}$ for the remainder of the section.

Definition 1.1. *For any set N , a condition $p \in \mathbb{P}$ is said to be strongly $(N, \mathbb{P})$ -generic if for any D which is a dense subset of the forcing poset $N \cap \mathbb{P}$, D is predense below p in $\mathbb{P}$.*

Note that if $\mathbb{P} \in N \prec H(\theta)$ for some infinite cardinal θ , if p is strongly $(N, \mathbb{P})$ -generic, then p is $(N, \mathbb{P})$ -generic in the sense that for any set $D \in N$ which is dense in $\mathbb{P}$, $D \cap N$ is predense below p .

Definition 1.2. *Let N be a set, $q \in \mathbb{P}$, and $s \in N \cap \mathbb{P}$. We will write $*_N^{\mathbb{P}}(q, s)$ to mean that for all $t \leq s$ in $N \cap \mathbb{P}$, q and t are compatible in $\mathbb{P}$.*

Note that if $*_N^{\mathbb{P}}(q, s)$ holds, then for all $s_1 \leq s$ in $N \cap \mathbb{P}$, $*_N^{\mathbb{P}}(q, s_1)$ also holds. And if $*_N^{\mathbb{P}}(q, s)$ holds and $q \leq q_0$, then $*_N^{\mathbb{P}}(q_0, s)$ holds.

Lemma 1.3. *Let N be a set and $p \in \mathbb{P}$. Then p is strongly $(N, \mathbb{P})$ -generic iff for all $q \leq p$ there is $s \in N \cap \mathbb{P}$ such that $*_N^{\mathbb{P}}(q, s)$ holds.*

Proof. See [3, Lemma 2.2]. □

Notation 1.4. *A set N is said to be suitable for $\mathbb{P}$ if for some regular uncountable cardinal θ with $\mathbb{P} \in H(\theta)$, $\mathbb{P} \in N \prec H(\theta)$.*

Notation 1.5. *Let N be suitable for $\mathbb{P}$ and p a strongly $(N, \mathbb{P})$ -generic condition. Let H be a generic filter on $N \cap \mathbb{P}$. In $V[H]$, define a forcing poset $(\mathbb{P}/p)/H$ with underlying set the conditions $q \leq p$ such that q is compatible in $\mathbb{P}$ with every condition in H , and ordered the same as $\mathbb{P}$.*

Lemma 1.6. *Let N be suitable for $\mathbb{P}$ and p a strongly $(N, \mathbb{P})$ -generic condition. Then for all $q \leq p$ and $s \in N \cap \mathbb{P}$, $*_N^{\mathbb{P}}(q, s)$ holds iff $s \Vdash_{N \cap \mathbb{P}} q \in (\mathbb{P}/p)/\dot{G}_{N \cap \mathbb{P}}$.*

Proof. Assume that $*_N^{\mathbb{P}}(q, s)$ holds. Suppose for a contradiction that there is $t \leq s$ in $N \cap \mathbb{P}$ which forces that q is not in $(\mathbb{P}/p)/\dot{G}_{N \cap \mathbb{P}}$. Then we can find $u_1 \leq t$ in $N \cap \mathbb{P}$ and $v \in N \cap \mathbb{P}$ which is incompatible with q in $\mathbb{P}$ such that $u_1 \Vdash_{N \cap \mathbb{P}} v \in \dot{G}_{N \cap \mathbb{P}}$. As u_1 and v are compatible in $N \cap \mathbb{P}$, fix $u_2 \leq u_1, v$ in $N \cap \mathbb{P}$. Since $u_2 \leq v$, u_2 and q are incompatible in $\mathbb{P}$. But $u_2 \leq u_1 \leq t \leq s$, which contradicts $*_N^{\mathbb{P}}(q, s)$.

Now assume that $*_N^{\mathbb{P}}(q, s)$ fails. Then there is $t \leq s$ in $N \cap \mathbb{P}$ which is incompatible with q in $\mathbb{P}$. As $t \Vdash_{N \cap \mathbb{P}} t \in \dot{G}_{N \cap \mathbb{P}}$, we have that $t \Vdash_{N \cap \mathbb{P}} q \notin (\mathbb{P}/p)/\dot{G}_{N \cap \mathbb{P}}$. Hence, s does not force in $N \cap \mathbb{P}$ that $q \in (\mathbb{P}/p)/\dot{G}_{N \cap \mathbb{P}}$. $\square$

Lemma 1.7. *Let N be suitable for $\mathbb{P}$ and p a strongly $(N, \mathbb{P})$ -generic condition.*

- (1) *If $q \leq p$, $s \in N \cap \mathbb{P}$, and q and s are compatible, then there is $t \leq s$ in $N \cap \mathbb{P}$ such that $*_N^{\mathbb{P}}(q, t)$;*
- (2) *if $D \subseteq \mathbb{P}$ is dense below p , then $N \cap \mathbb{P}$ forces that $D \cap (\mathbb{P}/p)/\dot{G}_{N \cap \mathbb{P}}$ is dense in $(\mathbb{P}/p)/\dot{G}_{N \cap \mathbb{P}}$;*
- (3) *if $D \subseteq \mathbb{P}$ is dense below p , then $N \cap \mathbb{P}$ forces that whenever $q \in (\mathbb{P}/p)/\dot{G}_{N \cap \mathbb{P}}$ and $s \in \dot{G}_{N \cap \mathbb{P}}$, then there is $r \in D \cap (\mathbb{P}/p)/\dot{G}_{N \cap \mathbb{P}}$ such that $r \leq q, s$.*

Proof. (1) Fix $r \leq q, s$ in $\mathbb{P}$. Then $r \leq p$. Since p is strongly $(N, \mathbb{P})$ -generic, there is $t \in N \cap \mathbb{P}$ such that $*_N^{\mathbb{P}}(r, t)$ holds. Moreover, by extending t in $N \cap \mathbb{P}$ if necessary, we may assume that t is either below s or is incompatible with s . Since $*_N^{\mathbb{P}}(r, t)$ holds and $r \leq q$, it follows that $*_N^{\mathbb{P}}(q, t)$ holds. We claim that $t \leq s$, which completes the proof. If not, then t is incompatible with s . But t and r are compatible and $r \leq s$, so t and s are compatible.

(2) Suppose for a contradiction that $s \in N \cap \mathbb{P}$, $q \in \mathbb{P}$, and s forces in $N \cap \mathbb{P}$ that $q \in (\mathbb{P}/p)/\dot{G}_{N \cap \mathbb{P}}$ but for all $r \leq q$ in D , r is not in $(\mathbb{P}/p)/\dot{G}_{N \cap \mathbb{P}}$. By Lemma 1.6, $*_N^{\mathbb{P}}(q, s)$ holds. In particular, q and s are compatible, so fix $r \leq q, s$. Since D is dense below p , by extending r if necessary we may assume that $r \in D$. Since $r \leq s$, in particular, r and s are compatible. So by (1), we can fix $t \leq s$ in $N \cap \mathbb{P}$ such that $*_N^{\mathbb{P}}(r, t)$ holds. Then t forces in $N \cap \mathbb{P}$ that $r \leq q$ is in $D \cap (\mathbb{P}/p)/\dot{G}_{N \cap \mathbb{P}}$, which contradicts that $t \leq s$.

(3) Suppose that $u \in N \cap \mathbb{P}$ forces that $q \in (\mathbb{P}/p)/\dot{G}_{N \cap \mathbb{P}}$ and $s \in \dot{G}_{N \cap \mathbb{P}}$. We will find $r \in D$ with $r \leq q, s$ and $z \leq u$ in $N \cap \mathbb{P}$ such that $z \Vdash_{N \cap \mathbb{P}} r \in (\mathbb{P}/p)/\dot{G}_{N \cap \mathbb{P}}$. Clearly u and s are compatible in $N \cap \mathbb{P}$, so fix $u_1 \leq u, s$ in $N \cap \mathbb{P}$. Since $*_N^{\mathbb{P}}(q, u)$ holds by Lemma 1.6, also $*_N^{\mathbb{P}}(q, u_1)$ holds, so u_1 and q are compatible. Fix $q_1 \leq u_1, q$ in $\mathbb{P}$. Then q_1 and u_1 are obviously compatible, so by (1) fix $t \leq u_1$ in $N \cap \mathbb{P}$ such that $*_N^{\mathbb{P}}(q_1, t)$ holds. By Lemma 1.6, t forces in $N \cap \mathbb{P}$ that $q_1 \in (\mathbb{P}/p)/\dot{G}_{N \cap \mathbb{P}}$.

Since D is forced to be dense in $(\mathbb{P}/p)/\dot{G}_{N \cap \mathbb{P}}$ by (2), we can find $z \leq t$ in $N \cap \mathbb{P}$ and $r \leq q_1$ in D such that z forces in $N \cap \mathbb{P}$ that $r \in (\mathbb{P}/p)/\dot{G}_{N \cap \mathbb{P}}$. Then $r \leq q_1 \leq q$, $r \leq q_1 \leq u_1 \leq s$, and $z \leq t \leq u_1 \leq u$. So $r \in D$, $r \leq q, s$, $z \leq u$, and $z \Vdash_{N \cap \mathbb{P}} r \in (\mathbb{P}/p)/\dot{G}_{N \cap \mathbb{P}}$. $\square$

Proposition 1.8. *Let N be suitable for $\mathbb{P}$ and p a strongly $(N, \mathbb{P})$ -generic condition. Suppose that G is a generic filter on $\mathbb{P}$ such that $p \in G$. Let $H := N \cap G$. Then:*

- (1) *H is a V -generic filter on $N \cap \mathbb{P}$;*
- (2) *$G' := G \cap (\mathbb{P}/p)/H$ is a $V[H]$ -generic filter on $(\mathbb{P}/p)/H$;*
- (3) *$V[G] = V[H][G']$.*

Proof. (1) If D is a dense subset of $N \cap \mathbb{P}$, then D is predense below p . Since $p \in G$, we can fix $z \in D \cap G$. But then $z \in G \cap N = H$. So $D \cap H \neq \emptyset$.

To see that H is a filter, note that H is obviously closed upwards in $N \cap \mathbb{P}$. Suppose that u and v are in H . Then u and v are in N . Let E be the set of $w \in N \cap \mathbb{P}$ which are either incompatible in $\mathbb{P}$ with at least one of u or v , or below them both. By the elementarity of N , it is easy to check that E is a dense subset of $N \cap \mathbb{P}$. Using the previous paragraph, fix $w \in E \cap H$. Since G is a filter and u and v are in G , w is compatible with both u and v , and hence by the definition of E , $w \leq u, v$.

(2) To see that G' is a filter, note that it is clearly upwards closed in $(\mathbb{P}/p)/H$. Let u and v be in G' . Since G is a filter, there is $w \leq u, v$ in G . So $w \leq p$. As $H \subseteq G$, w is compatible in $\mathbb{P}$ with every member of H . So $w \in G \cap (\mathbb{P}/p)/H = G'$.

Let D be a dense open subset of $(\mathbb{P}/p)/H$ in $V[H]$, and we will show that $D \cap G' \neq \emptyset$. Fix an $(N \cap \mathbb{P})$ -name $\dot{D}$ for a dense open subset of $(\mathbb{P}/p)/\dot{G}_{N \cap \mathbb{P}}$ such that $D = \dot{D}^H$.

Let E be the set of $s \leq p$ such that for some $u \in N \cap \mathbb{P}$ and $r \in \mathbb{P}$, $s \leq u, r$, and u forces in $N \cap \mathbb{P}$ that $r \in \dot{D}$. We claim that E is dense below p . So let $q \leq p$, and we will find $s \leq q$ in E .

Since p is strongly $(N, \mathbb{P})$ -generic, fix $t \in N \cap \mathbb{P}$ such that $*_{N}^{\mathbb{P}}(q, t)$ holds. Then $t \Vdash_{N \cap \mathbb{P}} q \in (\mathbb{P}/p)/\dot{H}$. Since $\dot{D}$ is forced to be dense, fix $u \leq t$ in $N \cap \mathbb{P}$ and $r \leq q$ such that $u \Vdash_{N \cap \mathbb{P}} r \in \dot{D}$. Then $*_{N}^{\mathbb{P}}(r, u)$ holds. In particular, r and u are compatible. Fix $s \leq r, u$. Then $s \leq q$ and $s \in E$, as witnessed by u and r .

Since E is dense below p , fix $s \in E \cap G$. Fix $u \in N \cap \mathbb{P}$ and r such that $s \leq u, r$, and $u \Vdash_{N \cap \mathbb{P}} r \in \dot{D}$. Then $s \leq p$ and $s \in G$ easily implies that $s \in (\mathbb{P}/p)/H$. So $s \in G'$. Since $s \leq u$ and $s \in G$, $u \in N \cap G = H$. Hence, $r \in \dot{D}^H = D$. As $s \leq r$ and $s \in G'$, $r \in G'$. Thus, $G' \cap D \neq \emptyset$.

(3) follows from the fact that G is the upwards closure in $\mathbb{P}$ of G' , which is easy to check (in fact, G' is just the set of members of G which are less than or equal to p). $\square$

Notation 1.9. We will write $*_{N}^{\mathbb{P}}(p, q, s)$ to mean the conjunction of $*_{N}^{\mathbb{P}}(p, s)$ and $*_{N}^{\mathbb{P}}(q, s)$.

Lemma 1.10. Let N be suitable for $\mathbb{P}$, p a strongly $(N, \mathbb{P})$ -generic condition, and D and E dense below p . Suppose that $*_{N}^{\mathbb{P}}(q, r, s)$ holds, where q and r are below p . Then there is $(p', q', s') \leq (p, q, s)$ such that $p' \in D$, $q' \in E$, p' and q' are below s , and $*_{N}^{\mathbb{P}}(p', q', s')$ holds.¹

Proof. Since $*_{N}^{\mathbb{P}}(p, s)$ holds, p and s are compatible, so fix $p' \leq p, s$ in D . Since $p' \leq s$, p' and s are obviously compatible, so by Lemma 1.7(1), there is $s_0 \leq s$ in $N \cap \mathbb{P}$ such that $*_{N}^{\mathbb{P}}(p', s_0)$. As $*_{N}^{\mathbb{P}}(q, s)$ holds and $s_0 \leq s$ in $N \cap \mathbb{P}$, s_0 and q are compatible, so fix $q' \leq q, s_0$ which is in E . Then q' and s_0 are obviously compatible, so by Lemma 1.7(1) fix $s' \leq s_0$ in $N \cap \mathbb{P}$ such that $*_{N}^{\mathbb{P}}(q', s')$. As $*_{N}^{\mathbb{P}}(p', s_0)$ holds and $s' \leq s_0$ is in $N \cap \mathbb{P}$, $*_{N}^{\mathbb{P}}(p', s')$ holds. To summarize, we have $(p', q', s') \leq (p, q, s)$, $p' \in D$, $q' \in E$, p' and q' are below s , and $*_{N}^{\mathbb{P}}(p', q', s')$ holds. $\square$

¹Throughout the paper, we will adopt the following notation. For a forcing poset $\mathbb{P}$ and tuples of conditions $(p_0, \dots, p_n)$ and $(q_0, \dots, q_n)$ in $\mathbb{P}$, we will write $(q_0, \dots, q_n) \leq (p_0, \dots, p_n)$ to mean that $q_i \leq p_i$ for all $i = 0, \dots, n$.

2. CHAINS OF MODELS AND INEFFABLE CARDINALS

Let κ be a regular uncountable cardinal which is fixed for the remainder of this section. Recall that an *ideal* I on κ is a collection of subsets of κ which includes the emptyset, is closed under subsets, and closed under finite unions. Let I^+ denote the collection of subsets of κ which are not in I . We say that I is *proper* if $I^+ \neq \emptyset$, or equivalently, $\kappa \notin I$. Let I^* denote the dual filter $\{\kappa \setminus A : A \in I\}$.

We will say that a proper ideal I on κ is *normal* if it contains every bounded subset of κ , and whenever $S \in I^+$ and $f : S \rightarrow \kappa$ is a regressive function, then there is $T \subseteq S$ in I^+ on which f is constant. By standard arguments, if I is normal, then I contains every nonstationary set, and is κ -complete, which means that any union of fewer than κ many sets in I is in I . In addition, if I is normal then I^* is closed under diagonal intersections, which means that whenever $\{C_i : i < \kappa\} \subseteq I^*$, then $\Delta\{C_i : i < \kappa\} := \{\alpha < \kappa : \forall i < \alpha, \alpha \in C_i\}$ is in I^* .

We will use ideals in the context of increasing chains of models.

Definition 2.1. *Let us say that a sequence $\langle N_i : i \in S \rangle$ is suitable if for some regular $\theta > \kappa$:*

- (1) *each N_i is an elementary substructure of $H(\theta)$, $\kappa \in N_i$, $N_i \cap \kappa \in \kappa$, and $|N_i| = |N_i \cap \kappa|$;*
- (2) *$N_i \in N_j$ for all $i < j$ in S ;*
- (3) *if $\delta \in S$ is a limit point of S , then $N_\delta = \bigcup\{N_i : i \in S \cap \delta\}$.*

Standard arguments show that if $\langle N_i : i \in S \rangle$ is suitable, then there is a club $C \subseteq \kappa$ such that for all $i \in C \cap S$, $N_i \cap \kappa = i$. In fact, this is true if (2) above is weakened to $N_i \cap \kappa \in N_j$.

Lemma 2.2. *Let I be a normal ideal on κ . Let $\langle N_i : i \in S \rangle$ be suitable, with union N , and $S \in I^+$. Suppose that f, g , and h are functions with domain S such that for all $i \in S$, $f(i) \in N_i$, $g(i) \in N \setminus N_i$, and $h(i)$ is a subset of $N \setminus N_i$ of size less than κ . Then there is $T \subseteq S$ in I^+ such that for all $i < j$ in T , $f(i) = f(j)$, $g(i) \in N_j$, and $h(i) \cap h(j) = \emptyset$.*

Proof. Fix a bijection $F : \kappa \rightarrow N$. A standard argument shows that there is a club $C_0 \subseteq \kappa$ such that for all $i \in C_0 \cap S$, $F[i] = N_i$. Now the function which maps $i \in S \cap C_0$ to $F^{-1}(f(i))$ is regressive, and hence fixed on a set $S' \subseteq S \cap C_0$ in I^+ . Therefore, if $i < j$ are in S' , then $F^{-1}(f(i)) = F^{-1}(f(j))$, and so applying F , $f(i) = f(j)$.

For each $i \in S'$, fix $\gamma_i < \kappa$ such that $h(i) \cup \{g(i)\} \subseteq N_{\gamma_i}$. Let C_1 be the club set of $\xi < \kappa$ such that for all $i \in S' \cap \xi$, $\gamma_i < \xi$. Let $T := S' \cap C_1$, which is in I^+ .

Consider $i < j$ in T . Then i and j are in S' , so $f(i) = f(j)$ as noted above. Also $\gamma_i < j$, so $h(i) \cup \{g(i)\} \subseteq N_j$. Hence, $g(i) \in N_j$. By the choice of h , $h(j) \cap N_j = \emptyset$. As $h(i) \subseteq N_j$, we have that $h(i) \cap h(j) = \emptyset$. $\square$

The cardinal κ is said to be *ineffable* if whenever $\langle A_\alpha : \alpha < \kappa \rangle$ is a sequence such that $A_\alpha \subseteq \alpha$ for all $\alpha < \kappa$, then there exists $A \subseteq \kappa$ such that the set $\{\alpha < \kappa : A \cap \alpha = A_\alpha\}$ is stationary in κ . We recall that if κ is ineffable, then κ is weakly compact. See [2] for more information about ineffable cardinals.

Given a sequence $\vec{A} = \langle A_\alpha : \alpha < \kappa \rangle$ as above, let us say that a set $S \subseteq \kappa$ is *coherent for $\vec{A}$* if for all $\alpha < \beta$ in S , $A_\alpha = A_\beta \cap \alpha$. Note that if S is coherent for $\vec{A}$, then the set $A := \bigcup\{A_\alpha : \alpha \in S\}$ satisfies that $A \cap \alpha = A_\alpha$ for all $\alpha \in S$. It easily

follows that S is coherent for $\vec{A}$ iff there is a set $A \subseteq \kappa$ such that for all $\alpha \in S$, $A \cap \alpha = A_\alpha$. And κ is ineffable iff for any sequence $\vec{A}$ there is a stationary coherent set.

Define the *ineffability ideal* on κ to be the collection of all sets $S \subseteq \kappa$ such that for some sequence $\vec{A}$ as above, S does not contain a stationary subset which is coherent for $\vec{A}$. It is easy to check that this collection is indeed an ideal which contains all nonstationary subsets of κ . Observe that κ is ineffable iff the ineffability ideal on κ is proper. It is also true that if κ is ineffable, then the ineffability ideal on κ is a normal ideal (see [2, Theorem 2.4]).

Assume for the rest of this section that κ is ineffable, and let J denote the ineffability ideal on κ . Note that a set S is in J^+ iff for any sequence $\vec{A}$, S contains a stationary subset which is coherent for $\vec{A}$.

We will use ineffability in the context of suitable sequences. This form of ineffability is described in the next lemma.

Lemma 2.3. *Let $\langle N_\alpha : \alpha \in S \rangle$ be suitable, with union N , and $S \in J^+$. Let $\langle B_\alpha : \alpha \in S \rangle$ be a sequence such that for all $\alpha \in S$, $B_\alpha \subseteq N_\alpha$. Then there exists a set $B \subseteq N$ and a stationary set $T \subseteq S$ such that for all $\alpha \in T$, $B \cap N_\alpha = B_\alpha$.*

Proof. Fix a bijection $F : \kappa \rightarrow N$. By standard arguments, there is a club $C \subseteq \kappa$ such that for all $\alpha \in S \cap C$, $F[\alpha] = N_\alpha$. Then $S \cap C \in J^+$.

Note that for all $\alpha \in S \cap C$, $A_\alpha := F^{-1}[B_\alpha] \subseteq F^{-1}[N_\alpha] = \alpha$. Applying the fact that $S \cap C \in J^+$ to the sequence $\langle A_\alpha : \alpha \in S \cap C \rangle$, there exists a set $A \subseteq \kappa$ and a stationary set $T \subseteq S \cap C$ such that for all $\alpha \in T$, $A \cap \alpha = A_\alpha$. Define $B := F[A]$. Then $B \subseteq N$, and for all $\alpha \in T$,

$$B \cap N_\alpha = F[A] \cap N_\alpha = F[A] \cap F[\alpha] = F[A \cap \alpha] = F[A_\alpha] = F[F^{-1}[B_\alpha]] = B_\alpha.$$

□

The next result describes a consequence of ineffability which will be used later in the paper. Since the proof involves some tedious arguments about names, in order to keep it as simple as possible we deal with names a bit loosely. The interested reader can fill in the complete details.

Proposition 2.4. *Let $\langle N_\alpha : \alpha \in S \rangle$ be a suitable chain of elementary substructures of $H(\theta)$, with union N , and $S \in J^+$, such that for all $\alpha \in S$, $N_\alpha^\omega \subseteq N_\alpha$. Let U be the set of $\alpha \in S$ satisfying the following property: for all countably closed forcing posets $\mathbb{P} \in N_\alpha$ and all nice $\mathbb{P}$ -names $\dot{T} \in N_\alpha$, if $H(\theta)$ models that $\mathbb{P}$ forces that $\kappa = \omega_2$ and $\dot{T}$ is a tree with underlying set κ with no chains of order type κ , then the forcing poset $N_\alpha \cap \mathbb{P}$ forces that $\alpha = \omega_2$ and $N_\alpha \cap \dot{T}$ is a tree with no chains of order type α . Then $S \setminus U$ is in J .*

Proof. Since the set of inaccessibles below κ is in J^* , we may assume without loss of generality that every member of S is inaccessible.

Suppose for a contradiction that $S \setminus U$ is in J^+ . For each $\alpha \in S \setminus U$, fix a counterexample $\mathbb{P}_\alpha$ and $\dot{T}_\alpha$ in N_α . By Lemma 2.2, we can find $W \subseteq S \setminus U$ in J^+ , $\mathbb{P}$, and $\dot{T}$ such that for all $\alpha \in W$, $\mathbb{P}_\alpha = \mathbb{P}$ and $\dot{T}_\alpha = \dot{T}$. For each $\alpha \in W$, since $\mathbb{P}$ is ω_1 -closed and N_α is countably closed, $N_\alpha \cap \mathbb{P}$ is countably closed, and obviously $N_\alpha \cap \dot{T}$ is an $(N_\alpha \cap \mathbb{P})$ -name. An easy argument using the elementarity of N_α shows that $N_\alpha \cap \dot{T}$ is an $(N_\alpha \cap \mathbb{P})$ -name for a tree.

For each $\alpha \in W$, either $N_\alpha \cap \mathbb{P}$ does not force that $\alpha = \omega_2$, or some condition in $N_\alpha \cap \mathbb{P}$ forces in $N_\alpha \cap \mathbb{P}$ that $N_\alpha \cap \dot{T}$ has a chain of order type α . So we can find $W_1 \subseteq W$ in J^+ such that all $\alpha \in W_1$ satisfy the first possibility, or all $\alpha \in W_1$ satisfy the second possibility.

Let us assume that for all $\alpha \in W_1$, there is $p_\alpha \in N_\alpha \cap \mathbb{P}$ which forces in $N_\alpha \cap \mathbb{P}$ that $N_\alpha \cap \dot{T}$ has a chain of order type α . We will omit the other case since the arguments are very similar to those in this case. By Lemma 2.2, we may assume without loss of generality that for some $p \in \mathbb{P}$, $p_\alpha = p$ for all $\alpha \in W_1$. For each $\alpha \in W_1$, fix a nice $(N_\alpha \cap \mathbb{P})$ -name $\dot{b}_\alpha$ for a subset of $N_\alpha \cap T$ which p forces is a chain of order type α . Note that $\dot{b}_\alpha \subseteq N_\alpha$.

Applying Lemma 2.3, fix $b \subseteq N$ and a stationary set $Z \subseteq W_1$ such that for all $\alpha \in Z$, $b \cap N_\alpha = \dot{b}_\alpha$. Then b is a $\mathbb{P}$ -name for a subset of $\dot{T}$. We claim that p forces that b is a chain of $\dot{T}$ of order type κ , which is a contradiction.

First let us see that p forces that b is a chain. Let $q \leq p$ and suppose that q forces that x and y are in b . Then for each of $z \in \{x, y\}$, the set D_z of $r \leq q$ such that $(r^+, \dot{z})$ is in b for some $r \leq r^+$ is dense below q . Let E be the set of conditions below q which decide in $\mathbb{P}$ if and how x and y compare in $\dot{T}$, and note that E is dense below q .

It easily follows that there is a club $C \subseteq \kappa$ such that for all $\alpha \in C \cap Z$, the sets $N_\alpha \cap D_x$, $N_\alpha \cap D_y$, and $N_\alpha \cap E$ are dense below q in $N_\alpha \cap \mathbb{P}$, and moreover, for $z \in \{x, y\}$, each $r \in D_z \cap N_\alpha$ has a witness r^+ as described above also in N_α . Fix $\alpha \in C \cap Z$ such that q , x , and y are in N_α . Then we can find $v \leq q$ in $N_\alpha \cap \mathbb{P}$ such that v is below members of these three dense subsets of $N_\alpha \cap \mathbb{P}$. Then clearly v forces in $N_\alpha \cap \mathbb{P}$ that x and y are in $N_\alpha \cap b = \dot{b}_\alpha$. Since $v \leq p$ and p forces in $N_\alpha \cap \mathbb{P}$ that $\dot{b}_\alpha$ is a chain, v forces in $N_\alpha \cap \mathbb{P}$ that x and y are comparable in $N_\alpha \cap \dot{T}$. By extending v further in $N_\alpha \cap \mathbb{P}$ if necessary, we may assume that v decides how x and y compare. Without loss of generality, assume that $v \Vdash_{N_\alpha \cap \mathbb{P}} x <_{N_\alpha \cap \dot{T}} y$.

We claim that $v \Vdash_{\mathbb{P}} x <_{\dot{T}} y$. If not, then by elementarity there is $w \leq v$ in $N_\alpha \cap \mathbb{P}$ such that $w \Vdash_{\mathbb{P}} x \not<_{\dot{T}} y$. But since $w \leq v$, $w \Vdash_{N_\alpha \cap \mathbb{P}} x <_{N_\alpha \cap \dot{T}} y$. In particular, we can find $w_1 \leq w$ in $N_\alpha \cap \mathbb{P}$ such that $(w_1^+, (\dot{x}, \dot{y})) \in N_\alpha \cap \dot{T}$ for some $w_1 \leq w_1^+$. But then w_1 forces in $\mathbb{P}$ that $x <_{\dot{T}} y$, which is a contradiction.

To see that p forces that b has order type κ , it suffices to show that for all $\beta < \kappa$, p forces that b contains an ordinal above β . Let $q \leq p$ and β be given. Fix $\alpha \in Z$ such that $N_\alpha \cap \kappa = \alpha$ and q and β are in N_α . Since p forces in $N_\alpha \cap \mathbb{P}$ that $\dot{b}_\alpha = N_\alpha \cap b$ has order type α , and $N_\alpha \cap \kappa = \alpha$ is inaccessible, we can find $\gamma > \beta$ in N_α and $r \leq q$ such that $(r^+, \dot{\gamma}) \in \dot{b}_\alpha = N_\alpha \cap b$ for some $r \leq r^+$ in $N_\alpha \cap \mathbb{P}$. Then $(r^+, \dot{\gamma}) \in b$ implies that r forces in $\mathbb{P}$ that $\gamma > \beta$ is in b . $\square$

3. TREES

We assume that the reader is familiar with the basic definitions and facts about trees. We introduce some notation. Let $(T, <_T)$ be a tree. A *chain* in T is a linearly ordered subset of T , and a *branch* of T is a maximal chain in T . For each $x \in T$, we let $\text{ht}_T(x)$ denote the height of x in T . For each ordinal δ , let $T(\delta) := \{x \in T : \text{ht}_T(x) = \delta\}$ denote level δ of T , and $T \restriction \delta := \{x \in T : \text{ht}_T(x) < \delta\}$. The *height* of T is the least ordinal θ such that $T(\theta) = \emptyset$. A branch b of T is *cofinal* if $b \cap T(\delta) \neq \emptyset$ for all δ less than the height of T .

For $x \in T$ and $\gamma \leq \text{ht}_T(x)$, let $p_T(x, \gamma)$ denote the unique node y of T such that $y \leq_T x$ and $\text{ht}_T(y) = \gamma$. A useful observation is that if x and y are in T ,

$\gamma \leq \min(\text{ht}_T(x), \text{ht}_T(y))$, and $p_T(x, \gamma) \neq p_T(y, \gamma)$, then x and y are not comparable in T , and for all $\gamma < \xi \leq \min(\text{ht}_T(x), \text{ht}_T(y))$, $p_T(x, \xi) \neq p_T(y, \xi)$.

Let κ be a regular uncountable cardinal which is fixed for the remainder of the section. A κ -tree is a tree of height κ such that for $\delta < \kappa$, the level $T(\delta)$ has size less than κ . A κ -tree is a κ -Aronszajn tree if it has no cofinal branch (or equivalently, no chain of order type κ). For a set $A \subseteq \kappa$, let $T \upharpoonright A$ denote the subtree of T consisting of all nodes x such that $\text{ht}_T(x) \in A$.

A κ -tree T is *normal* if:

- (1) for every $x \in T$ and $\text{ht}_T(x) < \gamma < \kappa$, there is $y \in T$ with $x <_T y$ and $\text{ht}_T(y) = \gamma$;
- (2) if x and y are distinct nodes of T with the same height δ , where δ is a limit ordinal, then there is $\xi < \delta$ such that $p_T(x, \xi) \neq p_T(y, \xi)$;
- (3) for every node x , there are y and z with $x <_T y$, $x <_T z$, and such that y and z are not comparable in T .

Definition 3.1. Let T and U be κ -trees. A function $f : T \rightarrow U$ is an isomorphism if f is a bijection and for all x and y in T , $x <_T y$ iff $f(x) <_U f(y)$. We say that T and U are isomorphic if there exists an isomorphism from T to U .

It is easy to verify that if $f : T \rightarrow U$ is an isomorphism, then for all $x \in T$, $\text{ht}_T(x) = \text{ht}_U(f(x))$.

Lemma 3.2. Let T and U be κ -trees. Suppose that $f : T \rightarrow U$ is a bijection satisfying that $\text{ht}_T(x) = \text{ht}_U(f(x))$ for all $x \in T$, and such that $x <_T y$ implies $f(x) <_U f(y)$ for all $x, y \in T$. Then f is an isomorphism.

Proof. Assume that x and y are in T and $f(x) <_U f(y)$. We will show that $x <_T y$. Let $\gamma := \text{ht}_T(x)$. Then $\gamma = \text{ht}_T(x) = \text{ht}_U(f(x)) < \text{ht}_U(f(y)) = \text{ht}_T(y)$.

Suppose for a contradiction that $x \not<_T y$. Let $x' := p_T(y, \gamma)$. Then $x' \neq x$. And $\text{ht}_U(f(x')) = \text{ht}_T(x') = \gamma$. But $x' <_T y$ implies that $f(x') <_U f(y)$. So $p_U(f(y), \gamma) = f(x')$. We are assuming that $f(x) <_U f(y)$, so $p_U(f(y), \gamma) = f(x)$. Hence, $f(x) = f(x')$, which contradicts that f is injective. $\square$

Definition 3.3. Let T and U be κ -trees. We say that T and U are club isomorphic if there exists a club set $C \subseteq \kappa$ such that $T \upharpoonright C$ and $U \upharpoonright C$ are isomorphic.

Lemma 3.4. Let T and U be normal κ -trees. Assume that there exists a cofinal set $A \subseteq \kappa$ such that $T \upharpoonright A$ and $U \upharpoonright A$ are isomorphic. Then T and U are club isomorphic.

Proof. Let $C := A \cup \lim(A)$. Then C is a club subset of κ . Fix an isomorphism $f : T \upharpoonright A \rightarrow U \upharpoonright A$. We will show that there is an isomorphism $g : T \upharpoonright C \rightarrow U \upharpoonright C$ such that $f \subseteq g$.

Let $\beta \in C \setminus A$ and $x \in T_\beta$, and we define $g(x)$. Note that $\beta \in \lim(A) \setminus A$. Let $\gamma := \min(A \setminus \beta)$. So $\beta < \gamma$. Using the normality of T , fix $x' \in T(\gamma)$ such that $x <_T x'$. Then $f(x') \in U(\gamma)$. Define $g(x) := p_U(f(x'), \beta)$.

Let us prove that g is well-defined. So consider another $x'' \in T(\gamma)$ such that $x <_T x''$. We claim that $p_U(f(x'), \beta) = p_U(f(x''), \beta)$. Suppose for a contradiction that $p_U(f(x'), \beta) \neq p_U(f(x''), \beta)$. Since U is normal and β is a limit ordinal, there is $\xi < \beta$ such that $p_U(f(x'), \xi) \neq p_U(f(x''), \xi)$. Moreover, as β is a limit point of A , without loss of generality we can assume that $\xi \in A \cap \beta$.

Fix y' and y'' in $T(\xi)$ such that $f(y') = p_U(f(x'), \xi)$ and $f(y'') = p_U(f(x''), \xi)$. Then $y' \neq y''$. As $f(y') <_U f(x')$ and $f(y'') <_U f(x'')$, it follows that $y' <_T x'$ and $y'' <_T x''$. But $x <_T x', x''$, and therefore $y' <_T x$ and $y'' <_T x$. This is impossible since y' and y'' are both on the same level of T and are different.

Now we prove that g is injective. Since f is injective and g preserves the heights of nodes, it suffices to show that if $x_0 \neq x_1$ are in $T(\beta)$, where $\beta \in \lim(A) \setminus A$, then $g(x_0) \neq g(x_1)$. Let $\gamma = \min(A \setminus \beta)$. Fix x'_0 and x'_1 above x_0 and x_1 in T , respectively, of height γ . Then $x'_0 \neq x'_1$, and by the definition of g , $g(x_0) = p_U(f(x'_0), \gamma)$ and $g(x_1) = p_U(f(x'_1), \gamma)$. Since f is injective, $f(x'_0) \neq f(x'_1)$. As T is normal and β is a limit ordinal, there is $\xi \in A \cap \beta$ such that $p_T(x_0, \xi) \neq p_T(x_1, \xi)$. Since f is injective, $f(p_T(x_0, \xi)) \neq f(p_T(x_1, \xi))$. But $f(p_T(x_0, \xi)) = f(p_T(x'_0, \xi)) = p_U(f(x'_0), \xi)$ and $f(p_T(x_1, \xi)) = f(p_T(x'_1, \xi)) = p_U(f(x'_1), \xi)$. So $g(x_0) = p_U(f(x'_0), \beta)$ and $g(x_1) = p_U(f(x'_1), \beta)$ must be different.

To see that g is surjective, let $\beta \in \lim(A) \setminus A$ and consider $y \in U(\beta)$. Let $\gamma := \min(A \setminus \beta)$. Using the normality of U , fix $y' \in U(\gamma)$ such that $y <_U y'$. Then $y' = f(x')$ for some $x' \in T(\gamma)$. Let $x := p_T(x', \beta)$. Then by the definition of g , $g(x) = y$.

So we have that $g : T \restriction C \rightarrow U \restriction C$ is a bijection which extends f , and it is easy to check by the definition of g that $x <_T y$ in $T \restriction C$ implies that $g(x) <_U g(y)$ in $U \restriction C$. We also have by the definition of g that g preserves the heights of nodes. It follows by Lemma 3.2 that g is an isomorphism. $\square$

We now assume for the remainder of the section that $\kappa = \omega_2$. Let us say that an ω_2 -tree T is *standard* if:

- (1) the nodes of T are ordinals in ω_2 ;
- (2) $x <_T y$ implies that $x < y$ as ordinals;
- (3) every level of T has size ω_1 , and every node in T has ω_1 many successors on the next level.

Standard arguments show that for every normal ω_2 -tree T , there is a club $C \subseteq \omega_2$ such that $T \restriction C$ is isomorphic to a standard normal ω_2 -tree.

An ω_2 -tree T is said to be *countably closed* if every countable chain in T has an upper bound. For a limit ordinal $\delta < \omega_2$, we say that level δ of T is *closed* if every cofinal branch of $T \restriction \delta$ has an upper bound. Note that T is countably closed iff for every limit ordinal $\delta < \omega_2$ with cofinality ω , level δ of T is closed.

The goal of the paper is to prove the consistency of the statement that CH holds and any two normal countably closed ω_2 -Aronszajn trees are club isomorphic. Note that by the preceding comments it suffices to show the consistency that any two standard normal countably closed ω_2 -Aronszajn trees are club isomorphic.

An ω_2 -tree T is said to be *special* if there exists a function $f : T \rightarrow \omega_1$ such that $x <_T y$ implies that $f(x) \neq f(y)$, for all x and y in T . Observe that a special ω_2 -tree is an ω_2 -Aronszajn tree.

The next result is well-known. For example, assuming CH, the construction suggested by Exercise III.5.44 of [4] can be used to build countably closed normal special ω_2 -trees, and also normal special ω_2 -trees which are not closed at any level.

Proposition 3.5. *Assume CH. Then there exists a countably closed normal special ω_2 -tree. There also exists a normal special ω_2 -tree which is not closed at any level.*

It is easy to see that if T and U are ω_2 -trees, $C \subseteq \omega_2$ is club, and $T \restriction C$ and $U \restriction C$ are isomorphic, then for all $\delta \in \lim(C)$, level δ of T is closed iff level δ

of U is closed. Therefore, the two trees described in Proposition 3.5 are not club isomorphic. Hence:

Corollary 3.6. *Assuming CH, there are normal ω_2 -Aronszajn trees which are not club isomorphic.*

Recall that an ω_2 -Suslin tree is an ω_2 -tree which has no chains or antichains of size ω_2 . Standard arguments show:

- (1) if S is a normal ω_2 -tree, then S is Suslin iff S has no antichains of size ω_2 ;
- (2) if S is an ω_2 -Suslin tree, then S is not special;
- (3) if there exists an ω_2 -Suslin tree, then there exists a normal ω_2 -Suslin tree.

Proposition 3.7. *Assume CH. Let S be a normal ω_2 -Aronszajn tree. Then there exists a countably closed normal ω_2 -Aronszajn tree U such that S is a subtree of U and for all $x \in S$, $\text{ht}_S(x) = \text{ht}_U(x)$.*

Proof. Using Proposition 3.5, fix a countably closed normal ω_2 -Aronszajn tree T . Without loss of generality, we assume that T has a unique node of height 0. For each branch b of S whose order type has countable cofinality, define $T_b := \{(b, x) : x \in T\}$. (Note that since b is a branch, it is maximal and hence it has no upper bound in S .) If there are no such branches, then S itself is countably closed, and we are done letting $U := S$. So assume that there are.

Define a tree U whose underlying set consists of the nodes of S together with nodes of the form (b, x) , where b is a branch of S whose order type has countable cofinality and $x \in T$. Let $x <_U y$ if either:

- (1) x and y are both in S and $x <_S y$, or
- (2) $x = (b, x_1)$ and $y = (b, y_1)$ for some b and $x_1 <_T y_1$, or
- (3) $x \in b$ and $y = (b, y_1)$ for some $y_1 \in T$.

In other words, we expand S by placing a copy of T above every branch b as above.

It is straightforward to check that U is a tree of height ω_2 . The nodes of U of height α consist of the nodes of S of height α , together with nodes of the form (b, y) , where $\alpha = \text{ot}(b) + \text{ht}_T(y)$. By CH, there are only ω_1 -many possibilities for b such that $\text{ot}(b) \leq \alpha$. Thus, U is an ω_2 -tree. It is easy to see that U is normal. Clearly S is a subtree of U , and for all $x \in S$, the predecessors of x in U are exactly the predecessors of x in S . Thus, $\text{ht}_S(x) = \text{ht}_U(x)$ for all $x \in S$.

Any chain c of U either lies entirely in S , or has a tail consisting of nodes of the form (b, y) for some fixed b . If the order type of c has countable cofinality, then in the first case it has an upper bound by the definition of U , and in the second case it has an upper bound since T is countably closed. Therefore, U is countably closed. If c is cofinal in U , then in the first case this contradicts that S is an Aronszajn tree, and in the second case this contradicts that T is an ω_2 -Aronszajn tree. Hence, U is an ω_2 -Aronszajn tree. $\square$

Recall that the ω_2 -Suslin hypothesis is the statement that there does not exist an ω_2 -Suslin tree.

Theorem 3.8. *Assume that CH holds and any two countably closed normal ω_2 -Aronszajn trees are club isomorphic. Then the ω_2 -Suslin hypothesis holds. In fact, any normal ω_2 -Aronszajn tree is special.*

Proof. Recall that if there exists an ω_2 -Suslin tree, then there exists a normal ω_2 -Suslin tree. So it suffices to prove that any normal ω_2 -Aronszajn tree S is special.

Applying Proposition 3.7, fix a countably closed normal ω_2 -Aronszajn tree U such that S is a subtree of U and $\text{ht}_S(x) = \text{ht}_U(x)$ for all $x \in S$. Using Proposition 3.5, fix a countably closed normal special ω_2 -tree T . Let $h : T \rightarrow \omega_1$ be a function such that $x <_T y$ implies that $h(x) \neq h(y)$ for all x and y in T .

We have that T and U are both countably closed normal ω_2 -Aronszajn trees. So by assumption we can fix a club $C \subseteq \omega_2$ and an isomorphism $f : U \restriction C \rightarrow T \restriction C$. Note that $S \restriction C$ is a normal ω_2 -Aronszajn tree, and since the nodes of S have the same height in S and U , $S \restriction C \subseteq U \restriction C$. Define $h^* : S \restriction C \rightarrow \omega_1$ by $h^*(x) = h(f(x))$.

Let x and y be in $S \restriction C$, and assume that $x <_S y$. Then $x <_U y$. As f is an isomorphism, $f(x) <_T f(y)$. So $h^*(x) = h(f(x)) \neq h(f(y)) = h^*(y)$. Thus, $S \restriction C$ is special. Now in general, if an ω_2 -tree is special on club many levels, then it is special, by a straightforward argument. So S is special. $\square$

4. THE FORCING POSET

Assume that CH holds and T and U are normal countably closed ω_2 -Aronszajn trees whose levels have size ω_1 . We will develop a forcing poset for adding a club isomorphism from T to U . This forcing poset is a natural generalization of the one defined by Abraham and Shelah [1, Section 5] for adding a club isomorphism between two normal ω_1 -Aronszajn trees.

Many of the lemmas and claims in this section have easy proofs, which we will sometimes omit.

Definition 4.1. *The forcing poset $\mathbb{P}(T, U)$ consists of all pairs (A, f) satisfying:*

- (1) *A is a countable subset of $\omega_2 \cap \text{cof}(\omega_1)$;*
- (2) *f is an injective function;*
- (3) *$\text{dom}(f)$ is a countable subset of $T \restriction A$ satisfying:*
 - (a) *if $x \in \text{dom}(f)$, then for all $\gamma < \text{ht}_T(x)$ in A , $p_T(x, \gamma) \in \text{dom}(f)$, and*
 - (b) *if $x \in \text{dom}(f)$ and $\text{ht}_T(x) < \beta \in A$, then there is $y \in \text{dom}(f)$ with $\text{ht}_T(y) = \beta$ and $x <_T y$;*
- (4) *for all $x \in \text{dom}(f)$, $\text{ht}_T(x) = \text{ht}_U(f(x))$;*
- (5) *for all x and y in $\text{dom}(f)$, if $x <_T y$ then $f(x) <_U f(y)$.*

Let $(B, g) \leq (A, f)$ if $A \subseteq B$ and $f \subseteq g$.

If $p \in \mathbb{P}(T, U)$, we will write $p = (A_p, f_p)$. If $B \subseteq A_p$, we will abbreviate $f_p \restriction (T \restriction B)$ as $f_p \restriction B$.

Observe that if $A \subseteq \omega_2 \cap \text{cof}(\omega_1)$ is countable, then $(A, \emptyset) \in \mathbb{P}(T, U)$.

Note that if $p \in \mathbb{P}(T, U)$, then $\text{ran}(f_p)$ is a countable subset of $U \restriction A_p$ satisfying (i) if $x \in \text{ran}(f_p)$, then for all $\gamma < \text{ht}_U(x)$ in A_p , $p_U(x, \gamma) \in \text{ran}(f_p)$, and (ii) if $x \in \text{ran}(f_p)$ and $\text{ht}_U(x) < \beta \in A_p$, then there is $y \in \text{ran}(f_p)$ with $\text{ht}_U(y) = \beta$ and $x <_U y$. As a consequence, one can easily check that the function which maps $p \in \mathbb{P}(T, U)$ to (A_p, f_p^{-1}) is an isomorphism from the forcing poset $\mathbb{P}(T, U)$ to $\mathbb{P}(U, T)$.

Lemma 4.2. *The forcing poset $\mathbb{P}(T, U)$ is countably closed. In fact, if $\langle p_n : n < \omega \rangle$ is a descending sequence of conditions, then (A, f) is the greatest lower bound, where $A := \bigcup \{A_{p_n} : n < \omega\}$ and $f := \bigcup \{f_{p_n} : n < \omega\}$.*

Notation 4.3. *If p is a condition and $\gamma < \omega_2$, let $p \restriction \gamma := (A_p \cap \gamma, f \restriction (A_p \cap \gamma))$.*

Lemma 4.4. *For any $p \in \mathbb{P}(T, U)$ and $\gamma < \omega_2$, $p \restriction \gamma$ is in $\mathbb{P}(T, U)$ and $p \leq p \restriction \gamma$.*

Lemma 4.5. *Let $p \in \mathbb{P}(T, U)$. Suppose that B is a countable subset of $\omega_2 \cap \text{cof}(\omega_1)$ such that $\sup(A_p) < \min(B)$. Then there is $q \leq p$ in $\mathbb{P}(T, U)$ satisfying:*

- (1) $A_q = A_p \cup B$;
- (2) $f_q \upharpoonright A_p = f_p$.

Proof. We first prove the lemma in the case that $B = \{\beta\}$ is a singleton. For each $x \in \text{dom}(f_p)$, construct a cofinal branch b_x in the tree $\text{dom}(f)$ which contains x . Specifically, if A_p has a maximum element, then just pick a node in $\text{dom}(f)$ with height $\max(A_p)$ which is above x , and let b_x be the set of nodes of $T \upharpoonright A_p$ less than or equal to that node. If A_p does not have a maximum element, then inductively build a chain of nodes above x with order type ω and cofinal in $\text{dom}(f)$, and let b_x be the downward closure of that chain in $T \upharpoonright A_p$.

Using the fact that T is countably closed and each branch b_x either has a top element or has an order type with countable cofinality, we can conclude that each b_x has an upper bound in T . Since T is normal, fix $m_x \in T$ which is an upper bound of b_x with height β . Moreover, since $\sup(A_p) < \beta$, each chain b_x has ω_1 many upper bounds in T_β . Hence, we can arrange that if $x \neq y$, then $m_x \neq m_y$ (even if it happens that $b_x = b_y$, which is possible).

For each $x \in \text{dom}(f_p)$, let $c_x := f_p[b_x]$, which is a cofinal branch of $\text{ran}(f_p)$ containing $f(x)$. Since U is countably closed, each chain c_x has an upper bound. As $\sup(A_p) < \beta$, each c_x has ω_1 many upper bounds of height β . So we can fix an upper bound n_x of c_x with height β in such a way that $x \neq y$ implies $n_x \neq n_y$.

Now define q by letting $A_q := A_p \cup \{\beta\}$, $\text{dom}(f_q) := \text{dom}(f_p) \cup \{m_x : x \in \text{dom}(f_p)\}$, $f_q(x) := f_p(x)$ for all $x \in \text{dom}(f_p)$, and $f_q(m_x) := n_x$ for all $x \in \text{dom}(f_p)$. It is easy to check that q is as required.

To prove the lemma in general, enumerate B in increasing order as $\langle \beta_i : i < \delta \rangle$, where $\delta < \omega_1$. Since B is countable and consists of ordinals of cofinality ω_1 , for all $i < \delta$ we have that $\sup(A_p \cup \{\beta_j : j < i\}) < \beta_i$.

Define by induction a descending sequence of conditions $\langle q_i : i \leq \delta \rangle$ as follows. Let $q_0 = p$. Suppose q_i is given, where $i < \delta$ and $A_{q_i} = A_p \cup \{\beta_j : j < i\}$. Then $\sup(A_{q_i}) < \beta_i$. By the case just proven, we can find $q_{i+1} \leq q_i$ satisfying that $\text{dom}(A_{q_{i+1}}) = \text{dom}(A_{q_i}) \cup \{\beta_i\}$ and $f_{q_{i+1}} \upharpoonright A_{q_i} = f_{q_i}$.

Suppose that $\delta_0 \leq \delta$ is a limit ordinal and q_i is defined for all $i < \delta_0$. Let q_{δ_0} be the greatest lower bound of $\langle q_i : i < \delta_0 \rangle$. This completes the construction. It is easy to check that $q := q_\delta$ is as required. $\square$

Lemma 4.6. *Let $p \in \mathbb{P}(T, U)$, $\beta \in A_p$, and $x \in T$ with $\text{ht}_T(x) = \beta$. Then there is $q \leq p$ such that $x \in \text{dom}(f_q)$.*

Proof. If $x \in \text{dom}(f_p)$, then we are done, so assume not. Then we also have that for all $y \in \text{dom}(f_p)$, $x \not\prec_T y$. Extending p if necessary using Lemma 4.5, we may assume without loss of generality that $\max(A_p)$ exists. Let $\beta := \max(A_p)$. We may also assume that $\text{ht}_T(x) = \beta$. For otherwise fix $x' \in T$ with $\text{ht}_T(x') = \beta$ and $x <_T x'$, and note that if $q \leq p$ satisfies that $x' \in \text{dom}(f_q)$, then also $x \in \text{dom}(f_q)$.

Define $b := \{y \in \text{dom}(f_p) : y <_T x\}$ and $c := f_p[b]$ (it is possible that b and c are empty). Let γ be the least member of A_p such that $p_T(x, \gamma) \notin \text{dom}(f_p)$, which exists since $x \notin \text{dom}(f_p)$. Fix an upper bound z_0 of c in U with height γ which is not in $\text{ran}(f_p)$. This is possible by the countable closure and normality of U , together with the fact that $\text{cf}(\gamma) = \omega_1$ whereas c is countable. Fix $z \in U$ with height β

which is above z_0 in U . Note that since $z_0 \notin \text{ran}(f_p)$, the set $\{y : z_0 \leq_U y \leq_U z\}$ is disjoint from $\text{ran}(f_p)$.

Define $A_q := A_p$ and let $\text{dom}(f_q)$ be the union of the disjoint sets $\text{dom}(f_p)$ and $\{y : y \leq_T x, \text{ht}_T(y) \in A \setminus \gamma\}$. Note that every member of $\text{dom}(f_q) \setminus \text{dom}(f_p)$ is an upper bound of b , and also $x \in \text{dom}(f_q)$. Define f_q so that $f_q \upharpoonright \text{dom}(f_p) = f_p$ and for all $y \in \text{dom}(f_q) \setminus \text{dom}(f_p)$, $f_q(y) := p_U(z, \text{ht}_T(y))$. It is straightforward to check that $q = (A_q, f_q)$ is as required. $\square$

Lemma 4.7. *Let $p \in \mathbb{P}(T, U)$, $\beta \in A_p$, and $y \in U$ with $\text{ht}_U(y) = \beta$. Then there is $q \leq p$ such that $y \in \text{ran}(f_q)$.*

Proof. Consider (A_p, f_p^{-1}) , which is in $\mathbb{P}(U, T)$. By the previous lemma applied in $\mathbb{P}(U, T)$, there is $r \leq (A_p, f_p^{-1})$ with $y \in \text{dom}(f_r)$. Then (A_r, f_r^{-1}) is below p in $\mathbb{P}(T, U)$ and satisfies that $y \in \text{ran}(f_r^{-1})$. $\square$

Since $\mathbb{P}(T, U)$ is countably closed, it preserves ω_1 . The issue of whether $\mathbb{P}(T, U)$ preserves ω_2 will be dealt with in Section 7.

Proposition 4.8. *Assume that $\mathbb{P}(T, U)$ preserves ω_2 . Then $\mathbb{P}(T, U)$ forces that T and U are club isomorphic.*

Proof. Let G be a generic filter on $\mathbb{P}(T, U)$. Define $A := \bigcup \{A_p : p \in G\}$ and $f := \bigcup \{f_p : p \in G\}$. An easy density argument using Lemma 4.5 shows that A is unbounded in ω_2 , and similarly Lemmas 4.6 and 4.7 can be used to show that $\text{dom}(f) = T \upharpoonright A$ and $\text{ran}(f) = U \upharpoonright A$.

The definition of the forcing poset $\mathbb{P}(T, U)$ implies that $f : T \upharpoonright A \rightarrow U \upharpoonright A$ is a bijection, $\text{ht}_T(x) = \text{ht}_U(f(x))$ for all $x \in \text{dom}(f)$, and $x <_T y$ in $\text{dom}(f)$ implies that $f(x) <_U f(y)$. By Lemma 3.2, $T \upharpoonright A$ and $U \upharpoonright A$ are isomorphic. By Lemma 3.4, T and U are club isomorphic. $\square$

Definition 4.9. *Let $p \in \mathbb{P}(T, U)$ and $\beta \in A_p$. We say that p is injective on β if for all distinct x and y in $\text{dom}(f_p)$ with height β , there exists $\gamma \in A_p \cap \beta$ such that $p_T(x, \gamma) \neq p_T(y, \gamma)$.*

Lemma 4.10. *Let $p \in \mathbb{P}(T, U)$ and $\beta \in A_p$ be a limit point of $\omega_2 \cap \text{cof}(\omega_1)$. Then there is $q \leq p$ such that q is injective on β .*

Proof. Since T and U are normal, $\text{cf}(\beta) = \omega_1$, and $\text{dom}(f_p)$ and $\text{ran}(f_p)$ are countable, we can find $\gamma < \beta$ with cofinality ω_1 such that:

- (1) $\sup(A_p \cap \beta) < \gamma$;
- (2) for all $x \neq y$ in $\text{dom}(f_p)$ with height β , $p_T(x, \gamma) \neq p_T(y, \gamma)$;
- (3) for all $x \neq y$ in $\text{ran}(f_p)$ with height β , $p_U(x, \gamma) \neq p_U(y, \gamma)$.

Now define q by letting $A_q := A_p \cup \{\gamma\}$,

$$\text{dom}(f_q) := \text{dom}(f_p) \cup \{p_T(x, \gamma) : x \in \text{dom}(f_p), \text{ht}_T(x) = \beta\},$$

$f_q \upharpoonright \text{dom}(f_p) := f_p$, and for all $x \in \text{dom}(f_p)$ with height β ,

$$f_q(p_T(x, \gamma)) := p_U(f_p(x), \gamma).$$

It is easy to check that q is a condition, $q \leq p$, and q is injective on β . $\square$

Lemma 4.11. *Let $\langle p_n : n < \omega \rangle$ be a descending sequence of conditions in $\mathbb{P}(T, U)$ and let $\beta \in A_{p_0}$. Suppose that for all $n < \omega$, p_n is injective on β . Then the greatest lower bound of $\langle p_n : n < \omega \rangle$ is injective on β .*

Lemma 4.12. *Let $p \in \mathbb{P}(T, U)$ and $\beta \in A_p$, and assume that p is injective on β . Suppose that ξ is an ordinal with cofinality ω_1 such that $\sup(A_p \cap \beta) < \xi < \beta$. Then:*

- (1) *for all distinct x and y in $\text{dom}(f_p)$ with height β , $p_T(x, \xi) \neq p_T(y, \xi)$;*
- (2) *for all distinct x and y in $\text{ran}(f_p)$ with height β , $p_U(x, \xi) \neq p_U(y, \xi)$.*

Proof. (1) is almost immediate. For (2), fix x' and y' with $f_p(x') = x$ and $f_p(y') = y$. Then x' and y' are distinct nodes in $\text{dom}(f)$ with height β . Since p is injective on β , fix $\gamma \in A_p \cap \beta$ such that $p_T(x', \gamma) \neq p_T(y', \gamma)$. As f_p is injective, $x_0 := f_p(p_T(x', \gamma))$ and $y_0 := f_p(p_T(y', \gamma))$ are distinct nodes in U of height γ . Also, $x_0 <_U x$ and $y_0 <_U y$. Since $x_0 \neq y_0$, $x_0 <_U p_U(x, \xi) <_U x$, and $y_0 <_U p_U(y, \xi)$, clearly $p_U(x, \xi) \neq p_U(y, \xi)$. $\square$

Lemma 4.13. *Let $p \in \mathbb{P}(T, U)$, $\beta \in A_p$, and assume that p is injective on β . Suppose that $X \subseteq \beta \cap \text{cof}(\omega_1)$ is countable and $\sup(A_p \cap \beta) < \min(X)$. Then there exists $q \leq p$ satisfying:*

- (1) $A_q = A_p \cup X$;
- (2) $f_q \upharpoonright A_p = f_p$.

Proof. We first handle the case in which $X = \{\xi\}$ is a singleton. Define $A_q := A_p \cup \{\xi\}$ and

$$\text{dom}(f_q) := \text{dom}(f_p) \cup \{p_T(x, \xi) : x \in \text{dom}(f_p), \text{ht}_T(x) = \beta\}.$$

Define $f_q \upharpoonright \text{dom}(f_p) := f_p$, and for all $x \in \text{dom}(f_p)$ with height β , $f_q(p_T(x, \xi)) := p_U(f_p(x), \xi)$.

We claim that f_q is well-defined and injective. If $x' \in \text{dom}(f_q)$ with height ξ , then by Lemma 4.12(1) there is a unique $x \in \text{dom}(f_p)$ with height β such that $x' = p_T(x, \xi)$. Thus, f_q is well-defined. If x' and y' are distinct nodes in $\text{dom}(f_q)$ with height ξ , then let x and y be the unique nodes above x' and y' , respectively, in $\text{dom}(f_p)$ with height β . Then obviously $x \neq y$, so $f_p(x) \neq f_p(y)$ since f_p is injective. By Lemma 4.12(2), $f_q(x') = p_U(f_p(x), \xi)$ and $f_q(y') = p_U(f_p(y), \xi)$ are distinct. Thus, f_q is injective. It easily follows that q is as required. Also note that q is injective on β .

To prove the lemma in general, enumerate X in increasing order as $\langle \xi_i : i < \delta \rangle$, where $\delta < \omega_1$. Define a descending sequence $\langle q_i : i \leq \delta \rangle$ as follows. Let $q_0 := p$. Let $i < \delta$ and assume that q_i is defined so that $A_{q_i} = A_p \cup \{\xi_j : j < i\}$ and q_i is injective on β . Since ξ_i has cofinality ω_1 , $\sup(A_{q_i} \cap \beta) < \xi_i$. So by the previous paragraph, we can find $q_{i+1} \leq q_i$ injective on β such that $\text{dom}(q_{i+1}) = \text{dom}(q_i) \cup \{\xi_i\}$ and $f_{q_{i+1}} \upharpoonright \text{dom}(f_{q_i}) = f_{q_i}$.

Let $\delta_0 \leq \delta$ be a limit ordinal, and assume that for all $i < \delta_0$, q_i is defined so that $A_{q_i} = A_p \cup \{\xi_j : j < i\}$ and q_i is injective on β . Let q_{δ_0} be the greatest lower bound of $\langle q_i : i < \delta_0 \rangle$. Then $A_{q_{\delta_0}} = A_p \cup \{\xi_i : i < \delta\}$ and q_{δ_0} is injective on β by Lemma 4.11. This completes the construction. It is easy to see that q_δ is as required. $\square$

Proposition 4.14. *Let (A, f) and (B, g) be in $\mathbb{P}(T, U)$. Assume that $\gamma < \xi$, where $\gamma \in A$ and $\xi \in B$, and the following properties hold:*

- (1) $A \subseteq \xi$;
- (2) $(A, f) \upharpoonright \gamma = (B, g) \upharpoonright \xi$;
- (3) (A, f) is injective on γ and (B, g) is injective on ξ .

Assume, moreover, that every node of $\text{dom}(f)$ with height γ is incomparable in T with every node of $\text{dom}(g)$ with height ξ , and every node of $\text{ran}(f)$ with height γ is incomparable in U with every node in $\text{ran}(g)$ with height ξ . Then (f, A) and (g, B) are compatible.

Proof. Since A is countable and $\text{cf}(\xi) = \omega_1$, $\sup(A) < \xi = \min(B \setminus \xi)$. By Lemma 4.5, we can find $(A', f') \leq (A, f)$ such that $A' = A \cup (B \setminus \xi)$ and $f' \upharpoonright A = f$. By property (2), $A \cap \gamma = B \cap \xi$, so $B \cap \xi \subseteq A$. Hence, $A' = A \cup B$.

Since $B \cap \xi \subseteq \gamma$, B is countable, and $\text{cf}(\gamma) = \omega_1$, it follows that $\sup(B \cap \xi) < \gamma$. So $A \setminus \gamma$ is a countable subset of $\xi \cap \text{cof}(\omega_1)$ and $\sup(B \cap \xi) < \gamma = \min(A \setminus \gamma)$. As (B, g) is injective on ξ , by Lemma 4.13 we can fix $(g', B') \leq (g, B)$ such that $B' = B \cup (A \setminus \gamma)$ and $g' \upharpoonright B = g$. By property (2), $A \cap \gamma = B \cap \xi \subseteq B$. Therefore, $B' = A \cup B$.

Claim 1: If $x \in \text{dom}(f')$ has height at least ξ and $y \in \text{dom}(g')$ has height at least ξ , then x and y are incomparable in T . Suppose for a contradiction that x and y are comparable in T . Then $p_T(x, \gamma) = p_T(y, \gamma)$, and so $p_T(x, \gamma) <_T p_T(y, \xi)$. As $f' \upharpoonright A = f$ and $\gamma \in A$, $p_T(x, \gamma) \in \text{dom}(f)$. And as $g' \upharpoonright B = g$ and $\xi \in B$, $p_T(y, \xi) \in \text{dom}(g)$. So $p_T(x, \gamma) \in \text{dom}(f)$ has height γ and $p_T(y, \xi) \in \text{dom}(g)$ has height ξ , and therefore these nodes are incomparable in T by assumption, giving a contradiction.

Claim 2: If $x \in \text{ran}(f')$ has height at least ξ and $y \in \text{ran}(g')$ has height at least ξ , then x and y are incomparable in U . The proof is similar to that of Claim 1.

Claim 3: Every node x in $\text{dom}(f')$ with $\text{ht}_T(x) \in A \setminus \gamma$ is incomparable in T with every node y in $\text{dom}(g')$ with $\text{ht}_T(y) \in A \setminus \gamma$. Fix z in $\text{dom}(g')$ with height ξ such that $y <_T z$. Since $g' \upharpoonright B = g$, $z \in \text{dom}(g)$. Suppose for a contradiction that x and y are comparable in T . Then $p_T(x, \gamma) = p_T(y, \gamma) <_T y <_T z$. Since $f' \upharpoonright A = f$, $p_T(x, \gamma) \in \text{dom}(f)$. So $p_T(x, \gamma) \in \text{dom}(f)$ has height γ , $z \in \text{dom}(g)$ has height ξ , so by assumption, $p_T(x, \gamma)$ and z are incomparable in T , which is a contradiction.

Claim 4: Every node x in $\text{ran}(f')$ with $\text{ht}_U(x) \in A \setminus \gamma$ is incomparable in U with every node y in $\text{ran}(g')$ with $\text{ht}_U(y) \in A \setminus \gamma$. The proof is similar to that of Claim 3.

Define (C, h) by letting $C := A \cup B$ and $h := f' \cup g'$. We claim that (C, h) is a condition which extends (A, f) and (B, g) . It is easy to see that if (C, h) is a condition, then it extends (A, f) and (B, g) . Clearly C is a countable subset of $\omega_2 \cap \text{cof}(\omega_1)$. By property (2), $f \upharpoonright (A \cap \gamma) = g \upharpoonright (B \cap \xi)$. As $A \subseteq \xi$ by property (1), it easily follows that $f \cup g$ is a function, since any node in $\text{dom}(f) \cap \text{dom}(g)$ has height in $B \cap \xi = A \cap \gamma$, and f and g agree on such nodes.

Since $g' \upharpoonright B = g$ and $f' \upharpoonright A = f$, Claims 1 and 3 imply that any node in $\text{dom}(f') \cap \text{dom}(g')$ is in $\text{dom}(f) \cap \text{dom}(g)$. Namely, any node in $\text{dom}(f') \cap \text{dom}(g')$ has height below ξ by Claim 1, and therefore by Claim 3 and the fact that $A' \cap \xi = A$ has height below γ . Hence, any such node has height in $A' \cap \gamma \subseteq A$ and in $B' \cap \gamma \subseteq B$. Therefore, any such node is in $\text{dom}(f' \upharpoonright A) = \text{dom}(f)$ and in $\text{dom}(g' \upharpoonright B) = \text{dom}(g)$. So $h = f' \cup g'$ is a function.

Note that $\text{ht}_T(x) = \text{ht}_U(h(x))$ for all $x \in \text{dom}(h)$, since this is true of f' and g' . Suppose that x and y are in $\text{dom}(h)$ and $h(x) = h(y)$. Then $\text{ht}_T(x) = \text{ht}_U(h(x)) = \text{ht}_U(h(y)) = \text{ht}_T(y)$, so $\text{ht}_T(x) = \text{ht}_T(y)$. We will show that $x = y$, which proves that h is injective. If x and y are either both in $\text{dom}(f')$ or both in $\text{dom}(g')$, then we are done since f' and g' are injective.

Assume that $x \in \text{dom}(f') \setminus \text{dom}(g')$ and $y \in \text{dom}(g') \setminus \text{dom}(f')$. The case with f' and g' switched follows by a symmetric argument. Then $f'(x) = g'(y)$. Claims 2 and 4 imply that the height of $f'(x) = g'(y)$ is less than γ . So $\text{ht}_T(x) = \text{ht}_T(y) < \gamma$. But then $y \in \text{dom}(g' \restriction \gamma) = \text{dom}(f \restriction \gamma)$, which contradicts our assumption that $y \notin \text{dom}(f')$.

So (C, h) satisfies properties (1), (2), and (4) of Definition 4.1. Since $\text{dom}(h) = \text{dom}(f') \cup \text{dom}(g')$, property (3) follows from the fact that (A', f') and (B', g') are conditions.

It remains to prove property (5). Let $x <_T y$ in $\text{dom}(h)$, and we will prove that $h(x) <_U h(y)$. If x and y are either both in $\text{dom}(f')$ or both in $\text{dom}(g')$, then we are done since (A', f') and (B', g') are conditions. There remains two cases, namely (I) $x \in \text{dom}(f') \setminus \text{dom}(g')$ and $y \in \text{dom}(g') \setminus \text{dom}(f')$, or (II) $x \in \text{dom}(g') \setminus \text{dom}(f')$ and $y \in \text{dom}(f') \setminus \text{dom}(g')$. We claim that neither case is possible.

Since $f' \restriction A = f$ and $A' = A \cup (B \setminus \xi)$, any node of $\text{dom}(f')$ with height below γ has height in $A \cap \gamma$ and is in $\text{dom}(f \restriction \gamma) = \text{dom}(g \restriction \gamma)$. And any node of $\text{dom}(g')$ with height below γ has height in $B' \cap \gamma \subseteq B$, and since $g' \restriction B = g$, is in $\text{dom}(g \restriction \gamma) = \text{dom}(f \restriction \gamma)$. It follows that in either case (I) or (II), $\text{ht}_T(x) \geq \gamma$. But $x <_T y$ implies that $p_T(x, \gamma) = p_T(y, \gamma)$. So $p_T(x, \gamma)$ is in both $\text{dom}(f')$ and $\text{dom}(g')$, and has height $\gamma \in A \setminus \gamma$, contradicting Claim 3. $\square$

5. THE FORCING ITERATION

In this section we will develop the forcing iteration which we will use to prove the consistency of the statement that any two countably closed normal ω_2 -Aronszajn trees are club isomorphic. This forcing will be a kind of countable support iteration of the forcing poset developed in Section 4, together with some rudimentary kind of side conditions.

For the remainder of the paper we fix an inaccessible cardinal κ such that $2^\kappa = \kappa^+$. We define by induction a sequence of forcing posets $\langle \mathbb{P}_i : i \leq \kappa^+ \rangle$. We maintain as inductive hypotheses the assumptions that each $\mathbb{P}_i$ is countably closed, preserves κ , is κ^+ -c.c., and forces that $\kappa = \omega_2$ if $i > 0$. The proof of the preservation of κ is complex and will be handled in Section 7. Our definition of the forcing posets will depend on these inductive hypotheses together with a fixed bookkeeping function which is in the background.

For each $\gamma < \kappa$, let I_γ denote the interval of ordinals $[\omega_1 \cdot \gamma, \omega_1 \cdot (\gamma + 1))$. We will assume that for any tree T we are working with, the nodes on level γ of T belong to I_γ .

We now begin the definition of the forcing iteration. We will let $\mathbb{P}_0$ denote the trivial forcing and $\mathbb{P}_1$ the Levy collapse $\text{Col}(\omega_1, < \kappa)$. However, we will write conditions in $\mathbb{P}_0$ and $\mathbb{P}_1$ in a specific form so they fit in with the later definitions.

So let $\mathbb{P}_0$ denote the trivial forcing whose single element is the pair $(\emptyset, \emptyset)$. Let $\mathbb{P}_1$ be the forcing whose conditions are either $(\emptyset, \emptyset)$, or $(p, \emptyset)$, where p is a function with domain $\{0\}$ such that $p(0) \in \text{Col}(\omega_1, < \kappa)$. Let $(q, \emptyset) \leq (p, \emptyset)$ in $\mathbb{P}_1$ if either $p = \emptyset$, or else p and q are nonempty and $q(0) \leq p(0)$ in $\text{Col}(\omega_1, < \kappa)$. Note that $\mathbb{P}_1$ is countably closed, and since κ is inaccessible, it is also κ -c.c. by a standard Δ -system lemma argument. We also define, for all $(p, \emptyset) \in \mathbb{P}_1$, $(p, \emptyset) \restriction 0 := (\emptyset, \emptyset)$, which is in $\mathbb{P}_0$.

Let $\mathbb{P}_\alpha := \mathbb{P}_1$ for all $\alpha \leq \kappa$. In other words, we do not force with anything between 1 and κ in the iteration.

Now assume that $\kappa \leq \beta < \kappa^+$ and $\mathbb{P}_\beta$ is defined, and we will define $\mathbb{P}_{\beta+1}$. We assume that our bookkeeping function provides us with a pair of nice $\mathbb{P}_\beta$ -names $\dot{T}_\beta$ and $\dot{U}_\beta$ for standard countably closed normal κ -Aronszajn trees. Without loss of generality, we will assume that for all $\gamma < \kappa$, $\mathbb{P}_\beta$ forces that the nodes of $\dot{T}_\beta$ on level γ consist of all even ordinals in I_γ , and the nodes of $\dot{U}_\beta$ on level γ consist of all odd ordinals in I_γ .

Define $\mathbb{P}_{\beta+1}$ to be the set of all pairs $p = (a_p, X_p)$ satisfying:

- (1) a_p is a function whose domain is a countable subset of $\beta + 1$;
- (2) X_p is a function whose domain is a countable subset of $\beta + 2$;
- (3) $p \restriction \beta := (a_p \restriction \beta, X_p \restriction (\beta + 1)) \in \mathbb{P}_\beta$;
- (4) if $\beta \in \text{dom}(a_p)$, then $a_p(\beta)$ is a $\mathbb{P}_\beta$ -name such that $p \restriction \beta$ forces that

$$a_p(\beta) = (A_{a_p(\beta)}, f_{a_p(\beta)}) \in \mathbb{P}(\dot{T}_\beta, \dot{U}_\beta);^2$$

- (5) if $\beta + 1 \in \text{dom}(X_p)$, then $X_p(\beta + 1)$ is a countable subset of

$$\{M \in P_\kappa(\beta + 1) : M \cap \kappa \in \kappa \cap \text{cof}(> \omega)\};$$

- (6) if $M \in X_p(\beta + 1)$ and $\gamma \in M \cap \text{dom}(a_p)$, then

$$p \restriction \gamma \Vdash_{\mathbb{P}_\gamma} M \cap \kappa \in A_{a_p(\gamma)}.$$

Let $q \leq p$ in $\mathbb{P}_{\beta+1}$ if

- (1) $q \restriction \beta \leq p \restriction \beta$ in $\mathbb{P}_\beta$;
- (2) if $\beta \in \text{dom}(a_p)$, then $\beta \in \text{dom}(a_q)$ and

$$q \restriction \beta \Vdash_{\mathbb{P}_\beta} a_q(\beta) \leq a_p(\beta) \text{ in } \mathbb{P}(\dot{T}_\beta, \dot{U}_\beta);$$

- (3) if $\beta + 1 \in \text{dom}(X_p)$, then $\beta + 1 \in \text{dom}(X_q)$ and $X_p(\beta + 1) \subseteq X_q(\beta + 1)$.

Now assume that $\kappa < \alpha < \kappa^+$ is a limit ordinal, and $\mathbb{P}_\beta$ is defined for all $\beta < \alpha$. Define $\mathbb{P}_\alpha$ to be the set of all pairs $p = (a_p, X_p)$ satisfying:

- (1) a_p is a function whose domain is a countable subset of α ;
- (2) X_p is a function whose domain is a countable subset of $\alpha + 1$;
- (3) for all $\beta < \alpha$, $p \restriction \beta := (a_p \restriction \beta, X_p \restriction (\beta + 1))$ is in $\mathbb{P}_\beta$;
- (4) if $\alpha \in \text{dom}(X_p)$, then $X_p(\alpha)$ is a countable subset of

$$\{M \in P_\kappa(\alpha) : M \cap \kappa \in \kappa \cap \text{cof}(> \omega)\};$$

- (5) if $M \in X_p(\alpha)$ and $\gamma \in M \cap \text{dom}(a_p)$, then

$$p \restriction \gamma \Vdash_{\mathbb{P}_\gamma} M \cap \kappa \in A_{a_p(\gamma)}.$$

Let $q \leq p$ if for all $\beta < \alpha$, $q \restriction \beta \leq p \restriction \beta$ in $\mathbb{P}_\beta$, and moreover, if $\alpha \in \text{dom}(X_p)$, then $\alpha \in \text{dom}(X_q)$ and $X_p(\alpha) \subseteq X_q(\alpha)$.

Finally, define $\mathbb{P}_{\kappa^+} := \bigcup \{\mathbb{P}_\alpha : \alpha < \kappa^+\}$, and let $q \leq p$ in $\mathbb{P}_{\kappa^+}$ if for all large enough $\alpha < \kappa^+$, $q \leq p$ in $\mathbb{P}_\alpha$.³

²Since we would like our forcings to be in $H(\kappa^{++})$, we will implicitly assume that $A_{a_p(\beta)}$ and $f_{a_p(\beta)}$ are nice names, and $a_p(\beta)$ is a canonical name for their pair.

³Let $p = (a_p, X_p)$ be a condition in $\mathbb{P}_\alpha$. We will often find it convenient to write $a_p(\beta)$ or $X_p(\gamma)$ for $\beta < \alpha$ and $\gamma \leq \alpha$ without necessarily knowing that $\beta \in \text{dom}(a_p)$ or $\gamma \in \text{dom}(X_p)$. In the case that they are not, by default $a_p(\beta)$ and $X_p(\gamma)$ will denote the empty set.

We will denote the order on $\mathbb{P}_\alpha$ as $\leq_\alpha$, for all $\alpha \leq \kappa^+$.

The definition of the forcing iteration was by necessity an induction. The next lemma provides a useful non-inductive description of conditions in $\mathbb{P}_\alpha$. The proof is a straightforward induction on α .

Lemma 5.1. *Let $\alpha \leq \kappa^+$. A pair (a, X) is in $\mathbb{P}_\alpha$ iff:*

- (1) *a is a function whose domain is a countable subset of α ;*
- (2) *X is a function whose domain is a countable subset of $(\alpha + 1) \cap \kappa^+$;*
- (3) *$(a \restriction \kappa, X \restriction (\kappa + 1)) = (a \restriction 1, \emptyset) \in \mathbb{P}_1$;*
- (4) *for all nonzero $\beta \in \text{dom}(a)$, $a(\beta)$ is a $\mathbb{P}_\beta$ -name such that, assuming that $(a \restriction \beta, X \restriction (\beta + 1))$ is in $\mathbb{P}_\beta$, then this condition forces in $\mathbb{P}_\beta$ that $a(\beta) \in \mathbb{P}(\dot{T}_\beta, \dot{U}_\beta)$;*
- (5) *for all $\gamma \in \text{dom}(X)$, $X(\gamma)$ is a countable subset of*

$$\{M \in P_\kappa(\gamma) : M \cap \kappa \in \kappa \cap \text{cof}(> \omega)\};$$

- (6) *for all $\gamma \in \text{dom}(X)$, if $M \in X(\gamma)$ and $\beta \in M \cap \text{dom}(a)$, then, assuming that $(a \restriction \beta, X \restriction (\beta + 1))$ is in $\mathbb{P}_\beta$, this condition forces in $\mathbb{P}_\beta$ that $M \cap \kappa \in A_{a(\beta)}$.*

Also $(b, Y) \leq_\alpha (a, X)$ iff

- (1) $\text{dom}(a) \subseteq \text{dom}(b)$ and $\text{dom}(X) \subseteq \text{dom}(Y)$;
- (2) for all $\gamma \in \text{dom}(X)$, $X(\gamma) \subseteq Y(\gamma)$;
- (3) assuming $0 \in \text{dom}(a)$, $a(0) \subseteq b(0)$;
- (4) for all nonzero $\beta \in \text{dom}(a)$, $(b \restriction \beta, Y \restriction (\beta + 1)) \Vdash_{\mathbb{P}_\beta} b(\beta) \leq a(\beta)$ in $\mathbb{P}(\dot{T}_\beta, \dot{U}_\beta)$.

Since $2^\kappa = \kappa^+$ and we are assuming as an inductive hypothesis that each $\mathbb{P}_\alpha$ is κ^+ -c.c., it is straightforward to check that $\mathbb{P}_\alpha \in H(\kappa^{++})$ for all $\alpha \leq \kappa^+$.

The proofs of the next two lemmas are straightforward.

Lemma 5.2. *Let $\beta < \alpha \leq \kappa^+$.*

- (1) $\mathbb{P}_\beta$ is a suborder of $\mathbb{P}_\alpha$;
- (2) $p \leq_\alpha p \restriction \beta$ for all $p \in \mathbb{P}_\alpha$;
- (3) if p and q are in $\mathbb{P}_\beta$ and $r \leq_\alpha p, q$, then $r \restriction \beta \leq_\beta p, q$.

Lemma 5.3. *For all $\alpha \leq \kappa^+$, the forcing poset $\mathbb{P}_\alpha$ is countably closed. In fact, given a descending sequence of conditions $\langle q_n : n < \omega \rangle$, let $r = (a_r, X_r)$ satisfy:*

- (1) $\text{dom}(a_r) := \bigcup \{\text{dom}(a_{q_n}) : n < \omega\}$;
- (2) $\text{dom}(X_r) := \bigcup \{\text{dom}(X_{q_n}) : n < \omega\}$;
- (3) for all $\beta \in \text{dom}(a_r)$, $a_r(\beta)$ is a $\mathbb{P}_\beta$ -name for the greatest lower bound in $\mathbb{P}(\dot{T}_\beta, \dot{U}_\beta)$ of $\langle a_{q_n}(\beta) : n < \omega \rangle$ as described in Lemma 4.2;
- (4) for all $\gamma \in \text{dom}(X_r)$, $X_r(\gamma) := \bigcup \{X_{q_n}(\gamma) : n < \omega\}$.

Then r is the greatest lower bound of $\langle q_n : n < \omega \rangle$.

Definition 5.4. *A condition $p \in \mathbb{P}_\alpha$ is said to be determined if for all nonzero $\gamma \in \text{dom}(a_p)$, there is a pair (A, f) in the ground model such that $a_p(\gamma)$ is the canonical $\mathbb{P}_\gamma$ -name for (A, f) .*

Note that if p is determined in $\mathbb{P}_\alpha$, then for all $\beta < \alpha$, $p \restriction \beta$ is determined in $\mathbb{P}_\beta$.

Lemma 5.5. *The set of determined conditions is dense in $\mathbb{P}_\alpha$. Also, if $\langle r_n : n < \omega \rangle$ is a descending sequence of determined conditions, then the greatest lower bound as described in Lemma 5.3 is also determined.*

Proof. By Lemma 5.3, $\mathbb{P}_\alpha$ does not add any new countable subsets of the ground model. Given a condition p , using a standard bookkeeping argument we can define by induction a descending sequence of conditions $\langle q_n : n < \omega \rangle$ below p so that for all $n < \omega$ and all nonzero $\gamma \in \text{dom}(a_{q_n})$, there is $m \geq n$ such that $q_m \restriction \gamma$ decides $a_{q_n}(\gamma)$ as $(A_{\gamma,n}, f_{\gamma,n})$. For each $\gamma \in \bigcup \{\text{dom}(a_{q_n}) : n < \omega\}$, let $A_\gamma := \bigcup \{A_{\gamma,n} : n < \omega\}$ and $f_\gamma := \bigcup \{f_{\gamma,n} : n < \omega\}$.

Let r be the greatest lower bound of $\langle q_n : n < \omega \rangle$. Define r' by letting $X_{r'} := X_r$, $\text{dom}(a_{r'}) := \text{dom}(a_r)$, $a_{r'}(0) := a_r(0)$, and for all nonzero $\gamma \in \text{dom}(a_r)$, letting $a_{r'}(\gamma)$ be a canonical $\mathbb{P}_\gamma$ -name for the pair (A_γ, f_γ) . Then easily $r \leq_\alpha r' \leq_\alpha r$ and r' is determined. The second statement follows by a similar but easier argument. $\square$

We will informally identify a determined condition p with the object (a, X) such that $\text{dom}(a) = \text{dom}(a_p)$, $X = X_p$, $a(0) = a_p(0)$, and for all nonzero $\gamma \in \text{dom}(a)$, $a(\gamma) = (A, f)$, where $a_p(\gamma)$ is the canonical $\mathbb{P}_\gamma$ -name for (A, f) .

Note that for all nonzero $\alpha < \kappa^+$, there are κ many determined conditions in $\mathbb{P}_\alpha$. Thus, $\mathbb{P}_\alpha$ has a dense subset of size κ and hence is κ^+ -c.c.

Lemma 5.6. *For all $\alpha \leq \kappa^+$, $\mathbb{P}_\alpha$ is κ^+ -c.c.*

Proof. By the preceding comments, it suffices to prove the statement for $\alpha = \kappa^+$. Suppose that $\langle p_i : i < \kappa^+ \rangle$ is a sequence of conditions in $\mathbb{P}_{\kappa^+}$. By extending these conditions if necessary, we may assume without loss of generality that each p_i is determined.

A straightforward argument using the Δ -system lemma shows that there is $Z \subseteq \kappa^+$ of size κ^+ and $\kappa < \beta < \kappa^+$ such that for all $i < j$ in Z :

- (1) $\text{dom}(a_{p_i}) \cap \text{dom}(a_{p_j}) \subseteq \beta$;
- (2) $\text{dom}(X_{p_i}) \cap \text{dom}(X_{p_j}) \subseteq \beta$;
- (3) $p_i \restriction \beta = p_j \restriction \beta$;
- (4) the intersection of the sets

$$Y_i := \left(\bigcup \text{ran}(X_{p_i}) \right) \cup \text{dom}(a_{p_i})$$

and

$$Y_j := \left(\bigcup \text{ran}(X_{p_j}) \right) \cup \text{dom}(a_{p_j})$$

is a subset of β .

Let $i < j$ in Z , and we will prove that p_i and p_j are compatible. Define $q = (a, X)$ as follows:

- (a) $\text{dom}(a) := \text{dom}(a_{p_i}) \cup \text{dom}(a_{p_j})$;
- (b) $\text{dom}(X) := \text{dom}(X_{p_i}) \cup \text{dom}(X_{p_j})$;
- (c) $a(\gamma) := a_{p_i}(\gamma)$ when $\gamma \in \text{dom}(a_{p_i})$;
- (d) $a(\gamma) := a_{p_j}(\gamma)$ when $\gamma \in \text{dom}(a_{p_j})$;
- (e) $X(\gamma) := X_{p_i}(\gamma)$ when $\gamma \in \text{dom}(X_{p_i})$;
- (f) $X(\gamma) := X_{p_j}(\gamma)$ when $\gamma \in \text{dom}(X_{p_j})$.

Note that (1), (2), and (3) above imply that q is well-defined. Also, if q is a condition, then easily $q \leq_{\kappa^+} p_i, p_j$.

We prove that q is a condition using Lemma 5.1. Note that for all $\xi < \kappa^+$, if $(a \restriction \xi, X \restriction (\xi + 1))$ is in $\mathbb{P}_\xi$ then it is clearly below $p_i \restriction \xi$ and $p_j \restriction \xi$. Properties (1)–(5) of Lemma 5.1 are thereby trivial. So it remains to show that for all $\gamma \in \text{dom}(X)$, $M \in X(\gamma)$, and $\xi \in M \cap \text{dom}(a)$, assuming that $(a \restriction \xi, X \restriction (\xi + 1))$ is a condition in $\mathbb{P}_\xi$, then this condition forces in $\mathbb{P}_\xi$ that $M \cap \kappa \in A_{a(\xi)}$.

This statement is trivial if $M \in X_{p_i}(\gamma)$ and $\xi \in M \cap \text{dom}(a_{p_i})$, or if $M \in X_{p_j}(\gamma)$ and $\xi \in M \cap \text{dom}(a_{p_j})$, since $(a \restriction \xi, X \restriction (\xi + 1))$ is below $p_i \restriction \xi$ and $p_j \restriction \xi$ assuming it is a condition. In particular, it is true if $\gamma < \beta$. Assume that $\gamma \geq \beta$. Then γ is either in $\text{dom}(X_{p_i})$ or $\text{dom}(X_{p_j})$, but not both. Without loss of generality, assume that $\gamma \in \text{dom}(X_{p_i})$. Then $M \in X_{p_i}(\gamma)$.

If $\xi \in \text{dom}(a_{p_i})$ then we are done, so assume that $\xi \in \text{dom}(a_{p_j}) \setminus \text{dom}(a_{p_i})$. Since $\xi \in M$, we have that $\xi \in Y_i$, and since $\xi \in \text{dom}(a_{p_j})$, we have that $\xi \in Y_j$. So $\xi \in Y_i \cap Y_j \subseteq \beta$. Hence, $\xi \in \text{dom}(a_{p_j}) \cap \beta$. Since $p_i \restriction \beta = p_j \restriction \beta$, $a_{p_i} \restriction \beta = a_{p_j} \restriction \beta$. So $\text{dom}(a_{p_i}) \cap \beta = \text{dom}(a_{p_j}) \cap \beta$. It follows that $\beta \in \text{dom}(a_{p_i})$, which is a contradiction. $\square$

Definition 5.7. Let $\kappa < \beta < \alpha \leq \kappa^+$. Define $D_{\beta, \alpha}$ as the set of $p \in \mathbb{P}_\alpha$ such that for all $\gamma \in \text{dom}(X_p) \setminus \beta$,

$$\{M \cap \beta : M \in X_p(\gamma)\} \subseteq X_p(\beta).$$

The next lemma is easy to prove.

Lemma 5.8. Let $\kappa < \beta < \alpha \leq \kappa^+$. Then $D_{\beta, \alpha}$ is dense in $\mathbb{P}_\alpha$. In fact, let $p \in \mathbb{P}_\alpha$ and $x \subseteq \alpha \setminus (\kappa + 1)$ be countable. Define q by letting $a_q := a_p$, $\text{dom}(X_q) := \text{dom}(X_p) \cup x$, for all $\gamma \in \text{dom}(X_p) \setminus x$, $X_q(\gamma) := X_p(\gamma)$, and for all $\gamma \in x$,

$$X_q(\gamma) := X_p(\gamma) \cup \{M \cap \gamma : \exists \xi \in \text{dom}(X_p) \setminus \gamma, M \in X_p(\xi)\}.$$

Then $q \in \mathbb{P}_\alpha$, $q \leq_\alpha p$, and for all $\beta \in x$, $q \in D_{\beta, \alpha}$.

Notation 5.9. Let $\beta < \alpha \leq \kappa^+$ and $p \in \mathbb{P}_\alpha$. Let $r \leq_\beta p \restriction \beta$. Define $r +_{\beta, \alpha} p$ to be the pair (a, X) satisfying:

- (1) $a \restriction \beta = a_r$ and $a \restriction [\beta, \alpha) = a_p \restriction [\beta, \alpha)$;
- (2) $X \restriction (\beta + 1) = X_r$ and $X \restriction (\beta + 1, \alpha] = X_p \restriction (\beta + 1, \alpha]$.

When β and α are understood from context, we will abbreviate $r +_{\beta, \alpha} p$ as $r + p$.

Lemma 5.10. Let $\beta < \alpha \leq \kappa^+$, $p \in \mathbb{P}_\alpha$, and $r \leq_\beta p \restriction \beta$. Suppose that $p \in D_{\beta, \alpha}$ in the case that $\kappa < \beta$. Then $r + p$ is in $\mathbb{P}_\alpha$ and is below p and r in $\mathbb{P}_\alpha$.

Proof. We use Lemma 5.1 to show that $r + p$ is a condition. It is then easy to see that $r + p$ is below p and r . In fact, for all $\xi \leq \alpha$, assuming that $(r + p) \restriction \xi := (a_{r+p} \restriction \xi, X_{r+p} \restriction (\xi + 1))$ is in $\mathbb{P}_\xi$, it is easily seen to be below $p \restriction \xi$ and $r \restriction \xi$ in $\mathbb{P}_\xi$.

The only nontrivial property to check from Lemma 5.1 is (6). Suppose that $\gamma \in \text{dom}(X_{r+p})$, $M \in X_{r+p}(\gamma)$, and $\xi \in M \cap \text{dom}(a_{r+p})$ is nonzero. Assume that $(r + p) \restriction \xi := (a_{r+p} \restriction \xi, X_{r+p} \restriction (\xi + 1))$ is a condition in $\mathbb{P}_\xi$ and is below $p \restriction \xi$ and $r \restriction \xi$. We claim that this condition forces in $\mathbb{P}_\xi$ that $M \cap \kappa \in A_{a_{r+p}(\xi)}$.

Assume first that $\gamma \leq \beta$. Then $\xi < \beta$. And $X_{r+p}(\gamma) = X_r(\gamma)$ and $a_{r+p} \restriction \gamma = a_r \restriction \gamma$. So $M \in X_r(\gamma)$ and $\xi \in M \cap \text{dom}(a_r)$. Since r is a condition, it follows that $r \restriction \xi = (a_{r+p} \restriction \xi, X_{r+p} \restriction (\xi + 1))$ forces that $M \cap \kappa \in A_{a_r(\xi)} = A_{a_{r+p}(\xi)}$. In the second case that $\xi \geq \beta$, a similar argument works using the fact that p is a condition.

Thirdly, assume that $\xi < \beta < \gamma$. Note that since $\xi \neq 0$ and $\text{dom}(a_p) \cap \kappa \subseteq \{0\}$ by the definition of the forcing poset $\mathbb{P}_\alpha$, $\kappa \leq \xi < \beta$. Since $p \in D_{\beta, \alpha}$, $r \leq_\beta p \restriction \beta$, and $M \in X_{r+p}(\gamma) = X_p(\gamma)$, it follows that $M \cap \beta \in X_p(\beta) \subseteq X_r(\beta)$. Also, $\xi \in \text{dom}(a_{r+p}) \cap \beta = \text{dom}(a_r) \cap \beta$. As r is a condition, $M \cap \beta \in X_r(\beta)$, and $\xi \in \text{dom}(a_r) \cap (M \cap \beta)$, it follows that $r \restriction \xi = (r + p) \restriction \xi$ forces that $M \cap \kappa \in A_{a_r(\xi)} = A_{a_{r+p}(\xi)}$. $\square$

Proposition 5.11. *Let $\beta < \alpha \leq \kappa^+$. Then $\mathbb{P}_\beta$ is a regular suborder of $\mathbb{P}_\alpha$.*

Proof. By Lemma 5.2(1,3), $\mathbb{P}_\beta$ is a suborder of $\mathbb{P}_\alpha$, and if p and q are in $\mathbb{P}_\beta$ and are compatible in $\mathbb{P}_\alpha$, then they are compatible in $\mathbb{P}_\beta$.

It remains to show that if A is a maximal antichain of $\mathbb{P}_\beta$, then A is predense in $\mathbb{P}_\alpha$. So let $p \in \mathbb{P}_\alpha$, and we will find a member of A which is compatible with p in $\mathbb{P}_\alpha$. In the case that $\kappa < \beta$, extend p to q in $D_{\beta,\alpha}$ by Lemma 5.8, and otherwise let $q = p$.

Since $q \restriction \beta$ is in $\mathbb{P}_\beta$ and A is a maximal antichain of $\mathbb{P}_\beta$, we can fix $r \in A$ and $s \in \mathbb{P}_\beta$ such that $s \leq_\beta q \restriction \beta, r$. By Lemma 5.10, $s + q$ is in $\mathbb{P}_\alpha$ and is below q and s in $\mathbb{P}_\alpha$. So it is also below q and r . Thus, p is compatible with a member of A . $\square$

In Section 7 we will prove that $\mathbb{P}_\alpha$ preserves κ , for all $\alpha < \kappa^+$, assuming that κ is ineffable. The next result shows that this implies that $\mathbb{P}_{\kappa^+}$ preserves κ^+ .

Proposition 5.12. *Assume that for all $\alpha < \kappa^+$, $\mathbb{P}_\alpha$ preserves κ . Then $\mathbb{P}_{\kappa^+}$ preserves κ , and in particular, $\mathbb{P}_{\kappa^+}$ forces that $\kappa = \omega_2$.*

Proof. As $\mathbb{P}_{\kappa^+}$ is countably closed by Lemma 5.3, it preserves ω_1 . Since $\mathbb{P}_1$ is a regular suborder of $\mathbb{P}_{\kappa^+}$ by Proposition 5.11 and $\mathbb{P}_1$ is forcing equivalent to the forcing $\text{Col}(\omega_1, < \kappa)$, the forcing poset $\mathbb{P}_{\kappa^+}$ collapses all uncountable cardinals less than κ to have size ω_1 . Hence, to show that $\mathbb{P}_{\kappa^+}$ forces that $\kappa = \omega_2$ it suffices to show that it preserves κ .

Suppose that $\dot{f}$ is a nice $\mathbb{P}_{\kappa^+}$ -name for a function from ω_1 to κ . Then there is a sequence $\langle A_x : x \in \omega_1 \times \kappa \rangle$ of antichains in $\mathbb{P}_{\kappa^+}$ so that

$$\dot{f} = \{(p, \check{x}) : x \in \omega_1 \times \kappa, p \in A_x\}.$$

Since $\mathbb{P}_{\kappa^+}$ is κ^+ -c.c. by Lemma 5.6, for each x there is $\beta_x < \kappa^+$ such that $A_x \subseteq \mathbb{P}_{\beta_x}$.

Let $\beta < \kappa^+$ be larger than β_x for all $x \in \omega_1 \times \kappa$. Then $\dot{f}$ is a $\mathbb{P}_\beta$ -name, and for any generic filter G on $\mathbb{P}_{\kappa^+}$, $\dot{f}^G = \dot{f}^{G \cap \mathbb{P}_\beta}$. Now $\mathbb{P}_\beta$ is a regular suborder of $\mathbb{P}_{\kappa^+}$ and $\mathbb{P}_\beta$ preserves κ . So if G is a generic filter on $\mathbb{P}_{\kappa^+}$, then $\dot{f}^G = \dot{f}^{G \cap \mathbb{P}_\beta} \in V[G \cap \mathbb{P}_\beta]$ is not surjective onto κ , and hence it is not surjective onto κ in $V[G]$. $\square$

We now show that the forcing iteration does what it was intended to do, namely, to add club isomorphisms between Aronszajn trees. This follows from the next lemma and Proposition 4.8.

Lemma 5.13. *Let $\kappa \leq \beta < \kappa^+$. Then the forcing posets $\mathbb{P}_{\beta+1}$ and $\mathbb{P}_\beta * \mathbb{P}(\dot{T}_\beta, \dot{U}_\beta)$ are forcing equivalent.*

Proof. Let D be the set of $q \in \mathbb{P}_{\beta+1}$ such that $\beta \in \text{dom}(a_q)$, and $q \in D_{\beta,\beta+1}$ if $\kappa < \beta$. We claim that D is dense in $\mathbb{P}_{\beta+1}$. Given $p \in \mathbb{P}_{\beta+1}$, extend p to p_1 such that $\beta \in \text{dom}(a_{p_1})$. Namely, if $\beta \in \text{dom}(a_p)$ already, let $p = p_1$; otherwise, add β and define $a_{p_1}(\beta) = (A, \emptyset)$, where $A = \{M \cap \kappa : M \in X_p(\beta + 1)\}$. Now apply Lemma 5.8 and extend p_1 to q in $D_{\beta,\beta+1}$, if $\kappa < \beta$.

Define $\pi : D \rightarrow \mathbb{P}_\beta * \mathbb{P}(\dot{T}_\beta, \dot{U}_\beta)$ by letting $\pi(p) = (p \restriction \beta) * a_p(\beta)$ for all $p \in D$. We claim that π is a dense embedding.

It is immediate from the definitions that if $q \leq_{\beta+1} p$ in D , then $\pi(q) \leq \pi(p)$ in $\mathbb{P}_\beta * \mathbb{P}(\dot{T}_\beta, \dot{U}_\beta)$. We claim that π is surjective, and in particular, the range of π is dense. So let $s = (s \restriction \beta, s(\beta)) \in \mathbb{P}_\beta * \mathbb{P}(\dot{T}_\beta, \dot{U}_\beta)$. Define p by letting $p \restriction \beta := s \restriction \beta$, $a_p(\beta) := s(\beta)$, and $X_p := X_{s \restriction \beta}$. It is easy to check that p is a condition and $\pi(p) = s$.

Let p and q be conditions in D , and suppose that $\pi(p)$ and $\pi(q)$ are compatible in $\mathbb{P}_\beta * \mathbb{P}(\dot{T}_\beta, \dot{U}_\beta)$. We will show that p and q are compatible in $\mathbb{P}_{\beta+1}$. Fix

$$r = (r \restriction \beta, r(\beta)) \leq \pi(p), \pi(q)$$

in $\mathbb{P}_\beta * \mathbb{P}(\dot{T}_\beta, \dot{U}_\beta)$. Then in particular,

$$r \restriction \beta \leq_\beta \pi(p) \restriction \beta = p \restriction \beta,$$

$$r \restriction \beta \leq_\beta \pi(q) \restriction \beta = q \restriction \beta,$$

and $r \restriction \beta$ forces in $\mathbb{P}_\beta$ that $r(\beta) \leq \pi(p)(\beta) = a_p(\beta)$ and $r(\beta) \leq \pi(q)(\beta) = a_q(\beta)$ in $\mathbb{P}(\dot{T}_\beta, \dot{U}_\beta)$.

Define $s \in \mathbb{P}_{\beta+1}$ as follows. Let $s \restriction \beta := r \restriction \beta$, $a_s(\beta) := r(\beta)$, $X_s \restriction \beta := X_r \restriction \beta$, and $X_s(\beta+1) := X_p(\beta+1) \cup X_q(\beta+1)$. It is straightforward to check that s satisfy properties (1)–(5) of being a condition in $\mathbb{P}_{\beta+1}$, and that if s is a condition, then $s \leq_{\beta+1} p, q$. It remains to prove property (6).

Assume that $M \in X_s(\beta+1)$ and $\gamma \in M \cap \text{dom}(a_s)$. We will show that $s \restriction \gamma$ forces in $\mathbb{P}_\gamma$ that $M \cap \kappa \in A_{a_s(\gamma)}$. Since $X_s(\beta+1) = X_p(\beta+1) \cup X_q(\beta+1)$, either $M \in X_p(\beta+1)$ or $M \in X_q(\beta+1)$. By symmetry, it suffices to consider the case that $M \in X_p(\beta+1)$.

First, assume that $\gamma < \beta$. Then $a_s(\gamma) = a_r(\gamma)$ and $\gamma \in \text{dom}(a_r) \cap (M \cap \beta)$. Since $p \in D_{\beta, \beta+1}$, $M \cap \beta \in X_p(\beta)$. As $r \restriction \beta \leq_\beta p \restriction \beta$, $M \cap \beta \in X_r(\beta)$. Since r is a condition, $r \restriction \gamma = s \restriction \gamma$ forces that $M \cap \kappa \in A_{a_r(\gamma)} = A_{a_s(\gamma)}$.

Secondly, assume that $\gamma = \beta$. Then $p \restriction \beta$ forces that $M \cap \kappa \in A_{a_p(\beta)}$. Since $r \restriction \beta = s \restriction \beta$ forces that $r(\beta) \leq a_p(\beta)$ in $\mathbb{P}(\dot{T}_\beta, \dot{U}_\beta)$, it forces that $M \cap \kappa \in A_{r(\beta)} = A_{a_s(\beta)}$. $\square$

Let us show how the material in this section can be used to prove the main result of the paper. Start with a model in which κ is ineffable and $2^\kappa = \kappa^+$. Assume, moreover, that the iterations defined in this section preserve κ , which will be verified in Section 7. Since $\mathbb{P}_{\kappa^+}$ is κ^+ -c.c., a standard nice name argument similar to the proof of Proposition 5.12 shows that any ω_2 -tree in a generic extension by $\mathbb{P}_{\kappa^+}$ appears in an intermediate generic extension by $\mathbb{P}_\beta$ for some $\beta < \kappa^+$. As $2^\kappa = \kappa^+$, standard arguments show that we can arrange our bookkeeping function to enumerate all nice names for standard countably closed normal ω_2 -Aronszajn trees in such a way that the iteration handles all possible pairs of such trees.

It follows that in a generic extension $V[G]$ by $\mathbb{P}_{\kappa^+}$, whenever T and U are standard countably closed normal ω_2 -Aronszajn trees such that for all $\gamma < \kappa$, the nodes of height γ in T are exactly the even ordinals in I_γ , and the nodes of height γ in U are exactly the odd ordinals in I_γ , then for some $\beta < \kappa^+$, $\dot{T}_\beta^{G \cap \mathbb{P}_\beta} = T$ and $\dot{U}_\beta^{G \cap \mathbb{P}_\beta} = U$. But $\mathbb{P}_{\beta+1}$ is a regular suborder of $\mathbb{P}_{\kappa^+}$ by Proposition 5.11, and $\mathbb{P}_{\beta+1}$ is forcing equivalent to $\mathbb{P}_\beta * \mathbb{P}(\dot{T}_\beta, \dot{U}_\beta)$ by Lemma 5.13. It follows by Proposition 4.8 that in $V[G]$ there is a club isomorphism from T to U .

By easy arguments, any countably closed normal ω_2 -Aronszajn tree in $V[G]$ is club isomorphic to trees of the type described in the previous paragraph. Hence, in $V[G]$ we have that any two countably closed normal ω_2 -Aronszajn trees are club isomorphic.

To complete the main result of the paper, it remains to show that the forcing posets defined in this section preserve κ , assuming that κ is ineffable. The last two sections of the paper are devoted to proving this.

6. PREPARATION FOR THE PRESERVATION OF κ

For the remainder of the paper, we will assume that κ is ineffable. Let J denote the ineffability ideal on κ as discussed in Section 2. Let us also identify a useful collection of models.

Definition 6.1. Let $\mathcal{Y}$ denote the collection of all sets N in $P_\kappa(H(\kappa^{++}))$ satisfying:

- (1) $N \prec (H(\kappa^{++}), \in, \trianglelefteq, J)$, where $\trianglelefteq$ is a fixed well-ordering of $H(\kappa^{++})$;
- (2) $N \cap \kappa$ is inaccessible, $|N| = N \cap \kappa$, and $N^{<(N \cap \kappa)} \subseteq N$.

The next lemma uses the standard fact that the set of inaccessible cardinals in κ is a member of J^* ; see [2, Proposition 2.5].

Lemma 6.2. The set $\mathcal{Y}$ is stationary in $P_\kappa(H(\kappa^{++}))$.

Proof. Let $F : H(\kappa^{++})^{<\omega} \rightarrow H(\kappa^{++})$. Build a $\in$ -increasing and continuous sequence $\langle N_i : i < \kappa \rangle$ of sets in $P_\kappa(H(\kappa^{++}))$ which are closed under F and are elementary substructures of $(H(\kappa^{++}), \in, \trianglelefteq, J)$. Since κ is inaccessible, a standard argument shows that there is a club $C \subseteq \kappa$ such that for all $\alpha \in C$, $|N_\alpha| = \alpha = N_\alpha \cap \kappa$. As κ is ineffable, it is Mahlo, so we can find an inaccessible α in C . Since α is inaccessible, it is easy to check that $N_\alpha^{<\alpha} \subseteq N_\alpha$. Then N_α is in $\mathcal{Y}$ and is closed under F . $\square$

Notation 6.3. Let $\alpha < \kappa^+$. A set $N \in \mathcal{Y}$ is said to be α -suitable if N is an elementary substructure of

$$(H(\kappa^{++}), \in, \trianglelefteq, J, \mathcal{Y}, \langle \mathbb{P}_\beta : \beta \leq \alpha \rangle).$$

Note that if N is α -suitable, then for all $\beta \in N \cap \alpha$, N is β -suitable.

Lemma 6.4. The collection of $N \in P_\kappa(H(\kappa^{++}))$ such that N is α -suitable is stationary.

Proof. Immediate from Lemma 6.2 and the definition of α -suitable. $\square$

Fix $\kappa < \alpha < \kappa^+$ for the remainder of the paper, and we will prove that $\mathbb{P}_\alpha$ preserves κ . It will then follow by Proposition 5.12 that $\mathbb{P}_{\kappa^+}$ preserves κ , which will complete the proof of the main result of the paper. We assume as an inductive hypothesis that for all $\beta < \alpha$, $\mathbb{P}_\beta$ preserves κ . We will identify two additional inductive hypotheses in Section 7.

Notation 6.5. Let N be α -suitable. Define $p(N, \alpha) := (\emptyset, X)$, where $\text{dom}(X) := \{\alpha\}$ and $X(\alpha) := \{N \cap \alpha\}$.

Easily, $p(N, \alpha)$ is a condition in $\mathbb{P}_\alpha$. Note that if $p \in \mathbb{P}_\alpha$, then $p \leq_\alpha p(N, \alpha)$ iff $N \cap \alpha \in X_p(\alpha)$.

Lemma 6.6. Let N be α -suitable, $p \leq_\alpha p(N, \alpha)$, $\beta \in N \cap \alpha$, and $p \in D_{\beta, \alpha}$. Then $p \restriction \beta \leq_\beta p(N, \beta)$.

Proof. Since $p \leq_\alpha p(N, \alpha)$, $N \cap \alpha \in X_p(\alpha)$. As $p \in D_{\beta, \alpha}$, $(N \cap \alpha) \cap \beta = N \cap \beta \in X_p(\beta) = X_{p \restriction \beta}(\beta)$. $\square$

Lemma 6.7. Let N be α -suitable. Then for all $p \in N \cap \mathbb{P}_\alpha$, p and $p(N, \alpha)$ are compatible in $\mathbb{P}_\alpha$.

Proof. By extending p in N if necessary, we may assume without loss of generality that p is determined. Note that by elementarity, $\text{dom}(a_p) \subseteq N$ and for all $\beta \in \text{dom}(a_p)$, $A_{a_p(\beta)} \subseteq N \cap \kappa$. Also, $\text{dom}(X_p) \subseteq N$ and for all $\gamma \in \text{dom}(X_p)$, $X_p(\gamma) \subseteq N$.

Define $q = (b, Y)$ as follows. Let $\text{dom}(b) := \text{dom}(a_p)$. Define $b(0) := a_p(0)$. For all nonzero $\beta \in \text{dom}(a_p)$, apply Lemma 4.5 to find a nice $\mathbb{P}_\beta$ -name $b(\beta)$ for an extension of $a_p(\beta)$ in $\mathbb{P}(\dot{T}_\beta, \dot{U}_\beta)$ such that $N \cap \kappa \in A_{b(\beta)}$. Let $\text{dom}(Y) := \text{dom}(X_p) \cup \{\alpha\}$, $Y(\beta) := X_p(\beta)$ for all $\beta \in \text{dom}(X_p)$ different from α , and $Y(\alpha) := X_p(\alpha) \cup \{N \cap \alpha\}$. It is straightforward to check using Lemma 5.1 that q is a condition, and clearly $q \leq_\alpha p, p(N, \alpha)$. $\square$

Definition 6.8. Let N be α -suitable. Define $D(N, \alpha)$ as the set of determined conditions $p \in \mathbb{P}_\alpha$ satisfying:

- (1) $p \leq_\alpha p(N, \alpha)$;
- (2) for all $\gamma \in \text{dom}(X_p)$, for all $M \in X_p(\gamma)$, if $M \cap \kappa < N \cap \kappa$ then $M \cap N \in X_p(\gamma)$;
- (3) for all nonzero $\beta \in N \cap \text{dom}(a_p)$, $p \restriction \beta$ forces that $a_p(\beta)$ is injective on $N \cap \kappa$ (in the sense of Definition 4.9).

Lemma 6.9. Let N be α -suitable. Then the set $D(N, \alpha)$ is dense below $p(N, \alpha)$.

Proof. Let $q \leq_\alpha p(N, \alpha)$, and we will find $r \leq_\alpha q$ in $D(N, \alpha)$. Without loss of generality, assume that q is determined.

We define by induction a descending sequence of conditions $\langle q_n : n < \omega \rangle$ in $\mathbb{P}_\alpha$. Let $q_0 := q$. Fix $n < \omega$, and assume that q_n is defined. If n is odd, then let q_{n+1} be an extension of q_n which is determined.

Assume that n is even and q_n is determined. Since $N \cap \alpha \in X_{q_n}(\alpha)$, we have that for all $\beta \in N \cap \text{dom}(a_{q_n})$, $N \cap \kappa \in A_{a_{q_n}(\beta)}$. Define q_{n+1} as follows. Let $X_{q_{n+1}} := X_{q_n}$ and $\text{dom}(a_{q_{n+1}}) := \text{dom}(a_{q_n})$. For each $\beta \in \text{dom}(a_{q_n}) \setminus N$, let $a_{q_{n+1}}(\beta) := a_{q_n}(\beta)$. For each $\beta \in N \cap \text{dom}(a_{q_n})$, apply Lemma 4.10 to find a $\mathbb{P}_\beta$ -name $a_{q_{n+1}}(\beta)$ for a condition in $\mathbb{P}(\dot{T}_\beta, \dot{U}_\beta)$ such that $q_n \restriction \beta$ forces that $a_{q_{n+1}}(\beta)$ is an extension of $a_{q_n}(\beta)$ which is injective on $N \cap \kappa$.

This completes the construction. Let r' be the greatest lower bound of the sequence $\langle q_n : n < \omega \rangle$. By Lemma 5.5, r' is determined. By construction, for all $\beta \in N \cap \text{dom}(a_{r'})$, for all large enough even $n < \omega$, $q_n \restriction \beta$, and hence $r' \restriction \beta$, forces that $a_{q_{n+1}}(\beta)$ is injective on $N \cap \kappa$. It easily follows by Lemma 4.11 that for all for all $\beta \in N \cap \text{dom}(a_{r'})$, $r' \restriction \beta$ forces that $a_{r'}(\beta)$ is injective on $N \cap \kappa$.

Now define r as follows. Let $a_r := a_{r'}$. Define $\text{dom}(X_r) := \text{dom}(X_{r'})$, and for all $\gamma \in \text{dom}(X_r)$,

$$X_r(\gamma) := X_{r'}(\gamma) \cup \{M \cap N : M \in X_{r'}(\gamma), M \cap \kappa < N \cap \kappa\}.$$

Using the fact that whenever $M \in X_{r'}(\gamma)$ and $M \cap \kappa < N \cap \kappa$, it follows that $M \cap N \cap \kappa = M \cap \kappa$, it is straightforward to check that $r \in \mathbb{P}_\alpha$. Also clearly $r \leq_\alpha r'$ and $r \in D(N, \alpha)$. $\square$

Lemma 6.10. Let N be α -suitable and x a countable subset of $(N \cap \alpha) \setminus (\kappa + 1)$. Then for any $p \leq_\alpha p(N, \alpha)$, there is $q \leq_\alpha p$ such that $q \in D(N, \alpha)$ and for all $\beta \in x$, $q \in D_{\beta, \alpha}$.

Proof. By extending p if necessary using Lemma 6.9, we may assume without loss of generality that $p \in D(N, \alpha)$. Now define q from p and x as described in the

statement of Lemma 5.8. Then $q \leq p$ and $q \in D_{\beta, \alpha}$ for all $\beta \in x$. Since $a_p = a_q$, it is routine to check that $q \in D(N, \alpha)$. $\square$

Lemma 6.11. *Let M and N be α -suitable with $M \cap \kappa = N \cap \kappa$. Then $p(M, \alpha) = p(N, \alpha)$ and $D(M, \alpha) = D(N, \alpha)$.*

Proof. Since $\alpha \in M \cap N$, $|\alpha| \leq \kappa$, and $M \cap \kappa = N \cap \kappa$, it follows by standard arguments that $M \cap \alpha = N \cap \alpha$. Now note that the definitions of $p(K, \alpha)$ and $D(K, \alpha)$ given in Notation 6.5 and Definition 6.8 depend only on $K \cap \alpha$, for any α -suitable K . $\square$

The proofs of the next two lemmas are straightforward.

Lemma 6.12. *Let N be α -suitable, $\beta \in N \cap \alpha$, and $p \in D(N, \alpha) \cap D_{\beta, \alpha}$. Then:*

- (1) $p \restriction \beta \in D(N, \beta)$.
- (2) If $u \leq_\beta p \restriction \beta$ and $u \in D(N, \beta)$, then $u + p \in D(N, \alpha)$.

Lemma 6.13. *Let N be α -suitable. Suppose that $\langle p_n : n < \omega \rangle$ is a descending sequence of conditions in $D(N, \alpha)$ with greatest lower bound q . Then $q \in D(N, \alpha)$.*

Definition 6.14. *Let N be α -suitable and p be determined. Define $p \restriction N$ to be the pair (a, X) satisfying:*

- (1) a is a function with domain equal to $\text{dom}(a_p) \cap N$;
- (2) $a(0) = a_p(0) \restriction (\omega_1 \times (N \cap \kappa))$;
- (3) for all nonzero $\gamma \in \text{dom}(a)$, $a(\gamma) = (A_{a_p(\gamma)} \cap (N \cap \kappa), f_{a_p(\gamma)} \restriction (N \cap \kappa))$;
- (4) X is a function with domain equal to $\text{dom}(X_p) \cap N$, and for all $\gamma \in \text{dom}(X)$, $X(\gamma) = X_p(\gamma) \cap N$.

Observe that by Lemma 4.4, for all nonzero $\gamma \in \text{dom}(a)$, $p \restriction \gamma$ forces that $a(\gamma) = a_p(\gamma) \restriction (N \cap \kappa)$ is in $\mathbb{P}(\dot{T}_\gamma, \dot{U}_\gamma)$. The object $p \restriction N$ is a member of N by the closure of N , but it is not necessarily a condition. If it is a condition, then it is determined and $p \leq_\alpha p \restriction N$. Usually we only consider $p \restriction N$ in the case that $p \leq_\alpha p(N, \alpha)$.

The proofs of the next two lemmas are straightforward.

Lemma 6.15. *Let N be α -suitable, $\beta \in N \cap \alpha$, and $p \in \mathbb{P}_\alpha$ determined. Then $(p \restriction \beta) \restriction N = (p \restriction N) \restriction \beta$. In particular, if $p \restriction N \in \mathbb{P}_\alpha$, then for all $\beta \in N \cap \alpha$, $(p \restriction \beta) \restriction N \in \mathbb{P}_\beta$.*

By $(p \restriction N) \restriction \beta$, we mean $(a \restriction \beta, X \restriction (\beta + 1))$, where $p \restriction N = (a, X)$. Of course if $p \restriction N$ is a condition, then this is the same as $(p \restriction N) \restriction \beta$ in the usual sense.

Lemma 6.16. *Let N be α -suitable and $p \in \mathbb{P}_\alpha$ be determined. Suppose that $p \restriction N \in \mathbb{P}_\alpha$. If $p \leq_\alpha s$, where $s \in N$ is determined, then $p \restriction N \leq_\alpha s$.*

Lemma 6.17. *Let M and N be α -suitable such that $M \cap \kappa = N \cap \kappa$. Let $p \in \mathbb{P}_\alpha$ be determined. Then $p \restriction M = p \restriction N$.*

Proof. As usual, $M \cap \alpha = N \cap \alpha$, and it is easily checked that properties (1), (2), and (3) in the definition of $p \restriction K$ in Definition 6.14 depend only on $K \cap \alpha$. For (4), since M and N are closed under subsets of size less than $M \cap \kappa = N \cap \kappa$, it follows that for all γ in $\text{dom}(X_p) \cap N = \text{dom}(X_p) \cap M$, the members of $X_p(\gamma) \cap N$ are exactly those sets in $X_p(\gamma)$ which are subsets of $M \cap \gamma = N \cap \gamma$ of size less than $M \cap \kappa = N \cap \kappa$, and similarly with $X_p(\gamma) \cap M$. Hence, $X_p(\gamma) \cap M = X_p(\gamma) \cap N$. $\square$

Definition 6.18. Let N be α -suitable. Define $\#_N^\alpha(p, s)$ to mean:

- (1) $p \in D(N, \alpha)$;
- (2) $s \in N \cap \mathbb{P}_\alpha$;
- (3) $p \restriction N = s$.

Also, we define $\#_N^\alpha(p, q, s)$ to mean the conjunction of $\#_N^\alpha(p, s)$ and $\#_N^\alpha(q, s)$.

Note that if $p \in D(N, \alpha)$, then $\#_N^\alpha(p, p \restriction N)$ holds iff $p \restriction N \in \mathbb{P}_\alpha$.

Recall that we are assuming in this section that $\kappa < \alpha$. But let us extend the definitions of $p(N, \alpha)$, $D(N, \alpha)$, $p \restriction N$, and $\#_N^\alpha$ in the case $\alpha \leq \kappa$ as follows. Assuming $\alpha \leq \kappa$, let $p(N, \alpha) := (\emptyset, \emptyset)$, $D(N, \alpha) := \mathbb{P}_\alpha$, and

$$p \restriction N := (a_p(0) \restriction (\omega_1 \times (N \cap \kappa)), \emptyset).$$

And let $\#_N^\alpha(p, s)$ mean that $p \in \mathbb{P}_\alpha$ and $p \restriction N = s$. Note that for $\alpha \leq \kappa$, for all $p \in \mathbb{P}_\alpha$, $p \restriction N \in \mathbb{P}_\alpha$ and $\#_N^\alpha(p, p \restriction N)$ holds.

Lemma 6.19. Let N be α -suitable and assume that for all $\beta \in N \cap \alpha$, $p(N, \beta)$ is $(N, \mathbb{P}_\beta)$ -generic. Then the set of $r \in \mathbb{P}_\alpha$ such that $\#_N^\alpha(r, r \restriction N)$ holds is dense below $p(N, \alpha)$.

Proof. We will prove the lemma by induction on $\alpha < \kappa^+$. Recall that we are assuming that $\kappa < \alpha$. However, the statement of the lemma is automatically true for $\alpha \leq \kappa$ as well, by the comments preceding the lemma. This fact will serve as the base case of our induction.

Now assume that $\kappa < \alpha$ and the lemma is true for all $\beta < \alpha$. Let $p \leq_\alpha p(N, \alpha)$, and we will find $r \leq_\alpha p$ satisfying $\#_N^\alpha(r, r \restriction N)$. Recall that if $r \in D(N, \alpha)$, then $\#_N^\alpha(r, r \restriction N)$ holds iff $r \restriction N \in \mathbb{P}_\alpha$.

Case 1: $\alpha = \beta + 1$ is a successor ordinal. By extending p if necessary using Lemma 6.10, we may assume without loss of generality that $p \in D_{\beta, \alpha} \cap D(N, \alpha)$. Define E as the dense open set of determined conditions in $\mathbb{P}_\beta$ which decide whether or not $a_p(\beta) \restriction (N \cap \kappa)$ is in $\mathbb{P}(\dot{T}_\beta, \dot{U}_\beta)$. By elementarity, $E \in N$. Since $p(N, \beta)$ is $(N, \mathbb{P}_\beta)$ -generic, $N \cap E$ is predense below $p(N, \beta)$.

By Lemma 6.6, $p \restriction \beta \leq_\beta p(N, \beta)$. Hence, we can find $u \leq_\beta p \restriction \beta$ which is below some $s \in N \cap E$. By extending u further if necessary using the inductive hypothesis, we may assume that $\#_N^\beta(u, u \restriction N)$ holds. Now define $r := u + p$.

We claim that $\#_N^\alpha(r, r \restriction N)$ holds. Since $\#_N^\beta(u, u \restriction N)$ holds, $u \in D(N, \beta)$. We also know that $p \in D(N, \alpha)$. By Lemma 6.12(2), it follows that $u + p = r$ is in $D(N, \alpha)$. So it suffices to show that $r \restriction N \in \mathbb{P}_\alpha$. Referring to the definition of $\mathbb{P}_{\beta+1}$ in Section 5, properties (1), (2), and (5) are immediate. For (3), by Lemma 6.15 we have that $(r \restriction N) \restriction \beta = (r \restriction \beta) \restriction N = u \restriction N \in \mathbb{P}_\beta$.

For (4), since $u \leq_\beta s \in N \cap E$ and E is open, by Lemma 6.16 we have that $(r \restriction N) \restriction \beta = u \restriction N \leq_\beta s$ and so $(r \restriction N) \restriction \beta \in E$. So $u \restriction N = (r \restriction N) \restriction \beta$ decides whether or not $a_p(\beta) \restriction (N \cap \kappa) = a_{r \restriction N}(\beta)$ is in $\mathbb{P}(\dot{T}_\beta, \dot{U}_\beta)$. But u is below $p \restriction \beta$ and $u \restriction N$ in $\mathbb{P}_\beta$, and $p \restriction \beta$ forces that $a_p(\beta) \restriction (N \cap \kappa)$ is in $\mathbb{P}(\dot{T}_\beta, \dot{U}_\beta)$. So $u \restriction N = (r \restriction \beta) \restriction N$ does as well. For (6), if $M \in X_{r \restriction N}(\beta + 1) = X_r(\beta + 1) \cap N$ and $\gamma \in M \cap \text{dom}(a_{r \restriction N}) = M \cap \text{dom}(a_r) \cap N = M \cap \text{dom}(a_r)$, then since r is a condition, we have that $M \cap \kappa \in A_{a_r(\gamma)} \cap N = A_{a_{r \restriction N}(\gamma)}$.

Case 2: α is a limit ordinal with uncountable cofinality. Since $N^\omega \subseteq N$, easily $\sup(N \cap \alpha)$ has uncountable cofinality. So we can fix $\beta \in N \cap \alpha$ which is strictly greater than $\sup(\text{dom}(a_p) \cap N)$ and $\sup(\text{dom}(X_p) \cap N)$.

Using Lemma 6.10, fix $q \leq_\alpha p$ in $D(N, \alpha) \cap D_{\beta, \alpha}$. By Lemma 6.6, $q \restriction \beta \leq_\beta p(N, \beta)$. By the inductive hypothesis, fix $r_0 \leq_\beta q \restriction \beta$ such that $\#_N^\beta(r_0, r_0 \restriction N)$ holds. Let $r := r_0 + q$. Then $r \leq_\alpha q \leq_\alpha p$.

We claim that $\#_N^\alpha(r, r \restriction N)$ holds. By Lemma 6.12(2), $r \in D(N, \alpha)$. It remains to show that $r \restriction N \in \mathbb{P}_\alpha$. But since β is strictly larger than any ordinal in $\text{dom}(a_q) \cap N$ and $\text{dom}(X_q) \cap N$, it easily follows that $r \restriction N = r_0 \restriction N$, which is in $\mathbb{P}_\beta$ and hence in $\mathbb{P}_\alpha$.

Case 3: α is a limit ordinal with cofinality ω . Fix an increasing and cofinal sequence $\langle \alpha_n : n < \omega \rangle$ in α which is a member (and hence a subset) of N . By Lemma 6.10, we can fix $q \leq_\alpha p$ such that $q \in D(N, \alpha)$ and $q \in D_{\alpha_n, \alpha}$ for all $n < \omega$. By Lemma 6.6, for all $n < \omega$, $q \restriction \alpha_n \leq_{\alpha_n} p(N, \alpha_n)$.

We define by induction a descending sequence of conditions $\langle q_n : n < \omega \rangle$ in $D(N, \alpha)$ such that for all $n < m$, $(q_m \restriction \alpha_n) \restriction N \in \mathbb{P}_{\alpha_n}$. Let $q_0 := q$. Fix $n < \omega$, and assume that q_n is defined so that $a_{q_n} \restriction [\alpha_n, \alpha) = a_q \restriction [\alpha_n, \alpha)$ and $X_{q_n} \restriction (\alpha_n, \alpha] = X_q \restriction (\alpha_n, \alpha]$. Since $q \in D_{\alpha_{n+1}, \alpha}$, $q \restriction \alpha_{n+1} \leq_{\alpha_{n+1}} p(N, \alpha_{n+1})$ by Lemma 6.6. As $q_n \restriction \alpha_{n+1} \leq_{\alpha_{n+1}} q \restriction \alpha_{n+1}$, $q_n \restriction \alpha_{n+1} \leq_{\alpha_{n+1}} p(N, \alpha_{n+1})$.

By the inductive hypothesis, we can fix a condition $q'_n \leq_{\alpha_{n+1}} q_n \restriction \alpha_{n+1}$ such that $\#_N^{\alpha_{n+1}}(q'_n, q'_n \restriction N)$ holds. Define $q_{n+1} := q'_n + q$. Note that since $q_{n+1} \restriction \alpha_{n+1} = q'_n \in D(N, \alpha_{n+1})$, it follows by Lemma 6.12(2) that $q_{n+1} \in D(N, \alpha)$. Also, clearly $a_{q_{n+1}} \restriction [\alpha_{n+1}, \alpha) = a_q \restriction [\alpha_{n+1}, \alpha)$ and $X_{q_{n+1}} \restriction (\alpha_{n+1}, \alpha] = X_q \restriction (\alpha_{n+1}, \alpha]$. Now for all $k \leq n$, $(q_{n+1} \restriction \alpha_k) \restriction N = (q'_n \restriction \alpha_k) \restriction N = (q'_n \restriction N) \restriction \alpha_k$, which is in $\mathbb{P}_{\alpha_k}$ since $q'_n \restriction N \in \mathbb{P}_{\alpha_{n+1}}$. So q_{n+1} is as required.

This completes the construction of $\langle q_n : n < \omega \rangle$. Let r be the greatest lower bound of this sequence. By Lemma 6.13, $r \in D(N, \alpha)$. It remains to show that $r \restriction N \in \mathbb{P}_\alpha$. Fix $n < \omega$. It is easy to check that $(r \restriction \alpha_n) \restriction N$ is the greatest lower bound of $\langle (q_m \restriction \alpha_n) \restriction N : n < m < \omega \rangle$, and hence is in $\mathbb{P}_{\alpha_n}$ since each $(q_m \restriction \alpha_n) \restriction N$ is in $\mathbb{P}_{\alpha_n}$.

Properties (1)–(4) in the definition of $\mathbb{P}_\alpha$ from Section 5 are now easy to verify for $r \restriction N$. For (5), suppose that $M \in X_{r \restriction N}(\alpha) = X_r(\alpha) \cap N = X_q(\alpha) \cap N$ and $\gamma \in \text{dom}(a_{r \restriction N}) \cap M = \text{dom}(a_r) \cap M$. Then for some $n < \omega$, $\gamma \in \text{dom}(a_{q_n}) \cap M$, so $M \cap \kappa \in A_{a_{q_n}(\gamma)} \cap N \subseteq A_{a_{r \restriction N}(\gamma)}$. $\square$

Lemma 6.20. *Let N be α -suitable. Suppose that $\langle p_n : n < \omega \rangle$ is a descending sequence of conditions satisfying $\#_N^\alpha(p_n, p_n \restriction N)$ for all $n < \omega$. Let q be the greatest lower bound of this sequence and s be the greatest lower bound of $\langle p_n \restriction N : n < \omega \rangle$. Then $s = q \restriction N$ and $\#_N^\alpha(q, s)$ holds.*

Proof. By Lemma 6.13, $q \in D(N, \alpha)$. It is routine to check that $s = q \restriction N$. Since $s \in \mathbb{P}_\alpha$, $\#_N^\alpha(q, s)$ holds. $\square$

Lemma 6.21. *Let N be α -suitable, $p \in \mathbb{P}_\alpha$, and $\beta \in N \cap \alpha$. Assume that $\#_N^\alpha(p, s)$ holds.*

- (1) *If $p \in D_{\beta, \alpha}$, then $\#_N^\beta(p \restriction \beta, s \restriction \beta)$ holds.*
- (2) *Suppose that p and s are in $D_{\beta, \alpha}$, $u \leq_\beta p \restriction \beta$, $t \leq_\beta s \restriction \beta$, and $\#_N^\beta(u, t)$ holds. Then $\#_N^\alpha(u + p, t + s)$ holds.*

Proof. (1) By Lemma 6.12(1), $p \restriction \beta \in D(N, \beta)$. By Lemma 6.15 and elementarity, $s \restriction \beta = (p \restriction N) \restriction \beta = (p \restriction \beta) \restriction N$ is in $N \cap \mathbb{P}_\beta$. So $\#_N^\beta(p \restriction \beta, s \restriction \beta)$ holds.

(2) By Lemma 6.12(2), $u+p$ is in $D(N, \alpha)$. And $t+s$ is in $N \cap \mathbb{P}_\alpha$ by elementarity. Now it is easy to check that $(u+p) \restriction N = (u \restriction N) + (p \restriction N) = t+s$. Thus, $\#_N^\alpha(u+p, t+s)$ holds. $\square$

Lemma 6.22. *Let N be α -suitable. Suppose that $\langle (p_n, q_n) : n < \omega \rangle$ is a descending sequence of conditions in $\mathbb{P}_\alpha$ and $\#_N^\alpha(p_n, q_n, s_n)$ holds for all $n < \omega$. Let p , q , and s be the greatest lower bounds of the sequences $\langle p_n : n < \omega \rangle$, $\langle q_n : n < \omega \rangle$, and $\langle s_n : n < \omega \rangle$ respectively. Then $\#_N^\alpha(p, q, s)$ holds.*

Proof. Immediate from Lemma 6.20. $\square$

Lemma 6.23. *Let N be α -suitable and assume that $\#_N^\alpha(p, q, s)$ holds. Let $x \subseteq N \cap \alpha$ be countable. Then there is $(p_0, q_0, s_0) \leq_\alpha (p, q, s)$ satisfying $\#_N^\alpha(p_0, q_0, s_0)$ and for all $\beta \in x$, p_0 , q_0 , and s_0 are in $D_{\beta, \alpha}$.*

Proof. Define p_0 , q_0 , and s_0 from p , q , and s as described in Lemma 5.8. Using property (2) of Definition 6.8, it is routine to check that $\#_N^\alpha(p_0, q_0, s_0)$ holds. $\square$

Lemma 6.24. *Let M and N be α -suitable, where $M \in N$. Assume that for all $\beta \in M \cap \alpha$, $p(M, \beta)$ is $(M, \mathbb{P}_\beta)$ -generic. Suppose that $r \in \mathbb{P}_\alpha$ is such that $\#_N^\alpha(r, r \restriction N)$ holds and $r \restriction N \in M$. Then there is $t \leq_\alpha r$ such that $\#_M^\alpha(t, t \restriction M)$ holds.*

Proof. Define s as follows. Let $\text{dom}(a_s) := \text{dom}(a_r)$ and $\text{dom}(X_s) := \text{dom}(X_r)$. Consider $\beta \in \text{dom}(a_s)$. If $\beta \notin M$, then let $a_s(\beta) := a_r(\beta)$. Suppose that $\beta \in M$. Then $\beta \in N$. Since $r \restriction N \in M$, we have that $A_{a_r(\beta)} \cap N \cap \kappa \subseteq M \cap \kappa$. As $r \in D(N, \alpha)$, $r \restriction \beta$ forces that $a_r(\beta)$ is injective on $N \cap \kappa$. By Lemma 4.13, we can fix a $\mathbb{P}_\beta$ -name $a_s(\beta)$ for an extension of $a_r(\beta)$ such that $M \cap \kappa \in A_{a_s(\beta)}$. Now consider $\gamma \in \text{dom}(X_s)$. If $\gamma \notin M$, then let $X_s(\gamma) := X_r(\gamma)$. Suppose that $\gamma \in M$. Let $X_s(\gamma) := X_r(\gamma) \cup \{M \cap \gamma\}$.

It is straightforward to check that s is a condition, $s \leq_\alpha r$, and $s \leq_\alpha p(M, \alpha)$. By Lemma 6.19, we can fix $t \leq_\alpha s$ such that $\#_M^\alpha(t, t \restriction M)$ holds. $\square$

Recall from Definition 1.2 that for a forcing poset $\mathbb{P}$ and a set N , $*_N^\mathbb{P}(p, s)$ means that $p \in \mathbb{P}$, $s \in N \cap \mathbb{P}$, and every extension of s in $N \cap \mathbb{P}$ is compatible with p .

Definition 6.25. *Let N be α -suitable. Define $*_N^\alpha(p, s)$ to mean:*

- (1) $p \in \mathbb{P}_\alpha$ is determined;
- (2) $p \leq_\alpha p(N, \alpha)$;
- (3) $s \in N \cap \mathbb{P}_\alpha$ is determined;
- (4) for all $t \leq_\alpha s$ in N , t and p are compatible in $\mathbb{P}_\alpha$.

Also, we define $*_N^\alpha(p, q, s)$ to mean the conjunction of $*_N^\alpha(p, s)$ and $*_N^\alpha(q, s)$.

Note that if $*_N^\alpha(p, s)$ holds, then for all determined $t \leq_\alpha s$ in N , $*_N^\alpha(p, t)$ also holds.

Lemma 6.26. *Let N be α -suitable. Then $*_N^\alpha(p, s)$ holds iff $*_N^{\mathbb{P}_\alpha}(p, s)$ holds, $p \leq_\alpha p(N, \alpha)$, and p and s are determined.*

Lemma 6.27. *Let N be α -suitable and assume that $p(N, \alpha)$ is strongly $(N, \mathbb{P}_\alpha)$ -generic. Suppose that $*_N^\alpha(p, q, s)$ holds, and D and E are sets of determined conditions which are dense below $p(N, \alpha)$. Then there is $(p', q', s') \leq_\alpha (p, q, s)$ such that $p' \in D$, $q' \in E$, p' and q' are below s , and $*_N^\alpha(p', q', s')$ holds.*

Proof. Note that N is suitable for $\mathbb{P}_\alpha$ in the sense of Notation 1.4. And $*_N^\alpha(p, q, s)$ implies $*_{N^\alpha}^\alpha(p, q, s)$ as in Definition 1.2. By Lemma 1.10, there is $(p', q', s') \leq_\alpha (p, q, s)$ such that $p' \in D$, $q' \in E$, p' and q' are below s , and $*_{N^\alpha}^\alpha(p', q', s')$ hold. Extending s' further in N if necessary, we may assume without loss of generality that s' is determined. By Lemma 6.26, $*_N^\alpha(p', q', s')$ holds and we are done. $\square$

Lemma 6.28. *Let N be α -suitable and assume that $*_N^\alpha(r, u)$ holds. Then there is v in $N \cap \mathbb{P}_\alpha$ satisfying:*

- (1) $v \leq_\alpha u$ is determined;
- (2) $\text{dom}(a_r) \cap N \subseteq \text{dom}(a_v)$;
- (3) $\text{dom}(X_r) \cap N \subseteq \text{dom}(X_v)$;
- (4) $a_r(0) \upharpoonright (\omega_1 \times (N \cap \kappa)) \subseteq a_v(0)$;
- (5) for all $\beta \in \text{dom}(a_r) \cap N$, $v \upharpoonright \beta$ forces in $\mathbb{P}_\beta$ that $a_r(\beta) \upharpoonright (N \cap \kappa)$ is in $\mathbb{P}(\dot{T}_\beta, \dot{U}_\beta)$ and $a_v(\beta) \leq a_r(\beta) \upharpoonright (N \cap \kappa)$ in $\mathbb{P}(\dot{T}_\beta, \dot{U}_\beta)$;
- (6) for all $\beta \in \text{dom}(X_r) \cap N$, $X_r(\beta) \cap N \subseteq X_v(\beta)$.

Proof. Since $*_N^\alpha(r, u)$ holds, we can fix a determined condition $v \leq_\alpha r, u$. Note that statements (1)–(6) hold for v , and the parameters mentioned in these statements are in N . So by elementarity, there is $v \in N \cap \mathbb{P}_\alpha$ satisfying (1)–(6). $\square$

Lemma 6.29. *Let N be α -suitable and assume that $p(N, \alpha)$ is strongly $(N, \mathbb{P}_\alpha)$ -generic. Suppose that $*_N^\alpha(p, q, s)$ holds. Then there is $(p', q', s') \leq_\alpha (p, q, s)$ such that $\#_N^\alpha(p', q', s')$ holds.*

Proof. We define by induction a descending sequence $\langle (p_n, q_n, s_n) : n < \omega \rangle$. Let $(p_0, q_0, s_0) := (p, q, s)$. Fix $n < \omega$ and assume that (p_n, q_n, s_n) has been defined so that p_n and q_n are in $D(N, \alpha)$ and $*_N^\alpha(p_n, q_n, s_n)$ holds.

Since $*_N^\alpha(p_n, q_n, s_n)$ holds, in particular, we have that $*_N^\alpha(p_n, s_n)$ holds. So we can fix $t_n \in N \cap \mathbb{P}_\alpha$ satisfying properties (1)–(6) of Lemma 6.28, where $r = p_n$, $u = s_n$, and $v = t_n$.

Since $t_n \leq_\alpha s_n$ is determined in N and $*_N^\alpha(q_n, s_n)$ holds, we also have that $*_N^\alpha(q_n, t_n)$ holds. So we can fix $w_n \in N \cap \mathbb{P}_\alpha$ satisfying properties (1)–(6) of Lemma 6.28, where $r = q_n$, $u = t_n$, and $v = w_n$.

Since $w_n \leq_\alpha s_n$ is in N and determined, we have that $*_N^\alpha(p_n, q_n, w_n)$ holds. By Lemma 6.27, we can fix $(p_{n+1}, q_{n+1}, s_{n+1}) \leq_\alpha (p_n, q_n, w_n)$ such that p_{n+1} and q_{n+1} are in $D(N, \alpha)$, are below w_n , and $*_N^\alpha(p_{n+1}, q_{n+1}, s_{n+1})$ holds.

This complete the induction. Now let p' , q' , and s' be the greatest lower bounds of the sequences $\langle p_n : n < \omega \rangle$, $\langle q_n : n < \omega \rangle$, and $\langle s_n : n < \omega \rangle$ respectively. We claim that $\#_N^\alpha(p', q', s')$ holds. By Lemma 6.13, p' and q' are in $D(N, \alpha)$, and clearly $s' \in N \cap \mathbb{P}_\alpha$. By construction, it is easy to check that $p' \upharpoonright N = q' \upharpoonright N = s'$. $\square$

Definition 6.30. *Let $\delta < \kappa$, and suppose that θ and τ are ordinals in I_δ which are either both even or both odd. Let $\dot{W}_\alpha$ denote $\dot{T}_\alpha$ if they are even and $\dot{U}_\alpha$ if they are odd. Let p and q be in $\mathbb{P}_\alpha$. We say that (p, q) δ -separates (θ, τ) in $\mathbb{P}_\alpha$ if there exists $\gamma < \delta$ and distinct ordinals θ' and τ' in I_γ such that*

$$p \Vdash_{\mathbb{P}_\alpha} \theta' <_{\dot{W}_\alpha} \theta, \quad \text{and} \quad q \Vdash_{\mathbb{P}_\alpha} \tau' <_{\dot{W}_\alpha} \tau.$$

Note that if (p, q) δ -separates (θ, τ) in $\mathbb{P}_\alpha$, then for all $(p', q') \leq_\alpha (p, q)$, (p', q') also δ -separates (θ, τ) in $\mathbb{P}_\alpha$.

Lemma 6.31. *Let N be α -suitable and let $\lambda := N \cap \kappa$. Suppose that $p(N, \alpha)$ is strongly $(N, \mathbb{P}_\alpha)$ -generic, the forcing poset $N \cap \mathbb{P}_\alpha$ forces that $\lambda = \omega_2$, and $N \cap \dot{T}_\alpha$ and $N \cap \dot{U}_\alpha$ are $(N \cap \mathbb{P}_\alpha)$ -names for trees with no chains of order type λ .*

*Assume that $*_N^\alpha(p, s)$ holds. Let $\theta \in I_\lambda$. Then there is $(p_0, p_1, t) \leq_\alpha (p, p, s)$ such that $*_N^\alpha(p_0, p_1, t)$ holds and (p_0, p_1) λ -separates (θ, θ) in $\mathbb{P}_\alpha$.*

Proof. Let $\dot{W}$ denote $\dot{T}_\alpha$ if θ is even, and $\dot{U}_\alpha$ if θ is odd.

Since $*_N^\alpha(p, s)$ holds, p and s are compatible. Forcing below them, fix a generic filter G on $\mathbb{P}_\alpha$ such that p and s are in G . Let $W := \dot{W}^G$ and $H := N \cap G$. Let $\mathbb{Q} := (\mathbb{P}_\alpha/p(N, \alpha))/H$, as described in Notation 1.5. Since $p \leq_\alpha p(N, \alpha)$ and $p(N, \alpha)$ is strongly $(N, \mathbb{P}_\alpha)$ -generic, we can apply Proposition 1.8 to conclude that H is a V -generic filter on $N \cap \mathbb{P}_\alpha$, $G' := G \cap \mathbb{Q}$ is $V[H]$ -generic filter on $\mathbb{Q}$, and $V[G] = V[H][G']$.

By the assumptions of the proposition, in $V[H]$ we have that $\lambda = \omega_2$ and $W' := (N \cap \dot{W})^H$ is a tree with no chains of order type λ . Since $H \subseteq G$, W' is obviously a subtree of W , and a straightforward argument using the elementarity of N and the $(N \cap \mathbb{P}_\alpha)$ -genericity of H shows in fact that $W' = W \upharpoonright \lambda$.

Let $b_\theta = \{x < \lambda : x <_W \theta\}$, and let $\dot{b}_\theta$ be a $\mathbb{P}_\alpha$ -name which is forced to satisfy this definition. Since b_θ is a chain in W' of order type λ , it follows that $b_\theta \notin V[H]$.

It follows by a standard argument that we can find $p'_0, p'_1 \leq p$ in $\mathbb{Q}$ and $\gamma < \lambda$ such that p'_0 and p'_1 decide in $\mathbb{Q}$ the node of $\dot{b}_\theta$ on level γ of $\dot{W}$ differently. Let θ_0 and θ_1 be the respective decisions of p'_0 and p'_1 for which ordinal of height γ is in $\dot{b}_\theta$. So $\theta_0 \neq \theta_1$ are in I_γ ,

$$p'_0 \Vdash_{\mathbb{Q}}^{V[H]} \theta_0 <_{\dot{W}} \theta, \text{ and } p'_1 \Vdash_{\mathbb{Q}}^{V[H]} \theta_1 <_{\dot{W}} \theta.$$

Fix $s' \leq s$ in H such that s' forces that p'_0 and p'_1 are in $\mathbb{Q}$ and that p'_0 and p'_1 forces the information above. Since $p'_0 \in \mathbb{Q} = (\mathbb{P}_\alpha/p(N, \alpha))/H$ and $s' \in H$, by Lemma 1.7(3) we can fix a determined condition $p_0 \in \mathbb{Q}$ such that $p_0 \leq_\alpha p'_0, s'$. Similarly, since $p'_1 \in \mathbb{Q}$ and $s \in H$, by Lemma 1.7(3) we can fix a determined condition $p_1 \in \mathbb{Q}$ such that $p_1 \leq_\alpha p'_1, s'$. Fix $t \leq s'$ in H which is determined and forces in $N \cap \mathbb{P}_\alpha$ that p_0 and p_1 are in $\mathbb{Q}$.

By Lemma 1.6, $*_N^\alpha(p_0, p_1, t)$ holds. Since p_0 and p_1 are below $p(N, \alpha)$ and p_0, p_1 , and t are determined, it follows that $*_N^\alpha(p_0, p_1, t)$ holds by Lemma 6.26. Using the fact that $p_0 \leq_\alpha s'$ and $p_1 \leq_\alpha s'$, it is easy to check that

$$p_0 \Vdash_{\mathbb{P}_\alpha}^V \theta_0 <_{\dot{W}} \theta, \text{ and } p_1 \Vdash_{\mathbb{P}_\alpha}^V \theta_1 <_{\dot{W}} \theta.$$

So (p_0, p_1) λ -separates (θ, θ) in $\mathbb{P}_\alpha$. $\square$

Proposition 6.32. *Let N be α -suitable and let $\lambda := N \cap \kappa$. Suppose that $p(N, \alpha)$ is strongly $(N, \mathbb{P}_\alpha)$ -generic, the forcing poset $N \cap \mathbb{P}_\alpha$ forces that $\lambda = \omega_2$, and $N \cap \dot{T}_\alpha$ and $N \cap \dot{U}_\alpha$ are $(N \cap \mathbb{P}_\alpha)$ -names for trees with no chains of order type λ .*

*Assume that $*_N^\alpha(p, q, s)$ holds. Let θ and τ be ordinals in I_λ which are either both even or both odd. Then there is $(p', q', s') \leq_\alpha (p, q, s)$ such that $*_N^\alpha(p', q', s')$ holds and (p', q') λ -separates (θ, τ) in $\mathbb{P}_\alpha$.*

Proof. Let $\dot{W}$ denote $\dot{T}_\alpha$ if θ and τ are both even, and $\dot{U}_\alpha$ if θ and τ are both odd. By Lemma 6.31, fix $(p_0, p_1, t) \leq_\alpha (p, p, s)$ such that $*_N^\alpha(p_0, p_1, t)$ holds and (p_0, p_1) λ -separates (θ, θ) as witnessed by a pair of distinct ordinals (θ_0, θ_1) in I_γ , where $\gamma < \lambda$.

Since $*_N^\alpha(q, s)$ holds and $t \leq_\alpha s$ is determined, also $*_N^\alpha(q, t)$ holds. Using Lemmas 1.6, 1.7(2), and 6.26, we can find $q' \leq_\alpha q$ determined, $t' \leq_\alpha t$ determined in $N \cap \mathbb{P}_\alpha$, and τ' such that q' decides $p_{\dot{W}}(\tau, \gamma)$ as τ' and $*_N^\alpha(q', t')$ holds.

We have that $\theta_0 \neq \theta_1$ and τ' are in I_γ . Let θ' be the ordinal in $\{\theta_0, \theta_1\}$ which is different from τ' , and let $p' = p_i$, where $\theta' = \theta_i$. Then $(p', q', t') \leq_\alpha (p, q, s)$, $*_N^\alpha(p', q', t')$ holds, and (p', q') λ -separates (θ, τ) as witnessed by (θ', τ') in I_γ . $\square$

Definition 6.33. Let N be α -suitable and p and q be determined conditions in $\mathbb{P}_\alpha$. Let $\lambda := N \cap \kappa$. We say that p and q are N -separated in $\mathbb{P}_\alpha$ if for all nonzero $\beta \in \text{dom}(a_p) \cap \text{dom}(a_q) \cap N$, for all $\theta \in (\text{dom}(f_{a_p(\beta)}) \cup \text{ran}(f_{a_p(\beta)})) \cap I_\lambda$ and $\tau \in (\text{dom}(f_{a_q(\beta)}) \cup \text{ran}(f_{a_q(\beta)})) \cap I_\lambda$, where θ and τ are either both even or both odd, the pair $(p \restriction \beta, q \restriction \beta)$ λ -separates (θ, τ) in $\mathbb{P}_\beta$.

Note that this definition also makes sense for $\alpha \leq \kappa$, but in that case any two conditions in $\mathbb{P}_\alpha$ are vacuously N -separated.

The next two lemmas have easy proofs, which we omit.

Lemma 6.34. Let N be α -suitable, $\beta \in N \cap \alpha$, and p and q in $D_{\beta, \alpha}$. Assume that

$$\text{dom}(a_p) \cap \text{dom}(a_q) \cap N \subseteq \beta$$

Let $p' \leq_\beta p \restriction \beta$ and $q' \leq_\beta q \restriction \beta$ in $\mathbb{P}_\beta$, and suppose that p' and q' are N -separated in $\mathbb{P}_\beta$. Then $p' + p$ and $q' + q$ are N -separated in $\mathbb{P}_\alpha$.

Lemma 6.35. Let N be α -suitable. Assume that $\text{cf}(\alpha) = \omega$ and $\langle \alpha_n : n < \omega \rangle$ is a strictly increasing sequence of ordinals cofinal in α which belongs to N . Let $\langle p_n : n < \omega \rangle$ and $\langle q_n : n < \omega \rangle$ be descending sequences of conditions in $\mathbb{P}_\alpha$ such that for all positive $n < \omega$, $p_n \restriction \alpha_n$ and $q_n \restriction \alpha_n$ are N -separated in $\mathbb{P}_{\alpha_n}$. Let p^* and q^* be the greatest lower bounds of $\langle p_n : n < \omega \rangle$ and $\langle q_n : n < \omega \rangle$ respectively. Then p^* and q^* are N -separated in $\mathbb{P}_\alpha$.

7. PRESERVATION OF κ

We now complete the argument that the forcing iteration from Section 5 preserves κ , under the assumption that κ is ineffable.

Notation 7.1. A sequence $\langle N_\gamma : \gamma \in S \rangle$, where $S \subseteq \kappa$, is said to be α -suitable if:

- (1) each N_γ is α -suitable;
- (2) $\gamma < \xi$ in S implies that $N_\gamma \in N_\xi$;
- (3) if $\delta \in S$ is a limit point of S , then $N_\delta = \bigcup \{N_\gamma : \gamma \in S \cap \delta\}$.

Note that an α -suitable sequence is suitable in the sense of Definition 2.1.

Recall that we are assuming as an inductive hypothesis that for all $\beta < \alpha$, $\mathbb{P}_\beta$ preserves κ . We will need to isolate two additional inductive hypotheses. Later we will verify that these hypotheses hold at α as well.

Inductive Hypothesis 7.2. Let $\beta < \alpha$. Suppose that $\langle N_\lambda : \lambda \in S \rangle$ is β -suitable, where $S \in J^+$. Let T be the set of $\lambda \in S$ such that $p(N_\lambda, \beta)$ is strongly $(N_\lambda, \mathbb{P}_\beta)$ -generic. Then $S \setminus T \in J$.

Inductive Hypothesis 7.3. Let $\beta < \alpha$. Suppose that $\langle N_\lambda : \lambda \in S \rangle$ is β -suitable, where $S \in J^+$. Let T be the set of $\lambda \in S$ such that whenever $\#_{N_\lambda}^\beta(p, q, s)$ holds, then there is $s' \leq_\alpha s$ such that $*_{N_\lambda}^\beta(p, q, s')$ holds. Then $S \setminus T \in J$.

Lemma 7.4. *Let $\langle N_\lambda : \lambda \in S \rangle$ be α -suitable, with union N , where $S \in J^+$. Let T be the set of $\lambda \in S$ which satisfy that whenever $\#_{N_\lambda}^\alpha(p, q, s)$ holds, then there exists $(p', q', s') \leq_\alpha (p, q, s)$ in $\mathbb{P}_\alpha$ satisfying:*

- (1) $\#_{N_\lambda}^\alpha(p', q', s')$;
- (2) p' and q' are N_λ -separated in $\mathbb{P}_\alpha$.

Then $S \setminus T \in J$, and in particular, $T \in J^+$.

Proof. The proof is by induction on $\alpha < \kappa^+$. Note that when $\alpha \leq \kappa$, the statement is vacuously true. So assume that $\kappa < \alpha$, and the lemma holds for $\mathbb{P}_\beta$ for all $\beta < \alpha$. For each $\beta < \alpha$, let $S_\beta := \{\lambda \in S : \beta \in N_\lambda\}$, which is a tail of S . Then the sequence $\langle N_\lambda : \lambda \in S_\beta \rangle$ is β -suitable, with union N , and $S_\beta \in J^+$. So by the inductive hypothesis, we can fix a set $C_\beta \in J^*$ such that for all $\lambda \in S_\beta \cap C_\beta$, whenever $\#_{N_\lambda}^\beta(p, q, s)$ holds, then there exists $(p', q', s') \leq_\beta (p, q, s)$ satisfying:

- (1) $\#_{N_\lambda}^\beta(p', q', s')$;
- (2) p' and q' are N_λ -separated in $\mathbb{P}_\beta$.

Define

$$C := \{\lambda < \kappa : \text{if } \lambda \in S, \text{ then } \forall \beta \in N_\lambda \cap \alpha, \lambda \in C_\beta\}.$$

Then $C \in J^*$ since J is normal. Specifically, if $C \notin J^*$, then $C' := \kappa \setminus C$ is a subset of S in J^+ . By Lemma 2.2, we can find $S' \subseteq C'$ in J^+ and $\beta < \alpha$ such that for all $\lambda \in S'$, $\lambda \notin C_\beta$. This is impossible, since $S' \in J^+$ and $C_\beta \in J^*$ imply that $S' \cap C_\beta \neq \emptyset$. So it suffices to show that $C \cap (S \setminus T) \in J$.

The proof splits into the three cases of whether α is a successor ordinal, α is a limit ordinal with countable cofinality, or α is a limit ordinal with uncountable cofinality.

Case 1: α is a limit ordinal with uncountable cofinality. We claim that $C \cap (S \setminus T) = \emptyset$. It suffices to show that if $\lambda \in C \cap S$, then $\lambda \in T$. So let $\lambda \in C \cap S$, and assume that $\#_{N_\lambda}^\alpha(p, q, s)$ holds. We will find $(p', q', s') \leq_\alpha (p, q, s)$ in $\mathbb{P}_\alpha$ satisfying that $\#_{N_\lambda}^\alpha(p', q', s')$ and p' and q' are N_λ -separated in $\mathbb{P}_\alpha$.

Since $N_\lambda^\omega \subseteq N_\lambda$ and $\text{cf}(\alpha) > \omega$, it easily follows that $\text{cf}(\sup(N_\lambda \cap \alpha)) > \omega$. So we can fix $\beta \in N_\lambda \cap \alpha$ such that

$$(\text{dom}(a_p) \cup \text{dom}(a_q)) \cap N_\lambda \subseteq \beta.$$

By Lemma 6.23, fix $(p_0, q_0, s_0) \leq_\alpha (p, q, s)$ in $D_{\beta, \alpha}$ such that $\#_{N_\lambda}^\alpha(p_0, q_0, s_0)$ holds.

By Lemma 6.21(1), $\#_{N_\lambda}^\beta(p_0 \restriction \beta, q_0 \restriction \beta, s_0 \restriction \beta)$ holds.

Since $\beta \in N_\lambda$ and $\lambda \in C$, we have that $\lambda \in C_\beta \cap S_\beta$. By the definition of C_β , we can fix $(p_1, q_1, s_1) \leq_\beta (p_0 \restriction \beta, q_0 \restriction \beta, s_0 \restriction \beta)$ in $\mathbb{P}_\beta$ satisfying that $\#_{N_\lambda}^\beta(p_1, q_1, s_1)$ holds and p_1 and q_1 are N_λ -separated in $\mathbb{P}_\beta$.

Define $p' := p_1 + p_0$, $q' := q_1 + q_0$, and $s' := s_1 + s_0$. Then $(p', q', s') \leq_\alpha (p_0, q_0, s_0) \leq_\alpha (p, q, s)$. By Lemma 6.21(2), $\#_{N_\lambda}^\alpha(p', q', s')$ holds, and by Lemma 6.34, p' and q' are N_λ -separated in $\mathbb{P}_\alpha$.

Case 2: $\text{cf}(\alpha) = \omega$. We claim that $C \cap (S \setminus T) = \emptyset$. It suffices to show that if $\lambda \in C \cap S$, then $\lambda \in T$. So let $\lambda \in C \cap S$, and assume that $\#_{N_\lambda}^\alpha(p, q, s)$ holds. We will find $(p', q', s') \leq_\alpha (p, q, s)$ satisfying that $\#_{N_\lambda}^\alpha(p', q', s')$ holds and p' and q' are N_λ -separated in $\mathbb{P}_\alpha$.

Since $\alpha \in N_\lambda$, by elementarity we can fix in N_λ a sequence $\langle \alpha_n : n < \omega \rangle$ which is increasing and cofinal in α . Since $\lambda \in C$, we have that for all $n < \omega$, $\lambda \in C_{\alpha_n} \cap S_{\alpha_n}$.

We define by induction a descending sequence $\langle (p_n, q_n, s_n) : n < \omega \rangle$ in $\mathbb{P}_\alpha$ below (p, q, s) satisfying that for all $n < \omega$:

- (1) $\#_{N_\lambda}^\alpha(p_n, q_n, s_n)$;
- (2) for all positive $n < \omega$, $p_n \restriction \alpha_n$ and $q_n \restriction \alpha_n$ are N_λ -separated in $\mathbb{P}_{\alpha_n}$.

If such a sequence can be constructed, then let p' , q' , and s' be the greatest lower bounds of $\langle p_n : n < \omega \rangle$, $\langle q_n : n < \omega \rangle$, and $\langle s_n : n < \omega \rangle$ respectively. By Lemma 6.22, $\#_{N_\lambda}^\alpha(p', q', s')$ holds, and by Lemma 6.35, p' and q' are N_λ -separated in $\mathbb{P}_\alpha$.

Let $p_0 := p$, $q_0 := q$, and $s_0 := s$. Now fix $n < \omega$ and assume that (p_n, q_n, s_n) is defined as required. By Lemma 6.23, we can fix $(p'_n, q'_n, s'_n) \leq_\alpha (p_n, q_n, s_n)$ such that $\#_{N_\lambda}^\alpha(p'_n, q'_n, s'_n)$ holds and for all $n < \omega$, p'_n , q'_n , and s'_n are in $D_{\alpha_{n+1}, \alpha}$. By Lemma 6.21(1), $\#_{N_\lambda}^{\alpha_{n+1}}(p'_n \restriction \alpha_{n+1}, q'_n \restriction \alpha_{n+1}, s'_n \restriction \alpha_{n+1})$ holds.

Since $\lambda \in C_{\alpha_{n+1}} \cap S_{\alpha_{n+1}}$, there is

$$(u_n, v_n, t_n) \leq_{\alpha_{n+1}} (p'_n \restriction \alpha_{n+1}, q'_n \restriction \alpha_{n+1}, s'_n \restriction \alpha_{n+1})$$

such that $\#_{N_\lambda}^{\alpha_{n+1}}(u_n, v_n, t_n)$ holds and u_n and v_n are N_λ -separated in $\mathbb{P}_{\alpha_{n+1}}$. Define $p_{n+1} := u_n + p'_n$, $q_{n+1} := v_n + q'_n$, and $s_{n+1} := t_n + s'_n$. By Lemma 6.21(2), $\#_{N_\lambda}^\alpha(p_{n+1}, q_{n+1}, s_{n+1})$ holds. And $p_{n+1} \restriction \alpha_{n+1} = u_n$ and $q_{n+1} \restriction \alpha_{n+1} = v_n$ are N_λ -separated in $\mathbb{P}_{\alpha_{n+1}}$. This completes the construction.

Case 3: $\alpha = \beta + 1$ is a successor ordinal. Applying Inductive Hypotheses 7.2 and 7.3 to the β -suitable sequence $\langle N_\lambda : \lambda \in S_\beta \rangle$, we can fix a set $D \in J^*$ such that for all $\lambda \in S_\beta \cap D$:

- (1) $p(N_\lambda, \beta)$ is strongly $(N_\lambda, \mathbb{P}_\beta)$ -generic;
- (2) whenever $\#_{N_\lambda}^\beta(p, q, s)$ holds, then there is $s' \leq_\beta s$ such that $*_{N_\lambda}^\beta(p, q, s')$ holds.

By the inductive hypotheses, $\mathbb{P}_\beta$ preserves κ , and hence forces that $\kappa = \omega_2$. By Proposition 2.4, we can fix a set $E \in J^*$ such that for all $\lambda \in S_\beta \cap E$, $N_\lambda \cap \mathbb{P}_\beta$ forces that $\lambda = \omega_2$, and $N_\lambda \cap \dot{T}_\beta$ and $N_\lambda \cap \dot{U}_\beta$ are $(N_\lambda \cap \mathbb{P}_\beta)$ -names for trees which have no chains of order type λ .

We claim that $C \cap D \cap E \cap (S \setminus T) = \emptyset$. It suffices to show that if $\lambda \in C \cap D \cap E \cap S$, then $\lambda \in T$. So let $\lambda \in C \cap D \cap E \cap S$, and assume that $\#_{N_\lambda}^\alpha(p, q, s)$ holds. We will find $(p', q', s') \leq_\alpha (p, q, s)$ satisfying that $\#_{N_\lambda}^\alpha(p', q', s')$ holds and p' and q' are N_λ -separated in $\mathbb{P}_\alpha$.

By Lemma 6.23, fix $(p_0, q_0, s_0) \leq_\alpha (p, q, s)$ in $D_{\beta, \alpha}$ such that $\#_{N_\lambda}^\alpha(p_0, q_0, s_0)$ holds. Let $\langle \theta_n : n < r_0 \rangle$, where $r_0 \leq \omega$, enumerate all ordinals in $\text{dom}(f_{a_{p_0}(\beta)}) \cup \text{ran}(f_{a_{p_0}(\beta)})$ in I_λ , and let $\langle \tau_n : n < r_1 \rangle$, where $r_1 \leq \omega$, enumerate all ordinals in $\text{dom}(f_{a_{q_0}(\beta)}) \cup \text{ran}(f_{a_{q_0}(\beta)})$ in I_λ .

Let $n \mapsto (n_0, n_1)$ denote a bijection from ω onto $\omega \times \omega$. We will define by induction a sequence $\langle (u_n, v_n, t_n) : n < \omega \rangle$ of conditions in $\mathbb{P}_\beta$ below $(p_0 \restriction \beta, q_0 \restriction \beta, s_0 \restriction \beta)$ satisfying that for all $n < \omega$:

- (1) $\#_{N_\lambda}^\beta(u_n, v_n, t_n)$ holds;
- (2) if $n_0 < r_0$ and $n_1 < r_1$ and θ_{n_0} and τ_{n_1} are either both even or both odd, then (u_{n+1}, v_{n+1}) λ -separates $(\theta_{n_0}, \tau_{n_1})$ in $\mathbb{P}_\beta$.

Let $u_0 := p_0 \restriction \beta$, $v_0 := q_0 \restriction \beta$, and $t_0 := s_0 \restriction \beta$. By Lemma 6.21(1), $\#_{N_\lambda}^\beta(u_0, v_0, t_0)$ holds.

Fix $n < \omega$ and assume that (u_n, v_n, t_n) is defined as required. If it is not the case that $n_0 < r_0$, $n_1 < r_1$, and θ_{n_0} and τ_{n_1} are either both even or both odd, then let $(u_{n+1}, v_{n+1}, t_{n+1}) := (u_n, v_n, t_n)$.

Otherwise, since $\lambda \in S_\beta \cap D$, we can fix $t_n^* \leq_\beta t_n$ such that $*_{N_\lambda}^\beta(u_n, v_n, t_n^*)$ holds. Also, $p(N_\lambda, \beta)$ is strongly $(N_\lambda, \mathbb{P}_\beta)$ -generic. By Proposition 6.32 (applied to β), there is $(u'_n, v'_n, t'_n) \leq_\beta (u_n, v_n, t_n^*)$ satisfying that $*_{N_\lambda}^\beta(u'_n, v'_n, t'_n)$ holds and (u'_n, v'_n) λ -separates $(\theta_{n_0}, \tau_{n_1})$ in $\mathbb{P}_\beta$. By Lemma 6.29 (applied to β), we can fix $(u_{n+1}, v_{n+1}, t_{n+1}) \leq_\beta (u'_n, v'_n, t'_n)$ such that $\#_{N_\lambda}^\beta(u_{n+1}, v_{n+1}, t_{n+1})$ holds. Then also (u_{n+1}, v_{n+1}) λ -separates $(\theta_{n_0}, \tau_{n_1})$.

This completes the construction. Let u, v , and t denote the greatest lower bounds in $\mathbb{P}_\beta$ of the sequences $\langle u_n : n < \omega \rangle$, $\langle v_n : n < \omega \rangle$, and $\langle t_n : n < \omega \rangle$ respectively. By Lemma 6.22, $\#_{N_\lambda}^\beta(u, v, t)$ holds.

Since $\lambda \in S_\beta \cap C_\beta$, there exists $(u', v', t') \leq_\beta (u, v, t)$ satisfying:

- (1) $\#_{N_\lambda}^\beta(u', v', t')$;
- (2) u' and v' are N_λ -separated in $\mathbb{P}_\beta$.

Let $p' := u' + p_0$, $q' := v' + q_0$, and $s' := t' + s_0$. Then $(p', q', s') \leq_\alpha (p, q, s)$, and by Lemma 6.21(2), $\#_{N_\lambda}^\alpha(p', q', s')$ holds. As u' and v' are N_λ -separated in $\mathbb{P}_\beta$, by our construction which handled separation at β , it is easy to check that p' and q' are N_λ -separated in $\mathbb{P}_\alpha$. $\square$

Proposition 7.5. *Let $\langle N_\lambda : \lambda \in S \rangle$ be α -suitable, with union N , where $S \in J^+$. Assume that for each $\lambda \in S$, we have fixed conditions p_λ , q_λ , and s_λ such that $\#_{N_\lambda}^\alpha(p_\lambda, q_\lambda, s_\lambda)$ holds. Then there exists $U \subseteq S$ in J^+ such that for all $\lambda < \mu$ in U , p_λ and q_μ are compatible in $\mathbb{P}_\alpha$.*

Proof. By Lemma 7.4, we can fix $T \subseteq S$ in J^+ such that for all $\lambda \in T$, there is $(p_\lambda^*, q_\lambda^*, s_\lambda^*) \leq_\alpha (p_\lambda, q_\lambda, s_\lambda)$ satisfying that $\#_{N_\lambda}^\alpha(p_\lambda^*, q_\lambda^*, s_\lambda^*)$ holds, and p_λ^* and q_λ^* are N_λ -separated in $\mathbb{P}_\alpha$. By intersecting T with a club if necessary, let us also assume that $N_\lambda \cap \kappa = \lambda$ for all $\lambda \in T$. It suffices to find $U \subseteq T$ in J^+ such that for all $\lambda < \mu$ in U , p_λ^* and q_μ^* are compatible in $\mathbb{P}_\alpha$.

Consider $\lambda \in T$. Define

$$J_\lambda := (\text{dom}(a_{p_\lambda^*}(0)) \cup \text{dom}(a_{q_\lambda^*}(0))) \setminus (\omega_1 \times \lambda)$$

Note that J_λ is a subset of $N \setminus N_\lambda$ of size less than κ . Define

$$K_\lambda := \bigcup \{M \cap N_\lambda : M \in \bigcup \text{ran}(X_{q_\lambda^*}), M \cap \kappa < \lambda\}.$$

Note that K_λ is the union of countably many subsets of N_λ each of size less than λ . Hence, K_λ is a subset of N_λ of size less than λ . As $N_\lambda^{<\lambda} \subseteq N_\lambda$, it follows that $K_\lambda \in N_\lambda$.

Since $\#_{N_\lambda}^\alpha(p_\lambda^*, q_\lambda^*, s_\lambda^*)$ holds, we have that $p_\lambda^* \restriction N_\lambda = s_\lambda^* = q_\lambda^* \restriction N_\lambda$. In particular,

$$\text{dom}(a_{p_\lambda^*}) \cap N_\lambda = \text{dom}(a_{s_\lambda^*}) = \text{dom}(a_{q_\lambda^*}) \cap N_\lambda.$$

Consider β in $\text{dom}(a_{s_\lambda^*})$. Enumerate all members of $\text{dom}(f_{a_{p_\lambda^*}(\beta)}) \cup \text{ran}(f_{a_{q_\lambda^*}(\beta)})$ in I_λ as

$$\langle \theta(\lambda, \beta, m) : m < r_{\lambda, \beta, 0} \rangle,$$

where $r_{\lambda, \beta, 0} \leq \omega$, and enumerate all members of $\text{dom}(f_{a_{q_\lambda^*}(\beta)}) \cup \text{ran}(f_{a_{q_\lambda^*}(\beta)})$ in I_λ as

$$\langle \tau(\lambda, \beta, n) : n < r_{\lambda, \beta, 1} \rangle,$$

where $r_{\lambda,\beta,1} \leq \omega$.

Now define a partial function

$$f_\lambda : \text{dom}(a_{s_\lambda^*}) \times r_{\lambda,\beta,0} \times r_{\lambda,\beta,1} \rightarrow \lambda \times \lambda$$

so that $f_\lambda(\beta, m, n)$ is defined iff $\theta(\lambda, \beta, m)$ and $\tau(\lambda, \beta, n)$ are either both even or both odd, in which case $f_\lambda(\beta, m, n)$ is equal to some witness (θ', τ') to the fact that the pair $(p_\lambda^* \upharpoonright \beta, q_\lambda^* \upharpoonright \beta)$ λ -separates $(\theta(\lambda, \beta, m), \tau(\lambda, \beta, n))$ in $\mathbb{P}_\beta$. Specifically, if $f_\lambda(\beta, m, n) = (\theta', \tau')$, then $\theta' \neq \tau'$, and for some $\gamma < \lambda$, θ' and τ' are in I_γ , $p_\lambda^* \upharpoonright \beta \Vdash_{\mathbb{P}_\beta} \theta' <_{\dot{W}} \theta$ and $q_\lambda^* \upharpoonright \beta \Vdash_{\mathbb{P}_\beta} \tau' <_{\dot{W}} \tau$, where $\dot{W} = \dot{I}_\beta$ in the case that $\theta(\lambda, \beta, m)$ and $\tau(\lambda, \beta, n)$ are both even, and $\dot{W} = \dot{U}_\beta$ in the case that $\theta(\lambda, \beta, m)$ and $\tau(\lambda, \beta, n)$ are both odd. In addition, define a function $g_\lambda : \text{dom}(a_{s_\lambda^*}) \rightarrow [\omega]^{<\omega}$ by letting $g_\lambda(\beta)$ be the set of $n < r_{\lambda,\beta,1}$ such that $\tau(\lambda, \beta, n)$ is even. Note that f_λ and g_λ are countable subsets of N_λ , and hence are members of N_λ .

To summarize, we have that s_λ^* , K_λ , f_λ , and g_λ are members of N_λ , and J_λ is a subset of $N \setminus N_\lambda$ of size less than κ . By Lemma 2.2, we can fix $U \subseteq T$ in J^+ such that for all $\lambda < \mu$ in U , $s_\lambda^* = s_\mu^*$, $K_\lambda = K_\mu$, $f_\lambda = f_\mu$, $g_\lambda = g_\mu$, $p_\lambda^* \in N_\mu$, and $J_\lambda \cap J_\mu = \emptyset$.

Fix $\lambda < \mu$ in U , and we will show that p_λ^* and q_μ^* are compatible in $\mathbb{P}_\alpha$. For notational simplicity, let $p := p_\lambda^*$ and $q := q_\mu^*$. We will define a lower bound $r = (a_r, X_r)$ of p and q .

Let the domain of a_r be equal to $\text{dom}(a_p) \cup \text{dom}(a_q)$ and the domain of X_r be equal to $\text{dom}(X_p) \cup \text{dom}(X_q)$. For each $\beta \in \text{dom}(X_r)$, define $X_r(\beta) := X_p(\beta) \cup X_q(\beta)$.

We will define $a_r(\gamma)$ for all $\gamma \in \text{dom}(a_r)$ by induction on γ . Assuming that $\beta \leq \alpha$ and $a_r(\gamma)$ is defined for all $\gamma \in \text{dom}(a_r) \cap \beta$, we will maintain two inductive hypotheses:

- (1) $r \upharpoonright \beta := (a_r \upharpoonright \beta, X_r \upharpoonright (\beta + 1))$ is in $\mathbb{P}_\beta$ and extends $p \upharpoonright \beta$ and $q \upharpoonright \beta$ in $\mathbb{P}_\beta$;
- (2) for all nonzero $\gamma \in \text{dom}(a_r) \cap \beta$ and $\xi \in \text{dom}(X_r)$ with $\gamma < \xi$, for all $M \in X_r(\xi)$ with $\gamma \in M$, $r \upharpoonright \gamma \Vdash_{\mathbb{P}_\gamma} M \cap \kappa \in A_{a_r(\gamma)}$.

To begin, let $a_r(0) := a_p(0) \cup a_q(0)$. Since $p \upharpoonright N_\lambda = s_\lambda^* = s_\mu^* = q \upharpoonright N_\mu$, we have that

$$a_p(0) \upharpoonright (\omega_1 \times \lambda) = a_{s_\lambda^*}(0) = a_{s_\mu^*}(0) = a_q(0) \upharpoonright (\omega_1 \times \mu).$$

Also, $\text{dom}(a_p(0)) \setminus (\omega_1 \times \lambda) \subseteq J_\lambda$, $\text{dom}(a_q(0)) \setminus (\omega_1 \times \mu) \subseteq J_\mu$, and $J_\lambda \cap J_\mu = \emptyset$. It easily follows that $a_r(0)$ is a function, and hence is in $\text{Col}(\omega_1, < \kappa)$, and $a_r(0)$ extends $a_p(0)$ and $a_q(0)$ in $\text{Col}(\omega_1, < \kappa)$. Thus, $r \upharpoonright 1 = (a_r \upharpoonright 1, \emptyset)$ is as required. We also have that $r \upharpoonright \beta = r \upharpoonright 1$ for all $1 \leq \beta \leq \kappa$, and these objects obviously satisfy the inductive hypotheses.

Assume that $\kappa < \beta \leq \alpha$ is a limit ordinal and $a_r(\gamma)$ is defined for all $\gamma \in \text{dom}(a_r) \cap \beta$. Assume that for all $\beta' < \beta$, $r \upharpoonright \beta'$ satisfies the inductive hypotheses. Let us check that $r \upharpoonright \beta = (a_r \upharpoonright \beta, X_r \upharpoonright (\beta + 1))$ is as required.

By inductive hypothesis (1), we know that for all $\beta' < \beta$,

$$r \upharpoonright \beta' = (a_r \upharpoonright \beta', X_r \upharpoonright (\beta' + 1))$$

is in $\mathbb{P}_{\beta'}$ and is below $p \upharpoonright \beta'$ and $q \upharpoonright \beta'$.

Let us show that $r \upharpoonright \beta$ is in $\mathbb{P}_\beta$. Referring to the definition of $\mathbb{P}_\beta$ from Section 5 in the case that β is a limit ordinal, requirements (1)–(4) are immediate. For requirement (5), we need to show that if $M \in X_r(\beta)$ and $\gamma \in M \cap \text{dom}(a_r) \cap \beta$

is nonzero, then $r \restriction \gamma \Vdash_{\mathbb{P}_\gamma} M \cap \kappa \in A_{a_r(\gamma)}$. But this follows immediately from inductive hypothesis (2) holding for $r \restriction (\gamma + 1)$.

Using inductive hypothesis (1) and the definition of X_r , it is simple to check that $r \restriction \beta$ extends $p \restriction \beta$ and $q \restriction \beta$ in $\mathbb{P}_\beta$. Also, inductive hypothesis (2) for $r \restriction \beta$ follows immediately from the fact that it holds for $r \restriction \beta'$ for all $\beta' < \beta$.

Now assume that $\kappa \leq \beta < \alpha$ and $a_r \restriction \beta$ is defined as required. We will define $a_r(\beta)$, and then show that $r \restriction (\beta + 1) = (a_r \restriction (\beta + 1), X_r \restriction (\beta + 2))$ satisfies the inductive hypotheses. We will consider four separate cases.

Case 1: $\beta \notin \text{dom}(a_p) \cup \text{dom}(a_q)$. In this case, $\beta \notin \text{dom}(a_r)$ and

$$r \restriction (\beta + 1) = (a_r \restriction \beta, X_r \restriction (\beta + 2)).$$

It is easy to check that $r \restriction (\beta + 1)$ is as required.

Case 2: $\beta \in \text{dom}(a_q) \setminus \text{dom}(a_p)$. Since $\text{dom}(a_q) \cap N_\mu = \text{dom}(a_{s_\mu^*}) = \text{dom}(a_{s_\lambda^*}) = \text{dom}(a_p) \cap N_\lambda$ and $\beta \notin \text{dom}(a_p)$, it follows that $\beta \notin N_\mu$. Define $a_r(\beta) := a_q(\beta)$.

Let us show that if $\xi > \beta$ is in $\text{dom}(X_r)$, $M \in X_r(\xi)$, and $\beta \in M$, then $r \restriction \beta \Vdash_{\mathbb{P}_\beta} M \cap \kappa \in A_{a_q(\beta)}$. This statement together with the inductive hypotheses for $r \restriction \beta$ easily imply that $r \restriction (\beta + 1)$ is as required.

If $M \in X_q(\xi)$, then we are done since q is a condition and $r \restriction \beta \leq_\beta q \restriction \beta$. Assume that $M \in X_p(\xi)$. Since $p \in N_\mu$, $M \in N_\mu$. So $M \subseteq N_\mu$. But then $\beta \in M \subseteq N_\mu$, so $\beta \in N_\mu$. This contradicts the observation above that $\beta \notin N_\mu$.

Case 3: $\beta \in \text{dom}(a_p) \setminus \text{dom}(a_q)$. Then $\beta \notin N_\lambda$, for otherwise $\beta \in \text{dom}(a_p) \cap N_\lambda = \text{dom}(a_{s_\lambda^*}) = \text{dom}(a_{s_\mu^*}) = \text{dom}(a_q) \cap N_\mu$, which contradicts that $\beta \notin \text{dom}(a_q)$.

Since $p \in N_\mu$, we have that $\sup(A_{a_p(\beta)}) < N_\mu \cap \kappa = \mu$. Let x be the set of ordinals of the form $M \cap \kappa$, where $M \in X_q(\xi)$ for some $\xi > \beta$ and $M \cap \kappa \geq \mu$. Then x is countable and consists of ordinals of uncountable cofinality which are greater than or equal to μ . By Lemma 4.5, we can fix a nice $\mathbb{P}_\beta$ -name $a_r(\beta)$ which $r \restriction \beta$ forces is an extension of $a_p(\beta)$ such that $x \subseteq A_{a_r(\beta)}$.

We will show that whenever $\xi > \beta$ is in $\text{dom}(X_r)$, $M \in X_r(\xi)$, and $\beta \in M$, then $r \restriction \beta \Vdash_{\mathbb{P}_\beta} M \cap \kappa \in a_r(\beta)$. This claim together with the inductive hypotheses easily imply that $r \restriction (\beta + 1)$ is as required.

Since $r \restriction \beta$ forces that $A_{a_p(\beta)} \subseteq A_{a_r(\beta)}$, if $M \in X_p(\xi)$ then we are done since p is a condition. Suppose that $M \in X_q(\xi)$. If $M \cap \kappa \geq \mu$, then $M \cap \kappa \in x$, so we are done by the choice of $a_r(\beta)$.

We claim that the remaining case $M \cap \kappa < \mu$ does not occur. Assume for a contradiction that $M \cap \kappa < \mu$. Then $M \cap N_\mu \subseteq K_\mu$ by the definition of K_μ . But $K_\mu = K_\lambda$. So $M \cap N_\mu \subseteq K_\lambda \subseteq N_\lambda$. Note that $\beta \in M \cap N_\mu$; namely, $\beta \in M$ by assumption, and $\beta \in N_\mu$, since $p \in N_\mu$. So $\beta \in N_\lambda$. But we observed above that $\beta \notin N_\lambda$, and we have a contradiction.

Case 4: $\beta \in \text{dom}(a_p) \cap \text{dom}(a_q)$. We claim that it suffices to show that $r \restriction \beta$ forces that $a_p(\beta)$ and $a_q(\beta)$ are compatible in $\mathbb{P}(\dot{T}_\beta, \dot{U}_\beta)$. For then we can choose a $\mathbb{P}_\beta$ -name $a_r(\beta)$ which $r \restriction \beta$ forces is below $a_p(\beta)$ and $a_q(\beta)$. As in the previous cases, it then suffices to verify that whenever $\xi > \beta$ is in $\text{dom}(X_r)$, $M \in X_r(\xi)$, and $\beta \in M$, then $r \restriction \beta \Vdash_{\mathbb{P}_\beta} M \cap \kappa \in a_r(\beta)$. Since $\beta \in \text{dom}(a_p) \cap \text{dom}(a_q)$, if $M \in X_p(\xi)$ then $p \restriction \beta$, and hence $r \restriction \beta$, forces that $M \cap \kappa \in A_{a_p(\beta)}$. And if $M \in X_q(\xi)$,

then $q \restriction \beta$, and hence $r \restriction \beta$, forces that $M \cap \kappa \in A_{a_q(\beta)}$. But $r \restriction \beta$ forces that $A_{a_p(\beta)} \cup A_{a_q(\beta)} \subseteq A_{a_r(\beta)}$. So in either case, $r \restriction \beta$ forces that $M \cap \kappa \in A_{a_r(\beta)}$.

We now complete the proof by showing that $r \restriction \beta$ forces that $a_p(\beta)$ and $a_q(\beta)$ are compatible. For notational simplicity, let $a_p(\beta) = (f, A)$, $a_q(\beta) = (g, B)$, and $s = s_\lambda^* = s_\mu^*$. Then $p \restriction N_\lambda = s = q \restriction N_\mu$. So $\beta \in \text{dom}(a_s)$ and $(A, f) \restriction \lambda = a_s(\beta) = (B, g) \restriction \mu$.

Since $\#_{N_\lambda}^\alpha(p, s)$ and $\#_{N_\mu}^\alpha(q, s)$ hold, in particular, $p \in D(N_\lambda, \alpha)$ and $q \in D(N_\mu, \alpha)$. As $p \in N_\mu$, by elementarity $\beta \in N_\mu$. Hence, $\beta \in \text{dom}(a_q) \cap N_\mu = \text{dom}(a_s) = \text{dom}(a_p) \cap N_\lambda$. So $\beta \in N_\lambda \cap N_\mu$. Also, since $p \in N_\mu$, by elementarity $A \subseteq \mu$.

The fact that $p \in D(N_\lambda, \alpha)$ and $\beta \in \text{dom}(a_p) \cap N_\lambda$ implies by definition that $p \restriction \beta \Vdash_{\mathbb{P}_\beta} a_p(\beta)$ is injective on $N_\lambda \cap \kappa = \lambda$. Similarly, $q \restriction \beta \Vdash_{\mathbb{P}_\beta} a_q(\beta)$ is injective on μ .

Since $p \in D(N_\lambda, \alpha)$ and $q \in D(N_\mu, \alpha)$, we have that $N_\lambda \cap \alpha \in X_p(\alpha)$ and $N_\mu \cap \alpha \in X_q(\alpha)$. As $\beta \in \text{dom}(a_p) \cap (N_\lambda \cap \alpha)$, p being a condition implies that $p \restriction \beta$ forces that $(N_\lambda \cap \alpha) \cap \kappa = N_\lambda \cap \kappa = \lambda \in A_{a_p(\beta)}$. Hence, $\lambda \in A_{a_p(\beta)} = A$. Similarly, $\mu \in B$.

To summarize, the following statements are forced by $r \restriction \beta$ to be true:

- (1) $\lambda < \mu$, $\lambda \in A$, and $\mu \in B$;
- (2) $A \subseteq \mu$;
- (3) $(A, f) \restriction \lambda = (B, g) \restriction \mu$;
- (4) (A, f) is injective on λ and (B, g) is injective on μ .

By Proposition 4.14, to show that $r \restriction \beta$ forces that $a_p(\beta)$ and $a_q(\beta)$ are compatible, it suffices to show that $r \restriction \beta$ forces that every node of $\text{dom}(f)$ in I_λ is incomparable in $\dot{T}_\beta$ with every node of $\text{dom}(g)$ in I_μ , and every node of $\text{ran}(f)$ in I_λ is incomparable in $\dot{U}_\beta$ with every node of $\text{ran}(g)$ in I_μ .

Consider $\theta \in \text{dom}(f)$ in I_λ and $\tau \in \text{dom}(g)$ in I_μ . Then θ and τ are both even. Fix $m < r_{\lambda, \beta, 0}$ and $n < r_{\mu, \beta, 1}$ such that $\theta = \theta(\lambda, \beta, m)$ and $\tau = \tau(\mu, \beta, n)$.

Since τ is even, by the definition of g_μ we have that $n \in g_\mu(\beta)$. As $g_\lambda = g_\mu$, $n \in g_\lambda(\beta)$. Therefore, $\tau(\lambda, \beta, n)$ is even. We also have that $\theta = \theta(\lambda, \beta, m)$ is even. By the definition of f_λ , $(\theta', \tau') := f_\lambda(\beta, m, n)$ is defined, θ' and τ' are distinct ordinals in I_γ for some $\gamma < \lambda$, and $p \restriction \beta \Vdash_{\mathbb{P}_\beta} \theta' <_{\dot{T}_\beta} \theta$. But $f_\lambda = f_\mu$. So we also have that $f_\mu(\beta, m, n) = (\theta', \tau')$, and therefore $q \restriction \beta \Vdash_{\mathbb{P}_\beta} \tau' <_{\dot{T}_\beta} \tau$.

Since $r \restriction \beta$ is below $p \restriction \beta$ and $q \restriction \beta$, it forces that $\theta' <_{\dot{T}_\beta} \theta$ and $\tau' <_{\dot{T}_\beta} \tau$. Since $\theta' \neq \tau'$ are both in I_γ , $r \restriction \beta$ forces that $p_{\dot{T}_\beta}(\theta, \gamma) = \theta' \neq p_{\dot{T}_\beta}(\tau, \gamma) = \tau'$. This implies that $r \restriction \beta$ forces that θ and τ are incomparable in $\dot{T}_\beta$, since otherwise they would have the same nodes below them on levels less than λ . By a symmetric argument, $r \restriction \beta$ also forces that if $\theta \in \text{ran}(f)$ is in I_λ and $\tau \in \text{ran}(g)$ is in I_μ , then θ and τ are incomparable in $\dot{U}_\beta$. $\square$

Proposition 7.6. *There exists a function $F : H(\kappa^{++})^{<\omega} \rightarrow H(\kappa^{++})$ such that for any α -suitable N which is closed under F , $p(N, \alpha)$ is $(N, \mathbb{P}_\alpha)$ -generic.*

Proof. The proof is by induction on α . Recall that for all $0 < \alpha \leq \kappa$, $\mathbb{P}_\alpha = \mathbb{P}_1$ is forcing equivalent to the Levy collapse $\text{Col}(\omega_1, < \kappa)$, which is κ -c.c. And $p(N, \alpha)$ is the maximum condition in $\mathbb{P}_\alpha$ and is $(N, \mathbb{P}_\alpha)$ -generic for any α -suitable N .

Now assume that $\kappa < \alpha < \kappa^+$ and the proposition holds for all ordinals below α . For each $\beta < \alpha$, fix a function $F_\beta : H(\kappa^{++})^{<\omega} \rightarrow H(\kappa^{++})$ such that for

any β -suitable N which is closed under F_β , $p(N, \beta)$ is $(N, \mathbb{P}_\beta)$ -generic. Define $F^* : \alpha \times H(\kappa^{++})^{<\omega} \rightarrow H(\kappa^{++})$ by $F^*(\beta, x) = F_\beta(x)$.

Let $\mathcal{X}$ denote the collection of all α -suitable sequences. Fix a Skolem function F for the structure $(H(\kappa^{++}), \in, \preceq, \mathcal{X}, F^*)$. Let N be α -suitable and closed under F , and we will show that $p(N, \alpha)$ is $(N, \mathbb{P}_\alpha)$ -generic. Let $\lambda := N \cap \kappa$. Note that by elementarity, for all $\beta \in N \cap \alpha$, N is closed under F_β , and therefore $p(N, \beta)$ is $(N, \mathbb{P}_\beta)$ -generic.

To prove that $p(N, \alpha)$ is $(N, \mathbb{P}_\alpha)$ -generic, fix a dense open subset D of $\mathbb{P}_\alpha$ which is a member of N . We will show that $D \cap N$ is predense below $p(N, \alpha)$. Let $r_0 \leq_\alpha p(N, \alpha)$, and we will prove that there exists a member of $D \cap N$ which is compatible in $\mathbb{P}_\alpha$ with r_0 . Since D is dense open, we can fix $r \leq_\alpha r_0$ in D . Moreover, by extending r further if necessary using Lemma 6.19, we may assume without loss of generality that $\#_N^\alpha(r, r \upharpoonright N)$ holds.

Suppose for a contradiction that there is no member of $D \cap N$ which is compatible with r . We claim that there exists a sequence $\langle (r_i, N_i) : i \in S \rangle$ in N satisfying:

- (1) the sequence $\langle N_i : i \in S \rangle$ is α -suitable;
- (2) $S \in J^+$;
- (3) for all $i \in S$, $N_i \cap \kappa = i$ if $i > 0$, N_i is closed under F^* , $r_i \in D$, and $\#_{N_i}^\alpha(r_i, r_i \upharpoonright N_i)$ holds;
- (4) for all $i < j$ in S , $r_i \in N_j$ and r_i and r_j are incompatible in $\mathbb{P}_\alpha$.

The definition of such a sequence is by induction. For the base case, we let $0 \in S$, and pick an α -suitable N_0 which is closed under F^* such that $r \upharpoonright N \in N_0$, and some r_0 such that $r_0 \in D$ and $\#_{N_0}^\alpha(r_0, r_0 \upharpoonright N_0)$ holds. Such objects clearly exist by the elementarity of N .

Let $\beta < \kappa$ and assume that $S \cap \beta$ and $\langle (r_i, N_i) : i \in S \cap \beta \rangle$ are defined and satisfy properties (1), (3), and (4) above for $S \cap \beta$ in place of S . Let $\gamma \geq \beta$ be the least ordinal below κ , if it exists, for which there is a pair (N, r) such that $\langle (r_i, N_i) : i \in S \cap \beta \rangle \cup \{(\gamma, (N, r))\}$ still satisfies properties (1), (3), and (4) for $(S \cap \beta) \cup \{\gamma\}$ in place of S . If such an ordinal γ does not exist, then let $S = S \cap \beta$ and we are done. Otherwise, define $S \cap (\gamma + 1) := (S \cap \beta) \cup \{\gamma\}$, and let (N_γ, r_γ) be the $\preceq$ -least pair such that $\langle (N_i, r_i) : i \in S \cap (\gamma + 1) \rangle$ satisfies properties (1), (3), and (4).

This completes the definition of $\langle (r_i, N_i) : i \in S \rangle$. Note that by elementarity, this sequence is in N . It remains to show that $S \in J^+$.

First let us note that S is unbounded in κ . If not, then by elementarity $\sup(S) < \lambda$. But recall that N is α -suitable and r is incompatible with every member of $D \cap N$, and in particular, with r_i for all $i \in S \cap \lambda$. And for all $i \in S \cap \lambda$, r_i and N_i are in N . Also, $\#_N^\alpha(r, r \upharpoonright N)$ holds. It easily follows that the sequence $\langle (r_i, N_i) : i \in S \rangle \cup \{(\lambda, (r, N))\}$ satisfies properties (1), (3), and (4) above, which contradicts that $\sup(S) < \lambda$.

So indeed S is cofinal in κ . It follows that the set $\lim(S)$ is club in κ , and hence is in J^* . Also, as we know, the set of inaccessible below κ is in J^* . We will show that every inaccessible limit point of S is in S , which implies that in fact $S \in J^*$.

Suppose for a contradiction there is an inaccessible limit point of S which is not in S . By elementarity, we can fix such an inaccessible μ in $N \cap \kappa$. Let $M := \bigcup \{N_\gamma : \gamma \in S \cap \mu\}$. Then $M \cap \kappa = \mu$. Note that M is closed under F^* . Therefore, if M is α -suitable, which we will check in a moment, it follows that for all $\beta \in M \cap \alpha$, $p(M, \beta)$ is $(M, \mathbb{P}_\beta)$ -generic.

We claim that there exists a condition $r_\mu \in \mathbb{P}_\alpha$ such that the following statements are satisfied:

- (a) $\langle N_i : i \in S \cap \mu \rangle \cup \{(\mu, M)\}$ is α -suitable;
- (b) $r_\mu \in D$ and $\#_M^\alpha(r_\mu, r_\mu \restriction M)$;
- (c) for all $i \in S \cap \mu$, $r_i \in M$ and r_i and r_μ are incompatible in $\mathbb{P}_\alpha$.

It follows from this claim and the definition of S that in fact $\mu \in S$, which contradicts the choice of μ .

By Notation 7.1 and the definition of M , statement (a) holds provided that M is α -suitable. By Definition 6.1 and Notation 6.3, that easily follows from the definition of M together with the fact that $M \cap \kappa = \mu$ is inaccessible. For part of statement (c), we know that for all $i \in S \cap \mu$, $r_i \in N_{\min(S \setminus (i+1))} \subseteq M$, so $r_i \in M$.

We have that $M \in N$ are both α -suitable and for all $\beta \in M \cap \alpha$, $p(M, \beta)$ is $(M, \mathbb{P}_\beta)$ -generic. And also $\#_N^\alpha(r, r \restriction N)$ holds and $r \restriction N \in M$. By Lemma 6.24, there is $r_\mu \leq_\alpha r$ such that $\#_M^\alpha(r_\mu, r_\mu \restriction M)$ holds. As D is dense open and $r \in D$, $r_\mu \in D$. And for all $i \in S \cap \mu$, r_i and r are incompatible, which implies that r_i and r_μ are incompatible. This completes the proof of the claim, and we have a contradiction.

To summarize, we have an α -suitable sequence $\langle N_i : i \in S \rangle$, where $S \in J^+$. And for each $i \in S$, we have a condition r_i such that $\#_{N_i}^\alpha(r_i, r_i \restriction N_i)$ holds, and for all $i < j$ in S , r_i and r_j are incompatible in $\mathbb{P}_\alpha$. Thus, the assumptions of Proposition 7.5 hold for the α -suitable sequence $\langle N_i : i \in S \rangle$ and the associated conditions r_i , r_i , and $r_i \restriction N_i$. It easily follows from the conclusion of Proposition 7.5 that there are $i < j$ in S such that r_i and r_j are compatible, which is a contradiction. $\square$

Corollary 7.7. *The forcing poset $\mathbb{P}_\alpha$ is κ -proper on a stationary set. In particular, $\mathbb{P}_\alpha$ preserves κ .*

Proof. Let $\theta > \kappa^+$ be a regular cardinal. Let F be the function described in Proposition 7.6. Since $\mathcal{V}$ is stationary in $P_\kappa(H(\kappa^{++}))$ by Lemma 6.2, there are stationarily many $M \in P_\kappa(H(\theta))$ such that $M \cap H(\kappa^{++})$ is α -suitable and closed under F .

Consider such a set M . We will prove that every member of $M \cap \mathbb{P}_\alpha$ has an extension which is $(M, \mathbb{P}_\alpha)$ -generic. Let $N := M \cap H(\kappa^{++})$. By the choice of F , $p(N, \alpha)$ is $(N, \mathbb{P}_\alpha)$ -generic. As $\mathbb{P}_\alpha$ is κ^+ -c.c. and is a member of $H(\kappa^{++})$, every maximal antichain of $\mathbb{P}_\alpha$ is a member of $H(\kappa^{++})$. Hence, every maximal antichain of $\mathbb{P}_\alpha$ which is a member of M is also a member of N . It easily follows that $p(N, \alpha)$ is $(M, \mathbb{P}_\alpha)$ -generic. Now if $p \in M \cap \mathbb{P}_\alpha$, then $p \in N \cap \mathbb{P}_\alpha$. Hence by Lemma 6.7, we can fix $r \leq_\alpha p, p(N, \alpha)$. Since $r \leq_\alpha p(N, \alpha)$ and $p(N, \alpha)$ is $(M, \mathbb{P}_\alpha)$ -generic, r is also $(M, \mathbb{P}_\alpha)$ -generic, and we are done. $\square$

This completes the proof that $\mathbb{P}_\alpha$ preserves κ . It remains to show that Inductive Hypotheses 7.2 and 7.3 hold for α .

Proposition 7.8. *Suppose that $\langle N_\lambda : \lambda \in S \rangle$ is α -suitable, with union N , where $S \in J^+$. Let T be the set of $\lambda \in S$ such that $p(N_\lambda, \alpha)$ is strongly $(N_\lambda, \mathbb{P}_\alpha)$ -generic. Then $S \setminus T \in J$.*

Proof. Suppose for a contradiction that $S \setminus T \in J^+$. For each $\lambda \in S \setminus T$, the fact that $p(N_\lambda, \alpha)$ is not strongly $(N_\lambda, \mathbb{P}_\alpha)$ -generic means that there exists a set D_λ which is a dense subset of $N_\lambda \cap \mathbb{P}_\alpha$ such that D_λ is not predense below $p(N_\lambda, \alpha)$. By extending members of D_λ to determined conditions, which is possible since

the determined conditions are dense in $N_\lambda \cap \mathbb{P}_\alpha$, we may assume without loss of generality that D_λ consists of determined conditions.

As $S \setminus T \in J^+$, by Lemma 2.3 there exists a set $D \subseteq N$ and a stationary set $U \subseteq S \setminus T$ such that for all $\lambda \in U$, $D \cap N_\lambda = D_\lambda$. Note that D is a dense subset of $\mathbb{P}_\alpha$. Namely, if $p \in \mathbb{P}_\alpha$, then for any large enough $\lambda \in U$, $p \in N_\lambda \cap \mathbb{P}_\alpha$. But then $D \cap N_\lambda = D_\lambda$, which is a dense subset of $N_\lambda \cap \mathbb{P}_\alpha$. Thus, we can find $q \leq_\alpha p$ in $D_\lambda = D \cap N_\lambda$. Also note that D consists of determined conditions.

Let F be a function as described in Proposition 7.6. For each $\lambda \in U$, let M_λ be the unique smallest elementary substructure of

$$(H(\kappa^{++}), \in, \trianglelefteq, J, \mathcal{Y}, \langle \mathbb{P}_\beta : \beta \leq \alpha \rangle, F)$$

such that $N_\lambda \cup \{D\} \subseteq M_\lambda$ and $M_\lambda^{<\lambda} \subseteq M_\lambda$. Standard arguments show that M_λ exists. The sequence $\langle M_\lambda : \lambda \in U \rangle$ is $\subseteq$ -increasing and continuous at limit points of U in U . So we can fix a club $C \subseteq \kappa$ such that for all $\lambda \in C \cap U$, $M_\lambda \cap \kappa = \lambda$.

We claim that the sequence $\langle M_\lambda : \lambda \in C \cap U \rangle$ is α -suitable. Observe that for all $\lambda \in C \cap U$, since λ is inaccessible we have that $|M_\lambda| = |N_\lambda| = \lambda = M_\lambda \cap \kappa$. So $M_\lambda^{<(M_\lambda \cap \kappa)} \subseteq M_\lambda$. Hence, M_λ is α -suitable. By the closure of M_λ , if $\mu < \lambda$ is in $C \cap U$ then $M_\mu \in M_\lambda$. So the sequence $\langle M_\lambda : \lambda \in C \cap U \rangle$ is $\in$ -increasing and continuous at limit points of U in U , and thus is α -suitable.

By the choice of F , for all $\lambda \in C \cap U$, $p(M_\lambda, \alpha)$ is $(M_\lambda, \mathbb{P}_\alpha)$ -generic. Since $M_\lambda \cap \kappa = N_\lambda \cap \kappa$, Lemma 6.11 implies that $p(M_\lambda, \alpha) = p(N_\lambda, \alpha)$.

Since $p(M_\lambda, \alpha)$ is $(M_\lambda, \mathbb{P}_\alpha)$ -generic and $D \in M_\lambda$ is a dense subset of $\mathbb{P}_\alpha$, $D \cap M_\lambda$ is predense below $p(M_\lambda, \alpha)$. So $D \cap M_\lambda$ is predense below $p(N_\lambda, \alpha)$. Since there are only κ many determined conditions in $\mathbb{P}_\alpha$ and $M_\lambda \cap \kappa = N_\lambda \cap \kappa$, it follows that M_λ and N_λ contain exactly the same determined conditions of $\mathbb{P}_\alpha$. Since D consists of determined conditions, we get that $D \cap M_\lambda = D \cap N_\lambda = D_\lambda$. So D_λ is predense below $p(N_\lambda, \alpha)$, which contradicts the choice of D_λ . $\square$

Proposition 7.9. *Let $\langle N_\lambda : \lambda \in S \rangle$ be α -suitable, with union N , where $S \in J^+$. Let T be the set of $\lambda \in S$ which satisfy that whenever $\#_{N_\lambda}^\alpha(p, q, s)$ holds, then there is $s' \leq_\alpha s$ such that $*_{N_\lambda}^\alpha(p, q, s')$ holds. Then $S \setminus T \in J$.*

Proof. Suppose for a contradiction that $S \setminus T \in J^+$. For each $\lambda \in S \setminus T$, we can fix p_λ , q_λ , and s_λ such that $\#_{N_\lambda}^\alpha(p_\lambda, q_\lambda, s_\lambda)$ holds, but for all $s' \leq_\alpha s_\lambda$ in $N_\lambda \cap \mathbb{P}_\alpha$, $*_{N_\lambda}^\alpha(p_\lambda, q_\lambda, s')$ does not hold.

By Proposition 7.5, fix $U \subseteq S \setminus T$ in J^+ such that for all $\lambda < \mu$ in U , p_λ and q_μ are compatible in $\mathbb{P}_\alpha$. By Proposition 7.8, fix $U_1 \subseteq U$ in J^+ such that for all $\lambda \in U_1$, $p(N_\lambda, \alpha)$ is strongly $(N_\lambda, \mathbb{P}_\alpha)$ -generic.

Consider $\lambda \in U_1$. Define H_λ as the set of $u \in N_\lambda \cap \mathbb{P}_\alpha$ such that either u and s_λ are incompatible in $\mathbb{P}_\alpha$, or else $u \leq_\alpha s_\lambda$ and u is incompatible in $\mathbb{P}_\alpha$ with either p_λ or q_λ .

We claim that H_λ is dense in $N_\lambda \cap \mathbb{P}_\alpha$. So let $p \in N_\lambda \cap \mathbb{P}_\alpha$. If p and s_λ are incompatible, then $p \in H_\lambda$ and we are done. Otherwise, by the elementarity of N_λ we can fix a determined condition $t \leq_\alpha p, s_\lambda$ in $N_\lambda \cap \mathbb{P}_\alpha$. Now by assumption, for all $s' \leq_\alpha s_\lambda$ in $N_\lambda \cap \mathbb{P}_\alpha$, $*_{N_\lambda}^\alpha(p_\lambda, q_\lambda, s')$ is false. Since $t \leq_\alpha s_\lambda$, in particular, $*_{N_\lambda}^\alpha(p_\lambda, q_\lambda, t)$ is false. By the definition of $*_{N_\lambda}^\alpha$, there is $u \leq_\alpha t$ in $N_\lambda \cap \mathbb{P}_\alpha$ such that u is incompatible in $\mathbb{P}_\alpha$ with either p_λ or q_λ . So $u \leq_\alpha p$ and $u \in H_\lambda$.

Since H_λ is dense in $N_\lambda \cap \mathbb{P}_\alpha$, we can fix a maximal antichain A_λ of the forcing poset $N_\lambda \cap \mathbb{P}_\alpha$ which is a subset of H_λ . Define a function $F_\lambda : A_\lambda \rightarrow 3$ as follows.

Given $u \in A_\lambda$, if u is incompatible with s_λ , then let $F_\lambda(u) := 0$. Otherwise, $u \leq_\alpha s_\lambda$ and u is incompatible with either p_λ or q_λ . If u is incompatible with p_λ , then let $F_\lambda(u) := 1$. Otherwise, u is compatible with p_λ but incompatible with q_λ , and we let $F_\lambda(u) := 2$.

This defines for each $\lambda \in U_1$ a set $F_\lambda \subseteq N_\lambda$. Since $U_1 \in J^+$, by Lemma 2.3 we can find a stationary set $W \subseteq U_1$ and a set $F \subseteq N$ such that for all $\lambda \in W$, $F \cap N_\lambda = F_\lambda$. It is easy to check that F is a function, $\text{dom}(F)$ is an antichain of $\mathbb{P}_\alpha$, and for all $\lambda \in W$, $\text{dom}(F) \cap N_\lambda = A_\lambda$.

Now we are ready to derive a contradiction. Fix $\lambda < \mu$ in W . Then p_λ and q_μ are compatible, so fix $w \in \mathbb{P}_\alpha$ which is below both of them. Then in particular, $w \leq_\alpha p(N_\lambda, \alpha), p(N_\mu, \alpha)$. Since A_μ is a maximal antichain of $N_\mu \cap \mathbb{P}_\alpha$ and $p(N_\mu, \alpha)$ is strongly $(N_\mu, \mathbb{P}_\alpha)$ -generic, A_μ is predense below $p(N_\mu, \alpha)$. As $w \leq_\alpha p(N_\mu, \alpha)$, we can find $t_\mu \in A_\mu$ and w_0 such that $w_0 \leq_\alpha w, t_\mu$. Then $w_0 \leq_\alpha w \leq_\alpha p(N_\lambda, \alpha)$. Since A_λ is a maximal antichain of $N_\lambda \cap \mathbb{P}_\alpha$ and $p(N_\lambda, \alpha)$ is strongly $(N_\lambda, \mathbb{P}_\alpha)$ -generic, A_λ is predense below $p(N_\lambda, \alpha)$. So we can find $t_\lambda \in A_\lambda$ and w_1 such that $w_1 \leq_\alpha w_0, t_\lambda$.

Since $A_\mu = \text{dom}(F) \cap N_\mu$ and $A_\lambda = \text{dom}(F) \cap N_\lambda$, t_μ and t_λ are in $\text{dom}(F)$. As $\text{dom}(F)$ is an antichain and $w_1 \leq_\alpha t_\lambda, t_\mu$, it follows that $t_\lambda = t_\mu$. Let $t^* := t_\lambda = t_\mu$.

Now $F \upharpoonright A_\lambda = F_\lambda$ and $F \upharpoonright A_\mu = F_\mu$. Therefore, $F(t^*) = F_\lambda(t_\lambda) = F_\mu(t_\mu)$. As $t^* = t_\lambda \in A_\lambda \subseteq H_\lambda$, t^* is either incompatible with or below s_λ . But $w_1 \leq_\alpha t^*$ and $w_1 \leq_\alpha w_0 \leq_\alpha w \leq_\alpha p_\lambda \leq_\alpha s_\lambda$. So $t_\lambda = t^* \leq_\alpha s_\lambda = s_\mu$. By the definition of F_λ , we have that $F(t^*) = F_\lambda(t_\lambda) \neq 0$. So either $F(t^*) = 1$ or $F(t^*) = 2$.

Assume that $F(t^*) = 1$. Then $F_\lambda(t_\lambda) = 1$. By the definition of F_λ , t_λ is incompatible with p_λ . But that is false, since $w_1 \leq_\alpha t_\lambda$ and $w_1 \leq_\alpha w_0 \leq_\alpha w \leq_\alpha p_\lambda$. So $F(t^*) = 2$. Hence, $F_\mu(t_\mu) = 2$. By the definition of F_μ , it follows that $t^* = t_\mu$ is incompatible with q_μ . But that is also false, since $w_0 \leq_\alpha t_\mu$ and $w_0 \leq_\alpha w \leq_\alpha q_\mu$. Thus, we have reached a contradiction. $\square$

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JOHN KRUEGER, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF NORTH TEXAS, 1155 UNION CIRCLE #311430, DENTON, TX 76203
E-mail address: jkrueger@unt.edu