

A FORCING AXIOM FOR A NON-SPECIAL ARONSZAJN TREE

JOHN KRUEGER

ABSTRACT. Suppose that T^* is an ω_1 -Aronszajn tree with no stationary antichain. We introduce a forcing axiom $\text{PFA}(T^*)$ for proper forcings which preserve these properties of T^* . We prove that $\text{PFA}(T^*)$ implies many of the strong consequences of PFA , such as the failure of very weak club guessing, that all of the cardinal characteristics of the continuum are greater than ω_1 , and the P -ideal dichotomy. On the other hand, $\text{PFA}(T^*)$ implies some of the consequences of diamond principles, such as the existence of Knaster forcings which are not stationarily Knaster.

In one of the earliest applications of the method of forcing in set theory, Solovay and Tennenbaum [11] proved the consistency of Suslin's hypothesis, which is the statement that there does not exist an ω_1 -Suslin tree. Baumgartner, Malitz, and Reinhardt [2] improved this by showing that MA_{ω_1} implies that every ω_1 -Aronszajn tree is special, a statement which implies Suslin's hypothesis. Shelah [10, Chapter IX] proved that Suslin's hypothesis does not imply that all ω_1 -Aronszajn trees are special by constructing a model in which there are no ω_1 -Suslin trees but there exists a non-special Aronszajn tree. In Shelah's model, the non-special Aronszajn tree satisfies that there are stationarily many levels on which the tree is special, and stationarily many levels on which there is no stationary antichain (recall that an antichain of a tree is stationary if the set of heights of its nodes forms a stationary set). Schlindwein [9] improved this result by constructing a model in which there are no ω_1 -Suslin trees but there exists an ω_1 -Aronszajn tree which has no stationary antichain.

Building on Woodin's work [19] on the $\mathbb{S}_{\max}$ forcing extension of $L(\mathbb{R})$, Larson [5] produced a maximal model of the existence of a minimal Suslin tree. Larson and Todorćević [6] proved the consistency of a positive solution to Katětov's problem, by creating a model in which every compact space whose square is T_5 is metrizable. Their method involved analyzing the maximal amount of MA_{ω_1} consistent with the existence of a Suslin tree. Later Todorćević [13] introduced a forcing axiom $\text{PFA}(S)$, which is a version of PFA but restricted to forcings which preserve the Suslinness of a particular coherent ω_1 -Suslin tree.

In this article we expand on these results by developing a forcing axiom for proper forcings which preserve a particular ω_1 -Aronszajn tree with no stationary antichain. Let T^* denote an ω_1 -Aronszajn tree with no stationary antichain. We let $\text{PFA}(T^*)$ be the statement that for any proper forcing poset $\mathbb{P}$ which forces that T^* is still an ω_1 -Aronszajn tree with no stationary antichain, for any sequence $\langle D_\alpha : \alpha < \omega_1 \rangle$ of dense subsets of $\mathbb{P}$, there exists a filter G such that $G \cap D_\alpha \neq \emptyset$

2020 *Mathematics Subject Classification*: Primary 03E35, 03E57; Secondary 03E05, 03E65.

Key words and phrases. Aronszajn tree, stationary antichain, $\text{PFA}(T^*)$, stationarily Knaster.

This material is based upon work supported by the National Science Foundation under Grant No. DMS-1464859.

for all $\alpha < \omega_1$. This forcing axiom is similar in spirit to the previously studied forcing axiom $\text{PFA}(S)$, but has some dramatically different consequences such as the non-existence of S -spaces. Both forcing axioms are useful for understanding the relationship between consequences of PFA , properties of ω_1 -trees, and the ω_1 -chain condition of forcing posets.

The consistency of $\text{PFA}(T^*)$ is established from a supercompact cardinal using forcing iteration preservation theorems of Shelah [10] and Schlindwein [9]. We then derive a variety of consequences of $\text{PFA}(T^*)$. These consequences include Suslin's hypothesis, the failure of very weak club guessing, that all of the standard cardinal characteristic of the continuum are greater than ω_1 , and the P -ideal dichotomy. We also prove that under $\text{PFA}(T^*)$, the forcing poset consisting of finite antichains of T^* ordered by reverse inclusion is Knaster but not stationarily Knaster, which is in contrast to MA_{ω_1} which implies that all Knaster forcings are stationarily Knaster. The existence of a Knaster forcing which is not stationarily Knaster is also shown to follow from $\diamond^*$. Thus, our forcing axiom $\text{PFA}(T^*)$ implies a significant portion of the consequences of PFA while also having consequences in common with diamond principles.

I would like to thank Sean Cox and Justin Moore for helpful discussions on the topics in this paper.

1. FORCINGS WHICH PRESERVE T^*

In this section, we develop the basic ideas concerning forcing posets which preserve a non-special Aronszajn tree.

Definition 1.1. *Let T be a tree with height ω_1 . A set $A \subseteq T$ is said to be a stationary antichain if A is an antichain of T and the set $\{\text{ht}_{T^*}(x) : x \in A\}$ is a stationary subset of ω_1 .*

Observe that any ω_1 -Suslin tree has no uncountable antichain, and hence no stationary antichain, but any special ω_1 -Aronszajn tree has a stationary antichain. In fact, if an ω_1 -tree has a special subtree whose nodes meet stationarily many levels, then by a pushing down argument it has a stationary antichain.

For the remainder of the paper, whenever we mention T^* we will assume that T^* is an ω_1 -Aronszajn tree with no stationary antichain.

Definition 1.2. *A forcing poset $\mathbb{P}$ is said to be T^* -preserving if $\mathbb{P}$ forces that T^* is an ω_1 -Aronszajn tree with no stationary antichain.*

Let T be an ω_1 -tree. We define a relation $<_{\overline{T}}$ by letting $x <_{\overline{T}} y$ if $\text{ht}_T(x) < \text{ht}_T(y)$ and $x \not\prec_T y$. Observe that $x <_{\overline{T}} y$ implies that x and y are not comparable in T . As a word of caution to the reader, the relation $<_{\overline{T}}$ is not transitive.

Definition 1.3. *Let N be a countable set such that $\delta := N \cap \omega_1$ is an ordinal. A node $x \in T_\delta$ is said to be (N, T) -generic if for any set $A \subseteq T$ which is a member of N and any relation $R \in \{<_T, <_{\overline{T}}\}$, if $x \in A$, then there exists $y R x$ such that $y \in A$.*

Lemma 1.4. *Let θ be a regular cardinal with $T \in H(\theta)$, and N a countable elementary substructure of $(H(\theta), \in, T)$. Let $\delta := N \cap \omega_1$. Assume that $x \in T_\delta$ is (N, T) -generic. Let $B \subseteq T$ be in N and $R \in \{<_T, <_{\overline{T}}\}$. If for all $y R x$, $y \in B$, then $x \in B$.*

Proof. Suppose for a contradiction that for all $y R x$, $y \in B$, but $x \notin B$. Define $A := T \setminus B$. Then $A \in N$ by elementarity, and $x \in A$. Since x is (N, T) -generic, there exists $y R x$ such that $y \in A$. Then $y \notin B$, which is a contradiction. $\square$

The next lemma basically appears in [9], but under the assumption that T^* is an ω_1 -Suslin tree.

Lemma 1.5. *Fix a regular cardinal θ such that $T^* \in H(\theta)$. Let N be a countable elementary substructure of $(H(\theta), \in, T^*)$. Then for all $x \in T_{N \cap \omega_1}^*$, x is (N, T^*) -generic.*

Proof. Let $\delta := N \cap \omega_1$. Assume that $x \in T_\delta^*$, and we will show that x is (N, T^*) -generic. Fix $A \in N$ and $R \in \{<_{T^*}, <_{\bar{T}^*}\}$. Suppose for a contradiction that $x \in A$, but for all $y R x$, $y \notin A$. Define B as the set of $z \in T^*$ such that $z \in A$ and for all $y R z$, $y \notin A$. Then $B \in N$ by elementarity, and $x \in B$. Since $x \in B$, the set $S := \{\text{ht}_{T^*}(z) : z \in B\}$ contains δ . But $\delta \in C$ for any club $C \subseteq \omega_1$ in N , so S is stationary by elementarity.

First, assume that R equals $<_{T^*}$. Consider z_1 and z_2 in B , where $\text{ht}_{T^*}(z_1) \leq \text{ht}_{T^*}(z_2)$. If z_1 and z_2 have the same height, then they are incomparable in T^* . If $\text{ht}_{T^*}(z_1) < \text{ht}_{T^*}(z_2)$, then since z_1 and z_2 are both in B , by the definition of B we must have that z_1 is not $<_{T^*}$ -below z_2 . Thus, any two members of B are incomparable, that is, B is an antichain. Since $\{\text{ht}_{T^*}(z) : z \in B\} = S$ is stationary, B is a stationary antichain, which contradicts that T^* has no stationary antichain.

Secondly, assume that R equals $<_{\bar{T}^*}$. Consider z_1 and z_2 in $B \cap N$. Since $x \in B$, by the definition of B it is not the case that z_1 and z_2 are $<_{\bar{T}^*}$ -below x . As they have lower heights than x , that means z_1 and z_2 are both $<_{T^*}$ -below x . This implies that z_1 and z_2 are comparable in T^* . By elementarity, it follows that any two members of B are comparable in T^* , that is, B is a chain. And since S is stationary, B is uncountable. This contradicts that T^* is an ω_1 -Aronszajn tree. $\square$

While the property of being T^* -preserving is very natural, in practice it will be helpful to have a related property which is more technical (see [10, Chapter IX] and [9]).

Definition 1.6. *Let T be an ω_1 -tree and $\mathbb{P}$ a forcing poset. Let θ be a regular cardinal such that T and $\mathbb{P}$ are members of $H(\theta)$ and N a countable elementary substructure of $(H(\theta), \in, T, \mathbb{P})$. A condition $q \in \mathbb{P}$ is said to be $(N, \mathbb{P}, T)$ -generic if q is $(N, \mathbb{P})$ -generic, and whenever $x \in T_{N \cap \omega_1}$ is (N, T) -generic, then q forces that x is $(N[\dot{G}_{\mathbb{P}}], T)$ -generic.*

Assume that q is $(N, \mathbb{P})$ -generic. Then the following property is easily seen to be equivalent to q being $(N, \mathbb{P}, T)$ -generic: for any node $x \in T_{N \cap \omega_1}$ which is (N, T) -generic, $\mathbb{P}$ -name $\dot{A} \in N$ for a subset of T , and relation $R \in \{<_T, <_{\bar{T}}\}$, whenever $r \leq q$ forces that $x \in \dot{A}$, then there exist $y R x$ and $s \leq r$ such that s forces that $y \in \dot{A}$.

Definition 1.7. *Let T be an ω_1 -tree and $\mathbb{P}$ a forcing poset. We say that $\mathbb{P}$ is T -proper if for all large enough regular cardinals θ with $\mathbb{P}$ and T in $H(\theta)$, there are club many N in $P_{\omega_1}(H(\theta))$ such that N is an elementary substructure of $(H(\theta), \in, T, \mathbb{P})$ and for all $p \in N \cap \mathbb{P}$, there is $q \leq p$ which is $(N, \mathbb{P}, T)$ -generic.*

Lemma 1.8. *Let $\mathbb{P}$ be a proper forcing poset. If $\mathbb{P}$ is T^* -preserving, then $\mathbb{P}$ is T^* -proper.*

Proof. Assume that $\mathbb{P}$ is T^* -preserving. Let θ be a large enough regular cardinal such that T^* and $\mathbb{P}$ are in $H(\theta)$, and let N be a countable elementary substructure of $(H(\theta), \in, T^*, \mathbb{P})$ such that every member of $N \cap \mathbb{P}$ has an $(N, \mathbb{P})$ -generic extension. Let $\delta := N \cap \omega_1$. Consider $p \in N \cap \mathbb{P}$. Fix $q \leq p$ which is $(N, \mathbb{P})$ -generic. We claim that q is $(N, \mathbb{P}, T^*)$ -generic.

Consider $x \in T_\delta^*$. Let G be a generic filter on $\mathbb{P}$ which contains q , and we claim that x is $(N[G], T^*)$ -generic. So $N[G]$ is an elementary substructure of $H(\theta)^{V[G]}$ which contains T^* . Since q is $(N, \mathbb{P})$ -generic, $N[G] \cap \omega_1 = N \cap \omega_1 = \delta$. Thus, $x \in T_{N[G] \cap \omega_1}^*$. As $\mathbb{P}$ is T^* -preserving, in $V[G]$ we have that T^* is still an ω_1 -Aronszajn tree with no stationary antichain. So we can apply Lemma 1.5 in $V[G]$ to conclude that x is $(N[G], T^*)$ -generic. $\square$

While the converse of Lemma 1.8 is not true in general, we will show below that if $\mathbb{P}$ is T^* -proper, then there is a $\mathbb{P}$ -name $\dot{\mathbb{Q}}$ for a forcing poset such that $\mathbb{P} * \dot{\mathbb{Q}}$ is proper and T^* -preserving.

The next lemma is proved in [9, Lemma 14].

Lemma 1.9. *If $\mathbb{P}$ is T^* -proper, then $\mathbb{P}$ forces that T^* is an ω_1 -Aronszajn tree.*

Throughout the paper, we will prove that a multitude of forcing posets are T^* -proper. As a warm-up and because we will need it in the next section, let us prove that every ω_1 -closed forcing is T^* -proper.

Definition 1.10. *A forcing poset $\mathbb{P}$ is ω_1 -generically closed if for all large enough regular cardinals θ , there are club many $N \in P_{\omega_1}(H(\theta))$ such that whenever $\langle p_n : n < \omega \rangle$ is a descending sequence of conditions in $N \cap \mathbb{P}$ which meets every dense open subset of $\mathbb{P}$ in N , then this sequence has a lower bound.¹*

A sequence as described in Definition 1.10 will be called an $(N, \mathbb{P})$ -generic sequence. Observe that countably closed forcings are ω_1 -generically closed.

Proposition 1.11. *Suppose that $\mathbb{P}$ is ω_1 -generically closed. Then $\mathbb{P}$ is T^* -proper.*

Proof. Fix a large enough regular cardinal θ and a countable elementary substructure N of $H(\theta)$ which contains $\mathbb{P}$ and T^* and satisfies that every descending $(N, \mathbb{P})$ -generic sequence has a lower bound. Let $\delta := N \cap \omega_1$.

Let $\langle x_n : n < \omega \rangle$, $\langle \dot{A}_n : n < \omega \rangle$, and $\langle D_n : n < \omega \rangle$ enumerate all of the nodes in T_δ^* , all $\mathbb{P}$ -names for subsets of T^* in N , and all dense open subsets of $\mathbb{P}$ in N . Fix a bijection $f : \omega \rightarrow \omega \times \omega \times 3$.

Consider $p \in N \cap \mathbb{P}$. We define by induction a descending sequence of conditions $\langle p_n : n < \omega \rangle$ in $N \cap \mathbb{P}$ as follows. Let $p_0 := p$. Assume that $n < \omega$ and p_n is defined. Let $f(n) = (m, k, j)$. If $j = 0$, fix $p_{n+1} \leq p_n$ in $N \cap D_m$ (in this case, we ignore k). Suppose that $j \neq 0$. If $j = 1$, let R be the relation $<_{T^*}$, and if $j = 2$, let R be the relation $<_{T^*}^-$.

Let B_n denote the set of $z \in T^*$ such that some extension of p_n forces that $z \in \dot{A}_k$. Then $B_n \in N$ by elementarity. If $x_m \notin B_n$, then let $p_{n+1} := p_n$. Suppose that $x_m \in B_n$. Since x_m is (N, T^*) -generic, there exists $y R x_m$ such that $y \in B_n$. Then $y \in N$. By the definition of B_n and elementarity, we can fix $p_{n+1} \leq p_n$ in $N \cap \mathbb{P}$ which forces that $y \in \dot{A}_k$.

This completes the definition of $\langle p_n : n < \omega \rangle$. Observe that this sequence is $(N, \mathbb{P})$ -generic. So we can find $q \in \mathbb{P}$ such that $q \leq p_n$ for all $n < \omega$. In particular,

¹See [10, Chapter V Section 1] for more information about this property.

$q \leq p$. Clearly, q is $(N, \mathbb{P})$ -generic. To see that it is $(N, \mathbb{P}, T^*)$ -generic, let $x \in T_\delta^*$, $\dot{A} \in N$ a $\mathbb{P}$ -name for a subset of T^* , and $R \in \{<_{T^*}, <_{\bar{T}^*}\}$. Suppose that $r \leq q$ and r forces that $x \in \dot{A}$.

Fix m and k such that $x = x_m$ and $\dot{A} = \dot{A}_k$. Let $j := 1$ if R equals $<_{T^*}$ and $j := 2$ if R equals $<_{\bar{T}^*}$. Fix n such that $f(n) = (m, k, j)$. Now $r \leq p_n$ and r forces that $x_m \in \dot{A}_k$. So r witnesses that $x_m \in B_n$. By construction, there exists $y \mathbin{R} x_m$ such that p_{n+1} forces that $y \in \dot{A}_k$. Since $r \leq p_{n+1}$, r forces that $y \in \dot{A}$. $\square$

Corollary 1.12. *Any ω_1 -closed forcing poset is T^* -proper.*

2. A FORCING AXIOM FOR T^*

We now introduce and prove the consistency of a forcing axiom for T^* .

The following forcing poset for adding a Baumgartner club will be used in a number of different contexts throughout the paper, beginning in this section. It is a slight variation of a forcing of Baumgartner [1, p. 926] for adding a club with finite conditions.

Definition 2.1. *Let $A \subseteq \omega_1$. Define $CU(A)$ to be the forcing poset whose conditions are functions f mapping into ω_1 satisfying:*

- (1) *$\text{dom}(f)$ is a finite subset of A ;*
- (2) *if $\alpha < \beta$ are in $\text{dom}(f)$, then $\alpha \leq f(\alpha) < \beta$;*

and ordered by $g \leq f$ if $f \subseteq g$.

It is not difficult to show that if A is stationary, then $CU(A)$ preserves ω_1 and forces that the set

$$\bigcup \{\text{dom}(f) : f \in \dot{G}_{CU(A)}\}$$

is a club subset of A .

The next result states that the property of being T^* -proper is satisfied by certain countable support forcing iterations.

Theorem 2.2. *Let T^* be an ω_1 -Aronszajn tree with no stationary antichain. Let*

$$\langle \mathbb{P}_i, \dot{Q}_j : i \leq \alpha, j < \alpha \rangle$$

be a countable support forcing iteration such that for all $\gamma < \alpha$, $\mathbb{P}_\gamma$ forces either:

- (1) *$\dot{Q}_\gamma$ is T^* -proper, or*
- (2) *there exists an antichain $I \subseteq T^*$ such that $\dot{Q}_\gamma = CU(\omega_1 \setminus S)$, where $S = \{\text{ht}_{T^*}(z) : z \in I\}$.*

Then $\mathbb{P}_\alpha$ is T^ -proper.*

This theorem was proven in [9, Theorem 21] under the assumption that T^* is an ω_1 -Suslin tree. However, exactly the same proof works assuming instead that T^* is an ω_1 -Aronszajn tree with no stationary antichain. In fact, the only place in the proof of [9, Theorem 21] where the Suslin property of the tree is used is in case 2 of the successor case, which uses [9, Lemma 8]. This lemma has the same conclusion as Lemma 1.5 above, but assumes instead that the tree is Suslin. Thus, the proof of Theorem 2.2 is exactly the same as the proof of [9, Theorem 21], except replacing its application of [9, Lemma 8] by our Lemma 1.5.

Proposition 2.3. *Let $\mathbb{P}$ be a T^* -proper forcing poset. Then there exists a $\mathbb{P}$ -name $\dot{Q}$ for a forcing poset such that $\mathbb{P} * \dot{Q}$ is proper and T^* -preserving.*

Proof. For simplicity, we assume that there exists a regular cardinal κ such that $\mathbb{P} * \text{Add}(\omega_1) * \text{Add}(\omega_2)$ forces that $\kappa = \omega_2$. If there does not, then the general case can be proven by a straightforward argument involving splitting this forcing into a maximal antichain of conditions deciding the value of ω_2 . Observe that this three-step forcing iteration forces that $2^\omega = \omega_1$ and $2^{\omega_1} = \omega_2$.

We define a forcing iteration $\langle \mathbb{P}_i, \dot{Q}_j : i \leq \kappa, j < \kappa \rangle$ as follows. The first forcing in the iteration is $\mathbb{P}$, and the second forcing is $\text{Add}(\omega_1) * \text{Add}(\omega_2)$. Assume that $2 \leq \alpha < \kappa$ and $\mathbb{P}_\alpha$ is defined. We consider a $\mathbb{P}_\alpha$ -name $\dot{I}_\alpha$ for an antichain of T^* , and let $\dot{Q}_\alpha$ be a $\mathbb{P}_\alpha$ -name for the forcing poset $\text{CU}(\omega_1 \setminus \dot{S}_\alpha)$, where $\dot{S}_\alpha$ is a $\mathbb{P}_\alpha$ -name for the set $\{\text{ht}_{T^*}(z) : z \in \dot{I}_\alpha\}$. Observe that after forcing with $\dot{Q}_\alpha$, $\dot{I}_\alpha$ is not a stationary antichain. At limit stages $\delta \leq \kappa$, we define $\mathbb{P}_\delta$ by taking the limit with countable support.

This completes the definition. Now $\mathbb{P}$ is T^* -proper, and since $\text{Add}(\omega_1)$ is forced to be ω_1 -closed, it is forced to be T^* -proper as well by Corollary 1.12. By Theorem 2.2, it follows that $\mathbb{P}_\kappa$ is T^* -proper. Hence, $\mathbb{P}_\kappa$ preserves ω_1 , and by Lemma 1.9, $\mathbb{P}_\kappa$ forces that T^* is an ω_1 -Aronszajn tree.

Now in $V^{\mathbb{P}_\kappa}$, which models $2^\omega = \omega_1$ and $2^{\omega_1} = \omega_2$, standard arguments show that the tail of the iteration is forcing equivalent to a countable support iteration of length ω_2 of the forcings $\text{CU}(\omega_1 \setminus \dot{S}_\alpha)$, which each have size ω_1 . By [10, Lemma 2.4, Chapter VIII], this iteration is ω_2 -c.c. Since κ equals ω_2 in $V^{\mathbb{P}_\kappa}$, it follows that $\mathbb{P}_\kappa$ forces that $\kappa = \omega_2$. The chain condition also implies that we can arrange by standard bookkeeping that all antichains of T^* appearing in $V^{\mathbb{P}_\kappa}$ are handled at some stage. It follows that $\mathbb{P}_\kappa$ forces that T^* has no stationary antichain. So in $V^{\mathbb{P}_\kappa}$, T^* is an ω_1 -Aronszajn tree with no stationary antichain.

Let $\dot{Q}$ be a $\mathbb{P}$ -name for the tail of the iteration after forcing with $\mathbb{P}$. Then $\mathbb{P} * \dot{Q}$ is forcing equivalent to $\mathbb{P}_\kappa$, which is proper and forces that T^* is an ω_1 -Aronszajn tree with no stationary antichain. Thus, $\mathbb{P} * \dot{Q}$ is T^* -preserving, completing the proof. $\square$

Definition 2.4. Assume that T^* is an ω_1 -Aronszajn tree with no stationary antichain. Define $\text{PFA}(T^*)$ to be the statement that for any proper T^* -preserving forcing poset $\mathbb{P}$, whenever $\langle D_\alpha : \alpha < \omega_1 \rangle$ is a sequence of dense subsets of $\mathbb{P}$, then there exists a filter G on $\mathbb{P}$ such that for all $\alpha < \omega_1$, $G \cap D_\alpha \neq \emptyset$.

In practice, we do not have a general way to verify that a proper forcing is T^* -preserving. Therefore, the following characterization of $\text{PFA}(T^*)$ will be useful.

Proposition 2.5. Assume that T^* is an ω_1 -Aronszajn tree with no stationary antichain. Then $\text{PFA}(T^*)$ is equivalent to the statement that for any T^* -proper forcing poset $\mathbb{P}$, whenever $\langle D_\alpha : \alpha < \omega_1 \rangle$ is a sequence of dense subsets of $\mathbb{P}$, then there exists a filter G on $\mathbb{P}$ such that for all $\alpha < \omega_1$, $G \cap D_\alpha \neq \emptyset$.

Proof. Let P denote the statement which we are claiming to be equivalent to $\text{PFA}(T^*)$. Since any proper T^* -preserving forcing poset is T^* -proper by Lemma 1.8, P obviously implies $\text{PFA}(T^*)$. Conversely, assume $\text{PFA}(T^*)$. Let $\mathbb{P}$ be a T^* -proper forcing poset and $\langle D_\alpha : \alpha < \omega_1 \rangle$ a sequence of dense subsets of $\mathbb{P}$. By Proposition 2.3, there exists a $\mathbb{P}$ -name $\dot{Q}$ such that $\mathbb{P} * \dot{Q}$ is proper and T^* -preserving.

For each $\alpha < \omega_1$, let E_α be the set of conditions $p * \dot{q}$ in $\mathbb{P} * \dot{Q}$ such that $p \in D_\alpha$. Then easily each E_α is dense in $\mathbb{P} * \dot{Q}$. Applying $\text{PFA}(T^*)$, fix a filter G' on $\mathbb{P} * \dot{Q}$ such that for all $\alpha < \omega_1$, $G' \cap E_\alpha \neq \emptyset$. Define $G := \{p \in \mathbb{P} : \exists \dot{q} \ p * \dot{q} \in G'\}$. Then G is a filter on $\mathbb{P}$ and for all $\alpha < \omega_1$, $G \cap D_\alpha \neq \emptyset$. $\square$

Theorem 2.6. *Suppose that T^* is an ω_1 -Aronszajn tree with no stationary antichain (for example, if T^* is an ω_1 -Suslin tree). Assume that there exists a supercompact cardinal κ . Then there is a forcing poset which forces that $\kappa = \omega_2$ and $\text{PFA}(T^*)$ holds.*

The proof is nearly identical to the usual construction of a model of PFA. We define a forcing iteration of length κ with countable support, where at each stage we force with either a forcing which is T^* -proper and has size less than κ , or with a forcing of the form $\text{CU}(\omega_1 \setminus S)$, where $S = \{\text{ht}_{T^*}(z) : z \in I\}$ for some antichain $I \subseteq T^*$. We use a Laver function as a bookkeeping mechanism to anticipate all possible T^* -proper forcings and all possible antichains of T^* in the final model. By Theorem 2.2, such a forcing iteration is T^* -proper. Also, standard arguments show that it is κ -c.c., and by Lemma 1.9 and the construction, T^* is still an ω_1 -Aronszajn tree with no stationary antichain after forcing with it. Since ω_1 -closed forcings are T^* -proper, and in particular the collapse $\text{Col}(\omega_1, \omega_2)$ is T^* -proper, the iteration forces that κ equals ω_2 . Now the same elementary embedding argument used in the original proof for PFA works the same in this context to prove $\text{PFA}(T^*)$.

3. STRONG PROPERNESS AND CLUB GUESSING

In this section, we will prove that any strongly proper forcing is T^* -proper. It will follow that $\text{PFA}(T^*)$ implies the failure of weak club guessing. We then show by a more intricate argument that $\text{PFA}(T^*)$ implies the failure of very weak club guessing.

We recall some definitions. Let $\mathbb{P}$ be a forcing poset. For a set N , a condition $q \in \mathbb{P}$ is said to be *strongly $(N, \mathbb{P})$ -generic* if for any dense subset D of the poset $N \cap \mathbb{P}$, D is predense below q . This property is equivalent to saying that for all $r \leq q$, there is $u \in N \cap \mathbb{P}$ such that for all $v \leq u$ in $N \cap \mathbb{P}$, r and v are compatible. We say that $\mathbb{P}$ is *strongly proper* if for all large enough regular cardinals θ with $\mathbb{P} \in H(\theta)$, for club many countable $N \in P_{\omega_1}(H(\theta))$, for all $p \in N \cap \mathbb{P}$, there is $q \leq p$ which is strongly $(N, \mathbb{P})$ -generic.

Proposition 3.1. *Suppose that $\mathbb{P}$ is a strongly proper forcing poset. Then $\mathbb{P}$ is T^* -proper.*

Proof. Fix a large enough regular cardinal θ . Let N be a countable elementary substructure of $(H(\theta), \in, T^*, \mathbb{P})$ such that every member of $N \cap \mathbb{P}$ has a strongly $(N, \mathbb{P})$ -generic extension. Let $\delta := N \cap \omega_1$. Consider $p \in N \cap \mathbb{P}$. Fix $q \leq p$ which is strongly $(N, \mathbb{P})$ -generic. Then in particular, q is $(N, \mathbb{P})$ -generic.

We claim that q is $(N, \mathbb{P}, T^*)$ -generic. Let $x \in T_\delta^*$, $\dot{A} \in N$ a $\mathbb{P}$ -name for a subset of T^* , and $R \in \{<_{T^*}, <_{T^*}^-\}$. Assume that $r \leq q$ and r forces that $x \in \dot{A}$. We will find $s \leq r$ and $y R x$ such that s forces that $y \in \dot{A}$.

Since q is strongly $(N, \mathbb{P})$ -generic, we can fix $u \in N \cap \mathbb{P}$ such that for all $v \leq u$ in $N \cap \mathbb{P}$, r and v are compatible. Define B as the set of $z \in T^*$ for which there exists a condition below u which forces that $z \in \dot{A}$. Note that $B \in N$ by elementarity. Since u and r are compatible, and any common extension of them forces that $x \in \dot{A}$, it follows that $x \in B$. As x is (N, T^*) -generic, there exists $y R x$ such that $y \in B$. Then $y \in N$. By the definition of B and elementarity, we can find $v \leq u$ in $N \cap \mathbb{P}$ which forces that $y \in \dot{A}$. By the choice of u , v and r are compatible. Fix $s \leq v, r$. Then $s \leq r$ and s forces that $y \in \dot{A}$. $\square$

Corollary 3.2. *PFA(T^*) implies that for any strongly proper forcing poset $\mathbb{P}$, whenever $\langle D_\alpha : \alpha < \omega_1 \rangle$ is a sequence of dense subsets of $\mathbb{P}$, then there exists a filter G on $\mathbb{P}$ such that for all $\alpha < \omega_1$, $G \cap D_\alpha \neq \emptyset$.*

Recall that a *ladder system* is a sequence $\vec{L} = \langle L_\alpha : \alpha \in \lim(\omega_1) \rangle$ such that for each α , L_α is a cofinal subset of α with order type ω . Let $\vec{L}$ be such a ladder system. Consider the following club guessing properties of $\vec{L}$, which are ordered in decreasing strength:

- for any club $C \subseteq \omega_1$, there exists $\alpha \in \lim(\omega_1)$ such that $L_\alpha \subseteq C$;
- for any club $C \subseteq \omega_1$, there exists $\alpha \in \lim(\omega_1)$ such that $L_\alpha \subseteq^* C$;²
- for any club $C \subseteq \omega_1$, there exists $\alpha \in \lim(\omega_1)$ such that $L_\alpha \cap C$ is infinite.

We say that *club guessing*, *weak club guessing*, or *very weak club guessing* holds if there exists a ladder system satisfying (1), (2), or (3) respectively.

It is well-known that PFA implies the failure of very weak club guessing ([1, Theorem 3.6]). We will prove that the same holds under PFA(T^*). We use the forcing poset $\text{CU}(A)$ described in Section 2.

Lemma 3.3. *For any club $D \subseteq \omega_1$, the forcing poset $\text{CU}(D)$ is strongly proper.*

Proof. Let $\mathbb{P} := \text{CU}(D)$. Consider any regular cardinal θ such that $\mathbb{P} \in H(\theta)$, and let N be a countable elementary substructure of $(H(\theta), \in, \mathbb{P}, D)$. Let $\delta := N \cap \omega_1$. Then $\delta \in D$. Consider $f \in N \cap \mathbb{P}$. Define $g := f \cup \{(\delta, \delta)\}$. If $h \leq g$, then since $\delta \in \text{dom}(h)$, for all $\alpha < \delta$ in $\text{dom}(h)$, $h(\alpha) < \delta$. It easily follows that $h \restriction \delta \in N \cap \mathbb{P}$, and for all $h_1 \leq h \restriction \delta$ in $N \cap \mathbb{P}$, $h_1 \cup h$ is a condition below h_1 and h . $\square$

Proposition 3.4. *PFA(T^*) implies that for any sequence $\langle A_\alpha : \alpha < \omega_1 \rangle$ of infinite subsets of ω_1 , there exists a club $E \subseteq \omega_1$ such that for all $\alpha < \omega_1$, $A_\alpha \setminus E$ is infinite.*

To prove the proposition, we apply PFA(T^*) to the forcing poset $\text{CU}(\omega_1)$, which is strongly proper and hence T^* -proper. The argument is essentially the same as that of [1, Theorem 3.4], which drew the same conclusion from PFA.

It easily follows from the last proposition that PFA(T^*) implies the failure of weak club guessing. But we can do better.

Theorem 3.5. *PFA(T^*) implies the failure of very weak club guessing.*

Proof. Fix a ladder system $\vec{L} = \langle L_\alpha : \alpha \in \lim(\omega_1) \rangle$, and we will prove that under PFA(T^*), $\vec{L}$ is not a very weak club guessing sequence. We may assume without loss of generality that for all $\alpha \in \lim(\omega_1)$, $\alpha \setminus L_\alpha$ is cofinal in α . For if all ladder systems with this property are not very weak club guessing sequences, then it is easy to argue that the same is true for all ladder systems.

Define $\mathbb{P}$ as the forcing poset whose conditions are pairs (f, x) such that $f \in \text{CU}(\omega_1)$ and $x \subseteq \lim(\omega_1)$ is finite, ordered by $(g, y) \leq (f, x)$ if $x \subseteq y$, $f \subseteq g$, and for all $\alpha \in x$,

$$(\text{dom}(g) \setminus \text{dom}(f)) \cap L_\alpha \cap \lim(\omega_1) = \emptyset.$$

We will write $p = (f_p, x_p)$ for any $p \in \mathbb{P}$.

Assuming that $\mathbb{P}$ preserves ω_1 , straightforward arguments prove that $\mathbb{P}$ introduces a club $C \subseteq \omega_1$ such that for all $\alpha \in \lim(\omega_1)$, $C \cap L_\alpha$ is finite, namely, the club of limit ordinals which belong to the club set $\bigcup \{\text{dom}(f_p) : p \in \dot{G}_{\mathbb{P}}\}$. So assuming

²Throughout the paper, $A \subseteq^* B$ will mean that $A \setminus B$ is finite.

that $\mathbb{P}$ is T^* -proper, a routine selection of dense sets shows that under $\text{PFA}(T^*)$, $\vec{L}$ is not a very weak club guessing sequence.

It remains to prove that $\mathbb{P}$ is T^* -proper. Fix a regular cardinal θ such that T^* and $\mathbb{P}$ are members of $H(\theta)$. Let $\mathcal{X}$ denote the set of countable elementary substructures of $(H(\theta), \in, T^*, \mathbb{P})$. Observe that by Lemma 1.5, if $M \in \mathcal{X}$, then for all $x \in T_{M \cap \omega_1}^*$, x is (M, T^*) -generic.

Let N be a countable elementary substructure of $(H(\theta), \in, T^*, \mathbb{P}, \mathcal{X})$. Observe that $N \in \mathcal{X}$. Let $\delta := N \cap \omega_1$. Consider $p \in N \cap \mathbb{P}$. Define q by $f_q := f_p \cup \{(\delta, \delta)\}$ and $x_q := x_p$. Then q is a condition and $q \leq p$. We claim that q is $(N, \mathbb{P}, T^*)$ -generic. So let $D \in N$ be a dense open subset of $\mathbb{P}$, $x \in T_\delta^*$, $\dot{A} \in N$ a $\mathbb{P}$ -name for a subset of T^* , and $R \in \{<_{T^*}, <_{\bar{T}^*}\}$.

Let $r \leq q$. By extending further if necessary, we may assume without loss of generality that $r \in D$ and r decides whether or not $x \in \dot{A}$. If r forces that $x \notin \dot{A}$, then replace $\dot{A}$ in what follows with the canonical $\mathbb{P}$ -name for T^* . Hence, without loss of generality, we may assume that r forces that $x \in \dot{A}$. Observe that since $\delta \in \text{dom}(f_r)$, $f_r \restriction \delta$ maps into δ , and so is a member of N .

Let B_0 denote the set of $z \in T^*$ for which there exists some $M \in \mathcal{X}$ such that D and $\dot{A}$ are members of M , $M \cap \omega_1 = \text{ht}_{T^*}(z)$, and there exists some condition $s \in D$ such that $f_s \restriction (M \cap \omega_1) = f_r \restriction \delta$, $M \cap \omega_1 \in \text{dom}(f_s)$, $x_s \cap M = x_r \cap \delta$, and s forces that $z \in \dot{A}$. Since all of the parameters mentioned in the definition of B_0 are members of N , $B_0 \in N$ by elementarity. Also, $x \in B_0$ as witnessed by N and r .

Since x is (N, T^*) -generic, we can fix $z <_{T^*} x$ which is in B_0 . Let M and s witness that $z \in B_0$. Since $z \in N$, we may assume by elementarity that M and s are in N . Let $\delta_0 := M \cap \omega_1$. Note that since $\text{ht}_{T^*}(z) = M \cap \omega_1 = \delta_0$, we have that z is (M, T^*) -generic. As $\delta_0 \in \text{dom}(f_s)$, $f_s \restriction \delta_0 = f_r \restriction \delta$ maps into δ_0 , and hence is a member of M . (As a point of clarity, we do not claim that r and s are compatible; the argument in what follows is more subtle.)

Fix a limit ordinal $\xi < \delta_0$ which is large enough so that:

- $\text{dom}(f_r) \cap \delta = \text{dom}(f_s) \cap \delta_0$ is a subset of ξ ;
- $\text{ran}(f_r \restriction \delta) = \text{ran}(f_s \restriction \delta_0) \subseteq \xi$;
- $x_r \cap \delta = x_s \cap \delta_0$ is a subset of ξ ;
- for all $\beta \in x_r \setminus \delta$, $L_\beta \cap \delta_0 \subseteq \xi$;
- for all $\beta \in x_s \setminus (\delta_0 + 1)$, $L_\beta \cap \delta_0 \subseteq \xi$.

This is possible since δ_0 is a limit of limit ordinals and all of the finitely many sets mentioned above are finite subsets of δ_0 .

Define X as the set of β satisfying:

- (a) $\beta < \xi$ is a limit ordinal;
- (b) for all $\gamma \in \text{dom}(f_r) \cap \delta$, it is not the case that $\gamma \leq \beta \leq f_r(\gamma)$ (in particular, $\beta \notin \text{dom}(f_r) \cap \delta$);
- (c) there exists $\alpha \in x_r \setminus \delta$ such that $\beta \in L_\alpha$.

Observe that since $f_r \restriction \delta = f_s \restriction \delta_0$, (b) implies that for all $\gamma \in \text{dom}(f_s) \cap \delta_0$, it is not the case that $\gamma \leq \beta \leq f_s(\gamma)$ (in particular, $\beta \notin \text{dom}(f_s) \cap \delta_0$). Note that X is a finite subset of δ_0 , and hence is a member of M .

For each $\beta \in X$, choose $\alpha_\beta < \beta$ large enough so that:

- (1) $x_r \cap \beta$, $\text{dom}(f_r) \cap \beta$, $\text{ran}(f_r) \cap \beta$, and $X \cap \beta$ are subsets of α_β ;
- (2) for all $\alpha \in x_r \cup x_s$ strictly larger than β , $L_\alpha \cap \beta \subseteq \alpha_\beta$;
- (3) $\alpha_\beta \notin L_\beta$.

This is possible since the finitely many sets described in (1) and (2) are finite, and by our assumption about the ladder system, $\beta \setminus L_\beta$ is cofinal in β . Observe that since $f_r \restriction \delta = f_s \restriction \delta_0$ and $x_r \cap \delta = x_s \cap \delta_0$, (1) implies that $x_s \cap \beta$, $\text{dom}(f_s) \cap \beta$, and $\text{ran}(f_s) \cap \beta$ are subsets of α_β . Also note that by the definition of X , for all $\gamma \in \text{dom}(f_r) \cap \beta = \text{dom}(f_s) \cap \beta$, $f_r(\gamma) = f_s(\gamma) < \beta$, and therefore $f_r(\gamma) = f_s(\gamma) < \alpha_\beta$.

Define $v = (f_v, x_v)$ by letting $x_v := x_r \cap \delta$ and

$$f_v := (f_r \restriction \delta) \cup \{(\alpha_\beta, \beta) : \beta \in X\}.$$

Then also $x_v = x_s \cap \delta_0$ and $f_v = (f_s \restriction \delta_0) \cup \{(\alpha_\beta, \beta) : \beta \in X\}$. Clearly, v is in M . For $\beta \in X$, for all $\gamma \in \text{dom}(f_r) \cap \beta$, $f_r(\gamma) < \alpha_\beta$, and also if $\beta_1 < \beta_2$ are in X , then $\beta_1 < \alpha_{\beta_2}$. It easily follows that $f_v \in \text{CU}(\omega_1)$, and hence $v \in \mathbb{P}$. So $v \in M \cap \mathbb{P}$.

We make two claims about v : (I) v and s are compatible, and (II) any extension of v in M is compatible with r . To see that v and s are compatible, let $t := (f_v \cup f_s, x_v \cup x_s)$. Since $f_s \restriction \delta_0 \subseteq f_v \in M$ and for all $\gamma \in \text{dom}(f_v) \cap \delta_0$, $f_v(\gamma) < \delta_0$, we have that $f_v \cup f_s$ is in $\text{CU}(\omega_1)$. Hence, t is a condition. To see that $t \leq v$, we only need to show that for all $\alpha \in x_v$, the set $\text{dom}(f_t) \setminus \text{dom}(f_v)$ is disjoint from $L_\alpha \cap \text{lim}(\omega_1)$. But $x_v \subseteq \delta_0$, whereas $\text{dom}(f_t) \setminus \text{dom}(f_v) \subseteq \omega_1 \setminus \delta_0$, so this is immediate.

To show that $t \leq s$, it is enough to show that for all $\alpha \in x_s$,

$$(\text{dom}(f_v) \setminus \text{dom}(f_s)) \cap L_\alpha \cap \text{lim}(\omega_1) = \emptyset.$$

So let $\alpha \in x_s$ and let ζ be a limit ordinal in $\text{dom}(f_v) \setminus \text{dom}(f_s)$, and we will prove that $\zeta \notin L_\alpha$. By the definition of v , $\zeta = \alpha_\beta$ for some $\beta \in X$. By property (2) in the definition of α_β , if $\beta < \alpha$ then $L_\alpha \cap \beta \subseteq \alpha_\beta$, and thus $\zeta \notin L_\alpha$. So assume that $\alpha \leq \beta$. If $\alpha = \beta$, then by (3) in the definition of α_β , $\zeta \notin L_\alpha$. Finally, if $\alpha < \beta$, then by (1), $\alpha < \zeta$. In any case, $\zeta \notin L_\alpha$. This completes the proof that v and s are compatible.

Now we claim that for all $w \leq v$ in $M \cap \mathbb{P}$, w and r are compatible. Fix $w \leq v$ in $M \cap \mathbb{P}$. Define $t := (f_w \cup f_r, x_w \cup x_r)$. Since $f_r \restriction \delta \subseteq f_v \subseteq f_w$, and for all $\gamma \in \text{dom}(f_w)$, $f_w(\gamma) < \delta$, we have that $f_t \in \text{CU}(\omega_1)$. So t is a condition. As $x_w \subseteq \delta$ and $\text{dom}(f_t) \setminus \text{dom}(f_w) \subseteq \omega_1 \setminus \delta$, it easily follows that $t \leq w$.

It remains to show that $t \leq r$. It suffices to prove that for all $\alpha \in x_r$, $\text{dom}(f_t) \setminus \text{dom}(f_r)$ is disjoint from $L_\alpha \cap \text{lim}(\omega_1)$. Fix $\alpha \in x_r$. Let $\zeta \in \text{dom}(f_t) \setminus \text{dom}(f_r)$ be a limit ordinal, and we will show that $\zeta \notin L_\alpha$. Note that $\zeta \in \text{dom}(f_w) \setminus \text{dom}(f_r)$.

We split the argument into two cases. First, assume that $\alpha < \delta$. Then $\alpha \in x_r \cap \delta \subseteq x_v$. Suppose for a contradiction that $\zeta \in L_\alpha$. Then we cannot have $\zeta \in \text{dom}(f_w) \setminus \text{dom}(f_v)$, since $w \leq v$. Thus, $\zeta \in \text{dom}(f_v)$. As $\zeta \notin \text{dom}(f_r)$, by the definition of v it must be the case that $\zeta = \alpha_\beta$ for some $\beta \in X$. Since $\zeta \in L_\alpha$, $\zeta < \alpha$. If $\alpha < \beta$, then by (1) in the definition of α_β , we have that $\alpha < \alpha_\beta = \zeta$, which is false. By (3) in the definition of α_β , if $\alpha = \beta$ then $\zeta \notin L_\alpha$. Thus, $\beta < \alpha$. But then by (2) in the definition of α_β , $L_\alpha \cap \beta \subseteq \alpha_\beta = \zeta$, so $\zeta \notin L_\alpha$. This contradiction proves that $\zeta \notin L_\alpha$ in the case that $\alpha < \delta$.

Secondly, assume that $\delta \leq \alpha$. Suppose for a contradiction that $\zeta \in L_\alpha$. We claim that $\zeta \in X$. We know that ζ is a limit ordinal. By the choice of ξ , the fact that $\zeta \in L_\alpha$ implies that $\zeta < \xi$. Let $\gamma \in \text{dom}(f_r) \cap \delta$, and we will prove that it is not the case that $\gamma \leq \zeta \leq f_r(\gamma)$. Suppose for a contradiction that $\gamma \leq \zeta \leq f_r(\gamma)$. Observe that $\gamma \in \text{dom}(f_v)$ by the definition of v . So we have that $\zeta \in \text{dom}(f_w)$ and $\gamma \leq \zeta \leq f_w(\gamma)$. Since $f_w \in \text{CU}(\omega_1)$, this implies that $\zeta = \gamma$. So $\zeta \in \text{dom}(f_r)$, which contradicts our assumption that ζ is in $\text{dom}(f_w) \setminus \text{dom}(f_r)$. Thus, indeed

it is not the case that $\gamma \leq \zeta \leq f_r(\gamma)$. Now by our current assumption, $\zeta \in L_\alpha$ where $\alpha \in x_r \setminus \delta$. This concludes the argument that $\zeta \in X$. By the definition of v , $f_v(\alpha_\zeta) = \zeta$, where $\alpha_\zeta < \zeta$. Since $w \leq v$, $f_w(\alpha_\zeta) = \zeta$. But $\zeta \in \text{dom}(f_w)$, so as $f_w \in \text{CU}(\omega_1)$, $f_w(\alpha_\zeta) < \zeta$, which is a contradiction. This completes the proof that $\zeta \notin L_\alpha$, and hence that $t \leq w, r$.

We can now complete the proof. Let B_1 be the set of nodes $y \in T^*$ such that for some $w \leq v$ in D , w forces that $y \in \dot{A}$. Then $B_1 \in M$ by elementarity, and $z \in B_1$ as witnessed by any common lower bound of v and s . Since z is (M, T^*) -generic, we can find $y R z$ in B_1 . Note that $y R z$ and $z <_{T^*} x$ imply that $y R x$. As y and B_1 are in M , by elementarity there is $w \in M \cap D$ such that $w \leq v$ and w forces that $y \in \dot{A}$. Since every extension of v in $M \cap \mathbb{P}$ is compatible with r , w and r are compatible. Fix $t \leq r, w$. Then $y R x$ and t forces that $y \in \dot{A}$. Also, $t \leq w$ and $w \in N \cap D$. $\square$

4. STATIONARILY KNASTER AND CARDINAL CHARACTERISTICS

We will prove in this section that $\text{PFA}(T^*)$ implies $\text{MA}_{\omega_1}(\sigma\text{-linked})$, and in particular, that all of the standard cardinal characteristics of the continuum are greater than ω_1 . This result will factor through a property of forcing posets called stationarily Knaster. We will also prove that under $\text{PFA}(T^*)$, there is a Knaster forcing which is not stationarily Knaster, whereas under MA_{ω_1} , every ω_1 -c.c. forcing is stationarily Knaster.

Let $\mathbb{P}$ be a forcing poset. A set $B \subseteq \mathbb{P}$ is *linked* if any two members of B are compatible, and is *centered* if any finitely many members of B have a lower bound. The forcing poset $\mathbb{P}$ is *Knaster* if whenever $\{p_i : i < \omega_1\}$ is a family of conditions in $\mathbb{P}$, then there exists an uncountable set $X \subseteq \omega_1$ such that the collection $\{p_i : i \in X\}$ is linked. Oftentimes in proving that a particular forcing is Knaster using pushing down arguments, a stronger form of the Knaster property can be derived.

Definition 4.1 ([12]). *A forcing poset $\mathbb{P}$ is stationarily Knaster if whenever $\{p_i : i \in S\}$ is a family of conditions in $\mathbb{P}$, where $S \subseteq \omega_1$ is stationary, then there is a stationary set $T \subseteq S$ such that the collection $\{p_i : i \in T\}$ is linked.*

Proposition 4.2. *Let $\mathbb{P}$ be a stationarily Knaster forcing poset. Then $\mathbb{P}$ is T^* -proper.*

Proof. Fix a regular cardinal θ such that T^* and $\mathbb{P}$ are members of $H(\theta)$. Let N be a countable elementary substructure of $(H(\theta), \in, T^*, \mathbb{P})$. Let $\delta := N \cap \omega_1$. Since $\mathbb{P}$ is ω_1 -c.c., standard arguments show that every condition in $\mathbb{P}$ is $(N, \mathbb{P})$ -generic. We will show that every condition in $\mathbb{P}$ is $(N, \mathbb{P}, T^*)$ -generic.

Let $p \in \mathbb{P}$, $x \in T_\delta^*$, $\dot{A} \in N$ a $\mathbb{P}$ -name for a subset of T^* , and $R \in \{<_{T^*}, <_{T^*}^-\}$. Assume that $q \leq p$ and q forces that $x \in \dot{A}$. We claim that there exists $y R x$ and $r \leq q$ such that r forces that $y \in \dot{A}$. Suppose for a contradiction that the claim fails. Then for all $y R x$, q forces that $y \notin \dot{A}$.

Define B as the set of $z \in T^*$ such that for some $s \in \mathbb{P}$, s forces that $z \in \dot{A}$, but for all $y R z$, s forces that $y \notin \dot{A}$. Then $B \in N$ by elementarity, and $x \in B$ as witnessed by q . Define $S := \{\text{ht}_{T^*}(z) : z \in B\}$, which is in N . Since $x \in B$, $\delta = \text{ht}_{T^*}(x)$ is in S . But δ is a member of every club subset of ω_1 which lies in N , so by elementarity, S is a stationary subset of ω_1 .

For each $\alpha \in S$, fix a node $z_\alpha \in B$ with height α and a condition s_α which witnesses that $z_\alpha \in B$. Then s_α forces that $z_\alpha \in \dot{A}$, but for all $y R z_\alpha$, s_α forces

that $y \notin \dot{A}$. We have that $\{s_\alpha : \alpha \in S\} \subseteq \mathbb{P}$ and $S \subseteq \omega_1$ is stationary. Since $\mathbb{P}$ is stationarily Knaster, we can find a stationary set $T \subseteq S$ such that for all $\alpha < \beta$ in T , s_α and s_β are compatible.

Consider $\alpha < \beta$ in T . Fix $r \leq s_\alpha, s_\beta$. Then r forces that z_α and z_β are in $\dot{A}$. But s_β , and hence r , forces that any $y R z_\beta$ is not in $\dot{A}$. So it must be the case that z_α is not R -below z_β .

We consider two cases. First, assume that R equals $<_{T^*}$. Then for all $\alpha < \beta$ in T , $z_\alpha \not\leq_{T^*} z_\beta$. It follows that z_α and z_β are incomparable in T^* . Thus, $\{z_\alpha : \alpha \in T\}$ is an antichain of T^* . But the collection of heights of nodes in this antichain is exactly T , which is stationary. So $\{z_\alpha : \alpha \in T\}$ is a stationary antichain, which contradicts that T^* has no stationary antichain.

Secondly, assume that R equals $<_{\bar{T}^*}$. Then for all $\alpha < \beta$ in T , it is not the case that $z_\alpha <_{\bar{T}^*} z_\beta$. Since $\text{ht}_{T^*}(z_\alpha) = \alpha < \beta = \text{ht}_{T^*}(z_\beta)$, this means that $z_\alpha <_{T^*} z_\beta$. Thus, $\{z_\alpha : \alpha \in T\}$ is an uncountable chain of T^* , which contradicts that T^* is an ω_1 -Aronszajn tree. $\square$

Corollary 4.3. *PFA(T^*) implies MA_{ω_1} (stationarily Knaster).*

Recall that a forcing poset is σ -centered if it is the union of countably many centered subsets, and is σ -linked if it is the union of countably many linked subsets. Obviously, σ -centered implies σ -linked.

Lemma 4.4. *If $\mathbb{P}$ is a σ -linked forcing poset, then $\mathbb{P}$ is stationarily Knaster.*

Proof. Let $\mathbb{P} = \bigcup \{X_n : n < \omega\}$, where each X_n is linked. Consider a collection $\{p_i : i \in S\}$ of conditions in $\mathbb{P}$, where $S \subseteq \omega_1$ is stationary. If we define a function which maps each $\alpha \in S$ to the least $n < \omega$ such that $p_\alpha \in X_n$, then there exists a stationary set $T \subseteq S$ on which this function is constant. Then for some $n < \omega$, $\{p_i : i \in T\} \subseteq X_n$, and hence $\{p_i : i \in T\}$ is linked. $\square$

Corollary 4.5. *PFA(T^*) implies MA_{ω_1} (σ -linked).*

We can now conclude that PFA(T^*) implies that all of the standard cardinal characteristics of the continuum are larger than ω_1 . Since the pseudo-intersection number $\mathfrak{p}$ is the smallest among such cardinal characteristics, it suffices to show that PFA(T^*) implies that $\mathfrak{p} > \omega_1$. But $\mathfrak{p} > \omega_1$ is equivalent to the forcing axiom $\text{MA}_{\omega_1}(\sigma\text{-centered})$ ([3]). And by Lemma 4.4, every σ -centered forcing poset is stationarily Knaster, and hence T^* -proper.

Corollary 4.6. *PFA(T^*) implies that $\mathfrak{p} > \omega_1$.*

It is a natural question whether the property of being Knaster is equivalent to being stationarily Knaster. Under MA_{ω_1} , the answer is yes.

Proposition 4.7. *Assume MA_{ω_1} . Then every ω_1 -c.c. forcing poset is stationarily Knaster.*

This easily follows from Lemma 4.4 and the well-known fact that under MA_{ω_1} , every ω_1 -c.c. forcing poset of size ω_1 is σ -centered.

The rest of this section is devoted to proving that under PFA(T^*), the forcing poset consisting of finite antichains of T^* ordered by reverse inclusion is Knaster but not stationarily Knaster.

Recall the result of Baumgartner-Malitz-Reinhardt [2] that if T is a tree of height ω_1 with no uncountable chains, and $\langle x_\alpha : \alpha < \omega_1 \rangle$ is a pairwise disjoint sequence of

finite subsets of T , then there exist $\alpha < \beta$ such that every node in x_α is incomparable in T with every node in x_β . This implies an apparently stronger conclusion.

Lemma 4.8. *Let T be a tree of height ω_1 with no uncountable chains, and let $\langle x_\alpha : \alpha < \omega_1 \rangle$ be a pairwise disjoint sequence of finite subsets of T . Then there is a countably infinite set $A \subseteq \omega_1$ such that for all $\alpha < \beta$ in A , every member of x_α is incomparable in T with every member of x_β .*

Proof. Define a function $F : [\omega_1]^2 \rightarrow 2$ by $F(\alpha, \beta) = 0$ if there is a member of x_α which is comparable in T with a member of x_β , and $F(\alpha, \beta) = 1$ otherwise. By the Dushnik-Miller theorem $\omega_1 \rightarrow (\omega_1, \omega)^2$, there exists either an uncountable subset of ω_1 all of whose pairs get value 0, or a countably infinite subset of ω_1 all of whose pairs get value 1. Note that the first possibility contradicts the result of Baumgartner-Malitz-Reinhardt. So the second possibility holds, which gives us the conclusion of the lemma. $\square$

The next three results show that under $\text{PFA}(T^*)$, we can strengthen “countably infinite” in the last lemma to “uncountable”.

Lemma 4.9. *Let T be an ω_1 -tree. Suppose that $\langle y_n : n < \omega \rangle$ is a sequence satisfying:*

- (1) *each y_n is a finite subset of T ;*
- (2) *for all $m < n$, for all $x \in y_m$ and $z \in y_n$, $\text{ht}_T(x) < \text{ht}_T(z)$ and x and z are incomparable in T .*

Let y be a finite subset of T such that for all $n < \omega$, every node of y_n has height less than the height of every node in y . Then there exists $n < \omega$ such that every member of y_n is incomparable in T with every member of y .

Proof. Suppose for a contradiction that the conclusion of the lemma fails. Define a function $f : \omega \rightarrow y$ by letting $f(n) \in y$ be some node which is above some member of y_n in T . Then there is some $z \in y$ such that for infinitely many n , $f(n) = z$. Fix $m < n$ such that $f(m) = f(n) = z$. Pick $z_m \in y_m$ and $z_n \in y_n$ such that $z_m <_T z$ and $z_n <_T z$. Since T is a tree, it follows that $z_m <_T z_n$, which contradicts our assumption about y_m and y_n . $\square$

Theorem 4.10. *Let T be a tree of height ω_1 with no uncountable chains, and let $\langle x_\alpha : \alpha < \omega_1 \rangle$ be a pairwise disjoint sequence of finite subsets of T . Then there exists a T^* -proper forcing poset which adds an uncountable set $A \subseteq \omega_1$ such that for all $\alpha < \beta$ in A , every member of x_α is incomparable in T with every member of x_β .*

Proof. By thinning out and relabelling if necessary, we may assume without loss of generality that for all $\alpha < \beta$, every node of x_α has height at least α and less than the height of any member of x_β .

Let us say that a set N is a *suitable model* if it is a countable elementary substructure of $H(\omega_2)$ which contains T as an element. Define a forcing poset $\mathbb{P}$ as follows. Conditions in $\mathbb{P}$ are pairs (a, X) satisfying:

- (1) $a \subseteq \omega_1$ is finite;
- (2) for all $i < j$ in a , every node in x_i is incomparable in T with every node in x_j ;
- (3) X is a finite $\in$ -chain of suitable models;
- (4) for all $i < j$ in a , there is $N \in X$ such that $i < N \cap \omega_1 < j$;

- (5) if $N \in X$ and $i \in a \setminus N$, then for any set $Z \subseteq \omega_1$ in N with $i \in Z$, there are uncountably many $j \in Z$ such that every member of x_i is incomparable in T with every member of x_j ;
- (6) if $N \in X$ and $i \in a \setminus N$, then for any function $f : \omega_1 \rightarrow \omega_1$ in N , $f(N \cap \omega_1) < i$;
- (7) if $a \neq \emptyset$, then $X \neq \emptyset$ and there is some $K \in X$ such that $K \cap \omega_1 < \min(a)$.

Let $(b, Y) \leq (a, X)$ if $a \subseteq b$ and $X \subseteq Y$. For any condition $p \in \mathbb{P}$, we will write $p = (a_p, X_p)$.

We begin with the claim that for all $p \in \mathbb{P}$, there are uncountably many $\xi < \omega_1$ such that for all $i \in a_p$, every member of x_i is incomparable in T with every member of x_ξ . If $a_p = \emptyset$, then the claim holds vacuously. Otherwise fix $n < \omega$ such that $|a_p| = n + 1$.

Let i^* be the largest member of a_p . By the definition of $\mathbb{P}$, there exists $N \in X$ such that $a_p \setminus \{i^*\} \subseteq N$ and $i^* \notin N$. Let Z be the set of $\xi < \omega_1$ such that for all $i \in a_p \setminus \{i^*\}$, every member of x_i is incomparable in T with every member of x_ξ (for clarity, if $n = 0$ then $Z = \omega_1$, which is fine). Note that $Z \in N$ by elementarity, and $i^* \in Z$. By the definition of $\mathbb{P}$, there are uncountably many $\xi \in Z$ such that every member of x_{i^*} is incomparable in T with every member of x_ξ , which proves the claim.

Assume for a moment that $\mathbb{P}$ is T^* -proper. Let $\dot{A}$ be a $\mathbb{P}$ -name for the set

$$\bigcup \{a_p : p \in \dot{G}_{\mathbb{P}}\}.$$

Then by the definition of $\mathbb{P}$, it is forced that for all $i < j$ in $\dot{A}$, every node in x_i is incomparable in T with every node in x_j .

We claim that $\dot{A}$ is forced to be uncountable, which completes the proof of the theorem under the assumption that $\mathbb{P}$ is T^* -proper. To prove this, consider a condition $p \in \mathbb{P}$ and $\gamma < \omega_1$. We will find $q \leq p$ so that $a_q \setminus \gamma \neq \emptyset$. Fix a suitable model N such that p and γ are members of N . Then easily $(a_p, X_p \cup \{N\})$ is a condition below p . Let Y be the set of $\xi < \omega_1$ such that for all $i \in a_p$, every member of x_i is incomparable in T with every member of x_ξ . Then $Y \in N$ by elementarity, and Y is uncountable by the claim above.

Let Y' be the set of $\xi \in Y$ which satisfy the property that for all $Z \subseteq \omega_1$ in N with $\xi \in Z$, there are uncountably many $j \in Z$ such that every member of x_ξ is incomparable in T with every member of x_j . We claim that Y' is uncountable. If not, then there exists $\beta < \omega_1$ such that for all $\xi \in Y \setminus \beta$, there is $Z_\xi \in N$ with $\xi \in Z_\xi$ and $\delta_\xi < \omega_1$ such that for all $j \in Z_\xi \setminus \delta_\xi$, there is a member of x_ξ which is comparable in T with some member of x_j . Since N is countable, we can fix $Z \in N$ such that there are uncountably many $\xi \in Y \setminus \beta$ for which $Z_\xi = Z$. Now we can construct by induction an increasing sequence of ordinals $\langle \xi_i : i < \omega_1 \rangle$ in $Y \setminus \beta$ such that for all $i < \omega_1$, $Z_{\xi_i} = Z$, and for all $i < j < \omega_1$, $\delta_{\xi_i} \leq \xi_j$. But now for all $i < j$, $\xi_j \in Z_{\xi_i} \setminus \delta_{\xi_i}$, so there is a member of x_{ξ_i} which is comparable in T with a member of x_{ξ_j} . Thus, the sequence $\langle x_{\xi_i} : i < \omega_1 \rangle$ is a counterexample to the result of Baumgartner-Malitz-Reinhart, which is a contradiction.

Since Y' is uncountable, we can fix $\xi \in Y'$ which is larger than $N \cap \omega_1$, γ , and $f(N \cap \omega_1)$ for all $f : \omega_1 \rightarrow \omega_1$ in N . Then by the definition of Y' , it is easy to check that $(a_p \cup \{\xi\}, X_p \cup \{N\})$ is a condition below p . This completes the proof that $\dot{A}$ is forced to be uncountable.

It remains to prove that $\mathbb{P}$ is T^* -proper. Fix a regular cardinal $\theta > \omega_2$ such that T^* , T , and $\mathbb{P}$ are members of $H(\theta)$, and let N be a countable elementary substructure of $(H(\theta), \in, T^*, T, \mathbb{P})$. Let $\delta := N \cap \omega_1$. Consider $p \in N \cap \mathbb{P}$. Define $q := (a_p, X_p \cup \{N \cap H(\omega_2)\})$. Then q is a condition and $q \leq p$.

We claim that q is $(N, \mathbb{P}, T^*)$ -generic. Fix a dense set $D \in N$, $x \in T_\delta^*$, a $\mathbb{P}$ -name $\dot{A} \in N$ for a subset of T^* , and a relation $R \in \{<_{T^*}, <_{\bar{T}^*}\}$. Let $r \leq q$. By extending further if necessary, we may assume that $r \in D$, r decides whether or not $x \in \dot{A}$, and $a_r \setminus \delta \neq \emptyset$. If r forces that $x \notin \dot{A}$, then replace $\dot{A}$ in what follows by the canonical $\mathbb{P}$ -name for T^* . Thus, we may assume without loss of generality that r forces that $x \in \dot{A}$.

Our goal is to find $t \leq r$ which is below some member of $N \cap D$ and a node y R x such that t forces that $y \in \dot{A}$. Let $r_N := (a_r \cap N, X_r \cap N)$. It is easy to check that r_N is in $N \cap \mathbb{P}$ and $r \leq r_N$. Let B_0 denote the set of nodes $z \in T^*$ for which there exists a condition $s \in D$ such that:

- (1) $s \leq r_N$;
- (2) $a_s \cap \text{ht}_{T^*}(z) = a_r \cap \delta$;
- (3) s forces that $z \in \dot{A}$.

Observe that $B_0 \in N$ by elementarity, and $x \in B_0$ as witnessed by r .

For each $z \in B_0$, let $B_0(z)$ denote the set of conditions $s \in D$ which witness that $z \in B_0$. The function $z \mapsto B_0(z)$ is in N by elementarity, and $r \in B_0(x)$. Define a function $F : B_0 \rightarrow \omega_1 + 1$ by

$$F(z) := \sup\{\min(a_s \setminus \text{ht}_{T^*}(z)) : s \in B_0(z)\}.$$

Note that $F \in N$ by elementarity. Define

$$X := \{z \in B_0 : F(z) < \omega_1\}.$$

Define $H : \omega_1 \rightarrow \omega_1$ by

$$H(\alpha) := \sup\{F(z) : z \in X, \text{ht}_{T^*}(z) = \alpha\}.$$

Then X and H are in N by elementarity.

Property (6) in the definition of r being a condition implies that $H(\delta) < \min(a_r \setminus \delta)$. In particular, x cannot be in X . For otherwise $F(x)$ would be in the set for which $H(\delta)$ is the supremum, which implies that $\min(a_r \setminus \delta) \leq F(x) \leq H(\delta)$, giving a contradiction. Therefore, $F(x) = \omega_1$.

Let $B_1 := \{z \in B_0 : F(z) = \omega_1\}$. Then $B_1 \in N$ by elementarity, and $x \in B_1$ as just proven. Since x is (N, T^*) -generic, there exists y R x such that $y \in B_1$. Let $\delta_0 := \text{ht}_{T^*}(y)$. Since F and y are in N and $F(y) = \omega_1$, it is straightforward to construct in N a sequence $\langle s_i : i < \omega_1 \rangle$ of conditions in $B_0(y)$ such that for all $i < j < \omega_1$, $\max(a_{s_i}) < \min(a_{s_j} \setminus \delta_0)$.

For each $i < \omega_1$, let $y_i := \bigcup\{x_\alpha : \alpha \in a_{s_i} \setminus \delta_0\}$. Then $\langle y_i : i < \omega_1 \rangle$ is a pairwise disjoint sequence of finite subsets of T . Note that for all $i < j$, every node in y_i has height in T less than the height of every node in y_j . Apply Lemma 4.8 inside N to find a set $z \subseteq \omega_1$ in N with order type ω such that for all $i < j$ in z , every member of y_i is incomparable in T with every member of y_j . Define $y := \bigcup\{x_i : i \in a_r \setminus \delta\}$. Note that since $z \in N$, for all $i \in z$, every node in y_i has height less than δ , whereas every node in y has height at least δ . So we can apply Lemma 4.9 to $\langle y_i : i \in z \rangle$ and y to find $\beta \in z$ such that every node in y_β is incomparable in T with every node in y .

Let $s := s_\beta$. We claim that r and s are compatible. Define $t := (a_r \cup a_s, X_r \cup X_s)$. We will show that t is in $\mathbb{P}$. Then clearly $t \leq r, s$. It is easy to check that t satisfies all of the properties of being in $\mathbb{P}$ except possibly (2).

Consider $i < j$ in a_t , and we will prove that every node in x_i is incomparable in T with every node in x_j . If i and j are either both in a_r or both in a_s , then the desired conclusion is true since r and s are conditions. Otherwise $i \in a_s \setminus a_r$ and $j \in a_r \setminus a_s$. Since $s \in B_0(y)$, $a_s \cap \delta_0 = a_r \cap \delta$. Thus, $i \in a_s \setminus \delta_0$ and $j \in a_r \setminus \delta$. So $x_i \subseteq y_\beta$ and $x_j \subseteq y$. By the choice of β , every node of x_i is incomparable in T with every node of x_j .

So we have that $t \in \mathbb{P}$ and $t \leq r, s$. In particular, t is below a member of $N \cap D$, namely s . Also $y R x$ and s forces that $y \in \dot{A}$. So $t \leq r$ and t forces that $y \in \dot{A}$, which completes the proof that q is $(N, \mathbb{P}, T^*)$ -generic. $\square$

Corollary 4.11. *Assume $PFA(T^*)$. Let T be a tree of height ω_1 with no uncountable chains, and let $\langle x_\alpha : \alpha < \omega_1 \rangle$ be a pairwise disjoint sequence of finite subsets of T . Then there exists an uncountable set $A \subseteq \omega_1$ such that for all $\alpha < \beta$ in A , every member of x_α is incomparable in T with every member of x_β .*

Proof. Let $\mathbb{P}$ be the T^* -proper forcing from Theorem 4.10, and let $\dot{A}$ be a $\mathbb{P}$ -name for an uncountable subset of ω_1 such that for all $\alpha < \beta$ in $\dot{A}$, every member of x_α is incomparable in T with every member of x_β . For each $\alpha < \omega_1$, let D_α be the dense set of conditions in $\mathbb{P}$ which decide the α -th member of $\dot{A}$. Let G be a filter on $\mathbb{P}$ which meets each dense set D_α . For each α , let γ_α be the unique ordinal such that for some $p \in D_\alpha \cap G$, p forces that γ_α is the α -th member of $\dot{A}$. By the fact that G is a filter, it is easy to check that for all $\alpha < \beta$, every member of x_{γ_α} is incomparable in T with every member of x_{γ_β} . $\square$

If we ignore T^* , then Theorem 4.10 shows more generally that there exists a proper forcing which adds a set A as described in the theorem. Consequently, the conclusion of Corollary 4.11 follows from PFA .

Corollary 4.12. *$PFA(T^*)$ implies Suslin's hypothesis.*

Proof. Let T be an ω_1 -tree. For each $\alpha < \omega_1$, choose a node t_α of T with height α , and define $x_\alpha := \{t_\alpha\}$. Apply Corollary 4.11 to the collection $\{x_\alpha : \alpha < \omega_1\}$ to find an uncountable set $A \subseteq \omega_1$ so that for all $\alpha < \beta$ in A , t_α and t_β are incomparable in T . Then $\{t_\alpha : \alpha \in A\}$ is an uncountable antichain of T . So T is not an ω_1 -Suslin tree. $\square$

Theorem 4.13. *Assume $PFA(T^*)$. Let $\mathbb{P}$ be the forcing poset whose conditions are finite antichains of T^* , ordered by reverse inclusion. Then $\mathbb{P}$ is Knaster but not stationarily Knaster.*

Proof. Note that conditions x and y of $\mathbb{P}$ are compatible iff $x \cup y$ is an antichain of T^* . For each $\alpha < \omega_1$, pick an element t_α of T^* with height α , and let $p_\alpha := \{t_\alpha\}$. Then p_α and p_β are compatible in $\mathbb{P}$ iff t_α and t_β are incomparable in T^* .

If $S \subseteq \omega_1$, then $\{p_\alpha : \alpha \in S\}$ is a linked subset of $\mathbb{P}$ iff for all $\alpha < \beta$ in S , t_α and t_β are incomparable in T^* iff $\{t_\alpha : \alpha \in S\}$ is an antichain of T^* . Since T^* has no stationary antichains, if S is stationary then $\{p_\alpha : \alpha \in S\}$ is not linked. It follows that $\mathbb{P}$ is not stationarily Knaster.

To see that $\mathbb{P}$ is Knaster assuming $\text{PFA}(T^*)$, let $\{q_\alpha : \alpha < \omega_1\}$ be a family of conditions in $\mathbb{P}$. By the Δ -system lemma, we may assume without loss of generality that there is a finite set $r \subseteq T^*$ such that for all $\alpha < \beta < \omega_1$, $q_\alpha \cap q_\beta = r$.

Let $x_\alpha := q_\alpha \setminus r$ for all $\alpha < \omega_1$. Then $\langle x_\alpha : \alpha < \omega_1 \rangle$ is a pairwise disjoint sequence of finite subsets of T^* . By Corollary 4.11, there exists an uncountable set $A \subseteq \omega_1$ such that for all $\alpha < \beta$ in A , every member of x_α is incomparable in T^* with every member of x_β . It immediately follows that for all $\alpha < \beta$ in A , $q_\alpha \cup q_\beta$ is an antichain of T^* , and hence a condition below q_α and q_β . $\square$

5. DIGRESSION: MORE ON STATIONARILY KNASTER

In the previous section we proved that under $\text{PFA}(T^*)$, there exists a forcing poset which is Knaster but not stationarily Knaster. We now prove that the same conclusion follows from $\diamond^*$. Afterwards, we introduce and explore the property of having stationary precaliber ω_1 .

Recall that $\diamond^*$ is the statement that there exists a sequence $\langle \mathcal{S}_\alpha : \alpha < \omega_1 \rangle$ satisfying:

- (1) each $\mathcal{S}_\alpha$ is a collection of countably many subsets of α ;
- (2) for all $X \subseteq \omega_1$, there exists a club $D \subseteq \omega_1$ such that for all $\alpha \in D$, $X \cap \alpha \in \mathcal{S}_\alpha$.

Theorem 5.1. *$\diamond^*$ implies that there exists a forcing poset which is Knaster but not stationarily Knaster.*

Proof. Fix a sequence $\langle \mathcal{F}_\alpha : \alpha < \omega_1 \rangle$ satisfying:

- (1) each $\mathcal{F}_\alpha$ is a collection of countably many functions from α to ω ;
- (2) for any function $F : \omega_1 \rightarrow \omega$, there exists a club $D \subseteq \omega_1$ such that for all $\alpha \in D$, $F \restriction \alpha \in \mathcal{F}_\alpha$.

Standard arguments show that $\diamond^*$ implies the existence of such a sequence.

For each limit ordinal $\alpha < \omega_1$, fix a cofinal subset L_α of α with order type ω . Let $\alpha < \omega_1$ be a limit ordinal, and consider the countable collection of functions $\{f \restriction L_\alpha : f \in \mathcal{F}_\alpha\}$. An easy diagonalization argument shows that there exists a function $g_\alpha : L_\alpha \rightarrow \omega$ satisfying that for all $f \in \mathcal{F}_\alpha$, there exists $\gamma < \alpha$ such that for all $\beta \in L_\alpha \setminus \gamma$, $f(\beta) < g_\alpha(\beta)$. In particular, for all $f \in \mathcal{F}_\alpha$, $g_\alpha \neq f \restriction L_\alpha$.

We now consider the forcing poset from [4] for adding a uniformisation of the colored ladder system $\langle (L_\alpha, g_\alpha) : \alpha \in \lim(\omega_1) \rangle$. Let $\mathbb{P}$ be the forcing poset whose conditions are pairs (x, h) satisfying:

- (1) $x \subseteq \lim(\omega_1)$ is finite;
- (2) $h : \bigcup\{L_\alpha : \alpha \in x\} \rightarrow \omega$;
- (3) for all $\alpha \in x$ there is $\gamma < \alpha$ such that for all $\beta \in L_\alpha \setminus \gamma$, $h(\beta) = g_\alpha(\beta)$.

Let $(x_2, h_2) \leq (x_1, h_1)$ if $x_1 \subseteq x_2$ and $h_1 \subseteq h_2$.

We claim that $\mathbb{P}$ is Knaster but not stationarily Knaster. To see that it is Knaster, we argue in a similar way as in [4]. Suppose that $\langle (x_i, h_i) : i < \omega_1 \rangle$ is a sequence of conditions in $\mathbb{P}$. Applying the Δ -system lemma and a standard thinning out argument, we can find an uncountable set $Z \subseteq \omega_1$ such that for some set r , for all $i < j$ in Z , $x_i \cap x_j = r$ and $\max(x_i) < \min(x_j \setminus r)$. By relabeling the sequence if necessary, we may assume without loss of generality that $Z = \omega_1$.

For each $i < \omega_1$, choose a sequence $\langle \gamma_\alpha^i : \alpha \in r \rangle$ such that for each $\alpha \in r$, $\gamma_\alpha^i < \alpha$ and for all $\beta \in L_\alpha \setminus \gamma_\alpha^i$, $h_i(\beta) = g_\alpha(\beta)$. By a pressing-down argument, we can find $Y_1 \subseteq \omega_1$ stationary so that for all $i < j$ in Y_1 :

- (1) $\gamma_\alpha^i = \gamma_\alpha^j$ for all $\alpha \in r$;
- (2) $h_i \upharpoonright (L_\alpha \cap \gamma_\alpha^i) = h_j \upharpoonright (L_\alpha \cap \gamma_\alpha^j)$ for all $\alpha \in r$;
- (3) $x_i \cap i = x_j \cap j = r$;
- (4) $\max(x_i) < j$.

Observe that (1) and (2) imply that for all $i < j$ in Y_1 , $h_i \upharpoonright \bigcup \{L_\alpha : \alpha \in r\} = h_j \upharpoonright \bigcup \{L_\alpha : \alpha \in r\}$.

For each $i \in Y_1$, let i^+ denote the least member of Y_1 strictly above i . Then by construction, for each $i \in Y_1$,

$$\min(x_{i^+} \setminus r) > \max(x_i) \geq i.$$

Consequently, the set $\bigcup \{L_\alpha \cap i : \alpha \in x_{i^+} \setminus r\}$ is a finite subset of i . So we can find a stationary set $Y_2 \subseteq Y_1$ and a set s such that for all $i \in Y_2$,

$$\bigcup \{L_\alpha \cap i : \alpha \in x_{i^+} \setminus r\} = s.$$

Now find $Y_3 \subseteq Y_2$ stationary such that for all $i < j$ in Y_3 , $i^+ < j$ and $h_{i^+} \upharpoonright s = h_{j^+} \upharpoonright s$. It is now easy to check that for all $i < j$ in Y_3 , $(x_{i^+} \cup x_{j^+}, h_{i^+} \cup h_{j^+})$ is a lower bound of (x_{i^+}, h_{i^+}) and (x_{j^+}, h_{j^+}) . This completes the proof that $\mathbb{P}$ is Knaster.

Now we prove that $\mathbb{P}$ is not stationarily Knaster. For each limit ordinal $\alpha < \omega_1$, define $p_\alpha := (\{\alpha\}, g_\alpha)$. Then obviously $p_\alpha \in \mathbb{P}$. Suppose for a contradiction that there exists a stationary set $T \subseteq \lim(\omega_1)$ such that for all $i < j$ in T , p_i and p_j are compatible. Then for all $i < j$ in T , $(\{i, j\}, g_i \cup g_j) \in \mathbb{P}$, and in particular, $g_i \cup g_j$ is a function. So for all $\gamma \in L_i \cap L_j$, $g_i(\gamma) = g_j(\gamma)$.

Define a function $F : \omega_1 \rightarrow \omega$ so that for all $\gamma < \omega_1$, $F(\gamma) = g_i(\gamma)$ if $i \in T$ and $\gamma \in L_i$, and if there is no such i , then $F(\gamma) = 0$. By the choice of T , F is well-defined. By the choice of the sequence $\langle \mathcal{F}_\alpha : \alpha \in \lim(\omega_1) \rangle$, there exists a club $D \subseteq \omega_1$ such that for all $\alpha \in D$, $F \upharpoonright \alpha \in \mathcal{F}_\alpha$. Since T is stationary, we can fix $\alpha \in T \cap D$. Then $F \upharpoonright \alpha = f$ for some $f \in \mathcal{F}_\alpha$. So

$$f \upharpoonright L_\alpha = F \upharpoonright L_\alpha = g_\alpha$$

But this contradicts the choice of g_α . □

Recall that a forcing poset $\mathbb{P}$ has *precaliber* ω_1 if whenever $X \subseteq \mathbb{P}$ is uncountable, then there exists an uncountable set $Y \subseteq X$ which is centered. With the difference between Knaster and stationarily Knaster in mind, we introduce the following variation of this idea.

Definition 5.2. *A forcing poset $\mathbb{P}$ has stationary precaliber ω_1 if whenever $\{p_i : i \in S\} \subseteq \mathbb{P}$, where $S \subseteq \omega_1$ is stationary, then there exists a stationary set $T \subseteq \omega_1$ such that $\{p_i : i \in T\}$ is centered.*

Lemma 5.3. *If $\mathbb{P}$ is a σ -centered forcing poset, then $\mathbb{P}$ has stationary precaliber ω_1 .*

Proof. Similar to the proof of Lemma 4.4. □

Proposition 5.4. *MA_{ω_1} implies that every ω_1 -c.c. forcing poset has stationary precaliber ω_1 .*

Proof. Similar to the proof of Proposition 4.7, but using Lemma 5.3 instead of Lemma 4.4. □

On the other hand, it is consistent that there exists a forcing poset which is stationarily Knaster but does not have stationary precaliber ω_1 .

Proposition 5.5. *Assume that $\mathfrak{t} = \omega_1$. Then there exists a forcing poset which is stationarily Knaster but does not have stationary precaliber ω_1 . In particular, this is true if CH holds.*

Proof. By [17, Theorem 1.3], there exists a forcing poset of size $\mathfrak{t}$ which is σ -linked but has no centered subsets of size $\mathfrak{t}$. Assuming $\mathfrak{t} = \omega_1$, it follows that there exists a forcing poset $\mathbb{P}$ of size ω_1 which is σ -linked but has no centered subsets of size ω_1 .

Since $\mathbb{P}$ is σ -linked, $\mathbb{P}$ is stationarily Knaster by Lemma 4.4. Fix a sequence $\langle p_i : i < \omega_1 \rangle$ of distinct conditions in $\mathbb{P}$. Then for any stationary set $T \subseteq \omega_1$, $\{p_i : i \in T\}$ is not centered, since $\mathbb{P}$ contains no centered subsets of size ω_1 . Thus, $\mathbb{P}$ does not have stationary precaliber ω_1 . $\square$

Since $\diamond^*$ implies CH, the following corollary is immediate from the previous results.

Corollary 5.6. *Assume $\diamond^*$. Then there exists a forcing poset which is Knaster but not stationarily Knaster, and there exists a forcing poset which is stationarily Knaster but does not have stationary precaliber ω_1 .*

6. P -IDEAL DICHOTOMY

In this section we will prove that $\text{PFA}(T^*)$ implies the P -ideal dichotomy. This will allow us to draw strong consequences from $\text{PFA}(T^*)$, such as the failure of square principles.

We recall the relevant definitions.

Definition 6.1. *Let S be an uncountable set and $\mathcal{I}$ an ideal whose members are countable subsets of S and which contains all finite subsets of S . We say that $\mathcal{I}$ is a P -ideal over S if whenever $\{A_n : n < \omega\} \subseteq \mathcal{I}$, then there is $B \in \mathcal{I}$ such that for all $n < \omega$, $A_n \subseteq^* B$.*

Definition 6.2. *The P -ideal dichotomy is the statement that whenever $\mathcal{I}$ is a P -ideal over an uncountable set S , then either there exists an uncountable set $T \subseteq S$ such that $[T]^\omega \subseteq \mathcal{I}$, or $S = \bigcup \{S_n : n < \omega\}$, where for each $n < \omega$, $[S_n]^\omega \cap \mathcal{I} = \emptyset$.*

Fix a P -ideal $\mathcal{I}$ over an uncountable set S , and we will prove that under $\text{PFA}(T^*)$, one of the two alternatives of the P -ideal dichotomy holds for $\mathcal{I}$. For a set $X \subseteq S$, we will write $X \perp \mathcal{I}$ to mean that $[X]^\omega \cap \mathcal{I} = \emptyset$. Note that for any finite set $x \subseteq S$, $x \perp \mathcal{I}$. If S is equal to a union $\bigcup \{S_n : n < \omega\}$, where for each n , $S_n \perp \mathcal{I}$, then we are done. So assume not.

We will prove that there exists a T^* -proper forcing poset which adds an uncountable set $T \subseteq S$ such that $[T]^\omega \subseteq \mathcal{I}$. There are several proper forcing posets in the literature for adding such a set. We will prove that the version from [16] is T^* -proper.

Recall that a set $K \subseteq \mathcal{I}$ is *cofinal* if for all $x \in \mathcal{I}$, there is $y \in K$ such that $x \subseteq^* y$. For a cofinal set $K \subseteq \mathcal{I}$ and $a \in \mathcal{I}$, define $K \restriction a := \{b \in K : a \subseteq b\}$.

Definition 6.3. *Define $\mathbb{P}$ as the forcing poset consisting of pairs (a, X) , where $a \in \mathcal{I}$ and X is a countable collection of cofinal subsets of $\mathcal{I}$, ordered by $(b, Y) \leq (a, X)$ if $a \subseteq b$, $X \subseteq Y$, and for all $K \in X$, the set $K \restriction (b \setminus a)$ is in Y .*

For any condition $p \in \mathbb{P}$, we will write $p = (a_p, X_p)$.

Observe that if $p \in \mathbb{P}$ and $K \subseteq \mathcal{I}$ is cofinal, then $(a_p, X_p \cup \{K\})$ is a condition below p .

The next two lemmas follow by straightforward arguments. See [16] for the details.

Lemma 6.4. *Let $p \in \mathbb{P}$. For any countable set $S_0 \subseteq S$, there is $q \leq p$ such that $a_q \setminus S_0 \neq \emptyset$.*

Lemma 6.5. *Suppose that $p \in \mathbb{P}$, $z \in S$, and p forces that $z \in a_s$ for some $s \in \dot{G}_{\mathbb{P}}$. Then $z \in a_p$.*

We will prove below that the forcing poset $\mathbb{P}$ is T^* -proper and countably distributive. Assume for a moment that this is true. Observe that since $\mathbb{P}$ is countably distributive, $\mathbb{P}$ is still a P -ideal over S in $V^{\mathbb{P}}$. Let $\dot{T}$ be a $\mathbb{P}$ -name for the set

$$\bigcup \{a_p : p \in \dot{G}_{\mathbb{P}}\}.$$

Then Lemma 6.4 implies that $\mathbb{P}$ forces that $\dot{T}$ is uncountable, and Lemma 6.5 implies that $\mathbb{P}$ forces that $[\dot{T}]^{\omega} \subseteq \mathcal{I}$.

The next two lemmas, which follow from arguments appearing in [16], will be used to show that $\mathbb{P}$ is proper.

Lemma 6.6. *Let $K \subseteq \mathcal{I}$ be cofinal and $x \in \mathcal{I}$. Then there exists a finite set $x_0 \subseteq x$ such that $K \restriction (x \setminus x_0)$ is cofinal.*

Lemma 6.7. *Let θ be a regular cardinal such that $S, \mathcal{I}$, and $\mathbb{P}$ are members of $H(\theta)$. Let N be a countable elementary substructure of $(H(\theta), \in, S, \mathcal{I}, \mathbb{P})$. Let $p \in N \cap \mathbb{P}$, $D \in N$ a dense subset of $\mathbb{P}$, and $d \in \mathcal{I}$ such that for all $a \in N \cap \mathcal{I}$, $a \subseteq^* d$. Then there exists $q \leq p$ in $N \cap D$ such that $a_q \setminus a_p \subseteq d$.*

The next two lemmas will help us to prove that $\mathbb{P}$ is T^* -proper.

Lemma 6.8. *Let θ be a regular cardinal such that $S, \mathcal{I}$, and $\mathbb{P}$ are members of $H(\theta)$, and N a countable elementary substructure of $(H(\theta), \in, S, \mathcal{I}, \mathbb{P})$. Suppose that $d \in \mathcal{I}$ satisfies that $d \subseteq N$ and for all $a \in N \cap \mathcal{I}$, $a \subseteq^* d$. Let $p \in N \cap \mathbb{P}$, $\dot{A} \in N$ a $\mathbb{P}$ -name for a subset of T^* , and $y \in N \cap T^*$. Then either:*

- (1) *there exists $q \leq p$ in $N \cap \mathbb{P}$ such that $a_q \setminus a_p \subseteq d$ and q forces that $y \in \dot{A}$,*
or
- (2) *there exists a cofinal set $J \subseteq \mathcal{I}$ in N such that $(a_p, X_p \cup \{J\})$ forces that $y \notin \dot{A}$.*

Proof. Assume that (1) fails, and we will prove (2). So there does not exist $q \leq p$ in $N \cap \mathbb{P}$ such that $a_q \setminus a_p \subseteq d$ and q forces that $y \in \dot{A}$. Let J be the set of $a \in \mathcal{I}$ such that whenever $q \leq p$ and $a_q \setminus a_p \subseteq a$, then q does not force that $y \in \dot{A}$. Observe that $J \in N$ by elementarity.

We claim that J is cofinal in $\mathcal{I}$. To prove the claim, it suffices to show that $d \in J$. For if this is true, then for all $b \in N \cap \mathcal{I}$, there is $c \in J$ (namely $c = d$) such that $b \subseteq^* c$. Hence, by elementarity J is cofinal in $\mathcal{I}$. Suppose for a contradiction that $d \notin J$. Then there exists $t \leq p$ such that $a_t \setminus a_p \subseteq d$ and t forces that $y \in \dot{A}$. Since $d \subseteq N$, $a_t \setminus a_p$ is a finite subset of N and hence is a member of N . By elementarity, there exists $q \in N \cap \mathbb{P}$ such that $q \leq p$, $a_q \setminus a_p = a_t \setminus a_p$, and q forces that $y \in \dot{A}$. Then $a_q \setminus a_p \subseteq d$. So q witnesses that (1) holds, which is a contradiction.

It remains to show that $(a_p, X_p \cup \{J\})$ forces that $y \notin \dot{A}$. If not, then there exists $q \leq (a_p, X_p \cup \{J\})$ which forces that $y \in \dot{A}$. By the definition of the order on $\mathbb{P}$, $J \restriction (a_q \setminus a_p)$ is in X_q , and therefore is cofinal and hence non-empty. Fix $a \in J$ such that $a_q \setminus a_p \subseteq a$. Then $q \leq p$, $a_q \setminus a_p \subseteq a$, and q forces that $y \in \dot{A}$, contradicting the fact that $a \in J$. $\square$

Lemma 6.9. *Let θ be a regular cardinal such that $S, \mathcal{I}$, and $\mathbb{P}$ are members of $H(\theta)$, and N a countable elementary substructure of $(H(\theta), \in, S, \mathcal{I}, \mathbb{P})$. Let $\delta := N \cap \omega_1$. Assume that $d \in \mathcal{I}$ satisfies that $d \subseteq N$ and for all $a \in N \cap \mathcal{I}$, $a \subseteq^* d$. Suppose that $x \in T_\delta^*$, $\dot{A} \in N$ is a $\mathbb{P}$ -name for a subset of T^* , $R \in \{<_{T^*}, <_{\bar{T}^*}\}$, and $p \in N \cap \mathbb{P}$. Then either:*

- (1) *there exists $y R x$ and $q \leq p$ in $N \cap \mathbb{P}$ such that $a_q \setminus a_p \subseteq d$ and q forces that $y \in \dot{A}$, or*
- (2) *there exists a cofinal set $J \subseteq \mathcal{I}$ such that $(a_p, X_p \cup \{J\})$ forces that $x \notin \dot{A}$.*

Proof. If (1) fails, then by Lemma 6.8 we have that for all $y R x$, there exists a cofinal set $J \subseteq \mathcal{I}$ such that $(a_p, X_p \cup \{J\})$ forces that $y \notin \dot{A}$. Let B be the set of $z \in T^*$ for which there exists a cofinal set $J \subseteq \mathcal{I}$ such that $(a_p, X_p \cup \{J\})$ forces that $z \notin \dot{A}$. Then $B \in N$ by elementarity, and for all $y R x$, $y \in B$. By Lemma 1.4, it follows that $x \in B$. So there exists a cofinal set $J \subseteq \mathcal{I}$ such that $(a_p, X_p \cup \{J\})$ forces that $x \notin \dot{A}$. $\square$

Proposition 6.10. *The forcing poset $\mathbb{P}$ is T^* -proper and countably distributive.*

Proof. Fix a regular cardinal θ such that $T^*, S, \mathcal{I}$, and $\mathbb{P}$ are members of $H(\theta)$. Let N be a countable elementary substructure of $H(\theta)$ which contains these parameters. Let $\delta := N \cap \omega_1$. Consider $p \in N \cap \mathbb{P}$. We will find $q \leq p$ which is $(N, \mathbb{P}, T^*)$ -generic. In fact, q will have the property that it is a member of every dense open subset of $\mathbb{P}$ which lies in N . Standard arguments show that this implies that $\mathbb{P}$ is countably distributive.

Since $\mathcal{I}$ is a P -ideal and N is countable, we can fix a set $d \in \mathcal{I}$ such that for all $a \in N \cap \mathcal{I}$, $a \subseteq^* d$. By intersecting d with N if necessary, we may also assume that $d \subseteq N$.

Let $\langle D_n : n < \omega \rangle$ enumerate all dense open subsets of $\mathbb{P}$ in N . Let $\langle (x_n, \dot{A}_n) : n < \omega \rangle$ list all pairs $(x, \dot{A})$, where $x \in T_\delta^*$ and $\dot{A} \in N$ is a $\mathbb{P}$ -name for a subset of T^* . Let $\langle (i_n, K_n) : n < \omega \rangle$ enumerate all pairs (i, K) such that $i < \omega$ and K is a cofinal subset of $\mathcal{I}$ in N .

We will define by induction on $n < \omega$ objects p_n, d_n , and a function F whose domain is a subset of ω . For bookkeeping, fix a surjection $f : \omega \rightarrow 3 \times \omega$ such that for each $(i, m) \in 3 \times \omega$, $f^{-1}(i, m)$ is infinite. We will maintain the following properties:

- (1) $p_0 = p$ and $d_0 = d$;
- (2) for all $n < \omega$, $p_n \in N \cap \mathbb{P}$ and $p_{n+1} \leq p_n$;
- (3) for all $n < \omega$, $d_n \in \mathcal{I}$, for all $a \in N \cap \mathcal{I}$, $a \subseteq^* d_n$, and $d_{n+1} \subseteq d_n$;
- (4) for all $n < \omega$, $a_{p_{n+1}} \setminus a_{p_n} \subseteq d_{n+1}$.

Let $p_0 := p$ and $d_0 := d$. Assume that $n < \omega$ and p_n, d_n , and $F \restriction n$ are defined. Let $f(n) = (i, m)$. We split the construction into the three cases depending on whether i is 0, 1, or 2.

Suppose that $i = 0$. First, we specify that $n \notin \text{dom}(F)$. Secondly, in order to define d_{n+1} , we consider the pair (i_m, K_m) . We only care if $i_m \leq n$ and $K_m \in X_{p_{i_m}}$.

If not, then let $d_{n+1} := d_n$. Otherwise, since $p_n \leq p_{i_m}$, $K_m \restriction (a_{p_n} \setminus a_{p_{i_m}}) \in X_{p_n}$, and in particular is cofinal. By Lemma 6.6, we can find $d_{n+1} \subseteq d_n$ which is equal to d_n minus finitely many elements such that

$$(K_m \restriction (a_{p_n} \setminus a_{p_{i_m}})) \restriction d_{n+1}$$

is cofinal in $\mathcal{I}$. By the choice of d_{n+1} , clearly for all $a \in N \cap \mathcal{I}$, $a \subseteq^* d_{n+1}$. Thirdly, apply Lemma 6.7 to fix $p_{n+1} \leq p_n$ in D_m such that $a_{p_{n+1}} \setminus a_{p_n} \subseteq d_{n+1}$.

Suppose that $i = 1$. Consider the pair $(x_m, \dot{A}_m)$. Then x_m is (N, T^*) -generic by Lemma 1.5. There are two possibilities. If there exists $y <_{T^*} x_m$ and $v \leq p_n$ in $N \cap \mathbb{P}$ such that $a_v \setminus a_{p_n} \subseteq d_n$ and v forces that $y \in \dot{A}_m$, then let $F(n)$ be equal such a y and let p_{n+1} be such a v . Also, define $d_{n+1} := d_n$.

If not, then by Lemma 6.9 there exists a cofinal set $J \subseteq \mathcal{I}$ such that $(a_{p_n}, X_{p_n} \cup \{J\})$ forces that $x_m \notin \dot{A}_m$. Define $F(n)$ to be such a J . (For clarity, J is not necessarily in N .) Now apply Lemma 6.6 to find $d_{n+1} \subseteq d_n$ which is equal to d_n minus finitely many elements such that $F(n) \restriction d_{n+1}$ is cofinal. Also, let $p_{n+1} := p_n$.

Suppose that $i = 2$. This case is exactly the same as the case $i = 1$, except replacing the relation $<_{T^*}$ with $<_{T^*}^-$.

This completes the construction. Let $X := \bigcup \{X_{p_n} : n < \omega\}$. Define $q = (a_q, X_q)$ by $a_q := \bigcup \{a_{p_n} : n < \omega\}$ and

$$X_q := X \cup \{K \restriction (a_q \setminus a_{p_i}) : i < \omega, K \in X_{p_i}\}.$$

We claim that q is a condition below each p_n .

Observe that by construction, for all $n < \omega$, $a_q \setminus a_{p_n} \subseteq d_{n+1}$. In particular, $a_q = a_{p_0} \cup (a_q \setminus a_{p_0}) \subseteq a_{p_0} \cup d_1$, which implies that a_q is in $\mathcal{I}$.

Clearly X is a countable collection of cofinal subsets of $\mathcal{I}$. Consider $i < \omega$ and $K \in X_{p_i}$, and we will show that $K \restriction (a_q \setminus a_{p_i})$ is cofinal. Fix m such that $(i_m, K_m) = (i, K)$. Let $n \geq i$ be such that $f(n) = (0, m)$. Then $i_m = i \leq n$ and $K_m = K \in X_{p_{i_m}}$. By construction,

$$(K_m \restriction (a_{p_n} \setminus a_{p_{i_m}})) \restriction d_{n+1}$$

is cofinal in $\mathcal{I}$. But we claim that $K \restriction (a_q \setminus a_{p_i})$ contains this cofinal set, and hence is itself cofinal. To see this, assume that $a \in K_m$ contains $a_{p_n} \setminus a_{p_{i_m}}$ and d_{n+1} . Then $a \in K_m = K$. We need to show that $a_q \setminus a_{p_i} \subseteq a$. Since $a_{p_i} \subseteq a_{p_n} \subseteq a_q$, we have that

$$a_q \setminus a_{p_i} = (a_q \setminus a_{p_n}) \cup (a_{p_n} \setminus a_{p_i}).$$

And both of the sets in this union are subsets of a , by the choice of a and since $a_q \setminus a_{p_n} \subseteq d_{n+1}$.

Thus, q is a condition, and easily by definition, $q \leq p_n$ for all $n < \omega$. By construction, for any dense open subset D of $\mathbb{P}$ in N , $q \in D$. So q is $(N, \mathbb{P})$ -generic.

In order to find an $(N, \mathbb{P}, T^*)$ -generic condition, we need to extend q further. But before we do this, let us prove a claim. Consider $n \in \text{dom}(F)$ such that $F(n) \subseteq \mathcal{I}$ is cofinal. We claim that $F := F(n) \restriction (a_q \setminus a_{p_n})$ is cofinal. Note that by construction, we are in the case that $f(n) = (i, m)$, where $i \in \{1, 2\}$, $(a_{p_n}, X_{p_n} \cup \{F(n)\})$ forces that $x_m \notin \dot{A}_m$, and $F(n) \restriction d_{n+1}$ is cofinal. But since $a_q \setminus a_{p_n} \subseteq d_{n+1}$, we have that

$$F(n) \restriction d_{n+1} \subseteq F(n) \restriction (a_q \setminus a_{p_n}) = F.$$

Thus, F contains a cofinal set, and hence is itself cofinal.

Define

$$X_1 := \{F(n) : n \in \text{dom}(F), F(n) \subseteq \mathcal{I} \text{ is cofinal}\}$$

and

$$X_2 := \{F(n) \upharpoonright (a_q \setminus a_{p_n}) : n \in \text{dom}(F), F(n) \subseteq \mathcal{I} \text{ is cofinal}\}.$$

Note that by the claim of the previous paragraph, every member of X_2 is cofinal. Define

$$r := (a_q, X_q \cup X_1 \cup X_2).$$

Then clearly $r \in \mathbb{P}$ and $r \leq q$. In particular, r is $(N, \mathbb{P})$ -generic.

It remains to show that r is $(N, \mathbb{P}, T^*)$ -generic. So let $x \in T_\delta^*$, $\dot{A} \in N$ a $\mathbb{P}$ -name for a subset of T^* , and $R \in \{<_{T^*}, <_{T^*}^-\}$. Let i be 1 if R equals $<_{T^*}$ and 2 if R equals $<_{T^*}^-$. Fix m such that $(x_m, \dot{A}_m) = (x, \dot{A})$. Fix $n < \omega$ such that $f(n) = (i, m)$.

Recall that at stage n we considered two possibilities. The first is that there exists $y R x_m$ and $v \leq p_n$ in $N \cap \mathbb{P}$ such that v forces that $y \in \dot{A}_m$ and $a_v \setminus a_{p_n} \subseteq d_n$. In that case, $F(n) R x_m$ and p_{n+1} forces that $F(n) \in \dot{A}_m$. Thus, $F(n) R x$ and r forces that $F(n) \in \dot{A}$, and we are done.

The second possibility is that there exists a cofinal set, which we defined as $F(n)$, such that $(a_{p_n}, X_{p_n} \cup \{F(n)\})$ forces that $x_m \notin \dot{A}_m$. By the definition of r , $F(n) \in X_r$ and $F(n) \upharpoonright (a_r \setminus a_{p_n}) = F(n) \upharpoonright (a_q \setminus a_{p_n}) \in X_r$. It follows that $r \leq (a_{p_n}, X_{p_n} \cup \{F(n)\})$. Thus, r forces that $x \notin \dot{A}$, which concludes the proof. $\square$

The next theorem now follows by a straightforward argument.

Theorem 6.11. *PFA(T^*) implies the P -ideal dichotomy.*

Corollary 6.12. *PFA(T^*) implies that for all regular cardinals $\lambda > \omega_1$, $\square(\lambda)$ fails. In particular, it implies that for all cardinals $\kappa \geq \omega_1$, $\square_\kappa$ fails.*

Corollary 6.13. *PFA(T^*) implies the singular cardinal hypothesis.*

Corollary 6.14. *PFA(T^*) implies that for all regular uncountable cardinals κ and λ , if there exists a (κ, λ^*) -gap in $P(\omega)$, then $\kappa = \lambda = \omega_1$.*

These statements are immediate, since the conclusions of the corollaries are consequences of the P -ideal dichotomy ([16], [18]).

It is well-known that the P -ideal dichotomy combined with $\mathfrak{p} > \omega_1$ has many consequences which do not follow from the P -ideal dichotomy alone. Since PFA(T^*) implies $\mathfrak{p} > \omega_1$ by Corollary 4.6, we can derive additional conclusions. We mention two important examples (see [14]).

Corollary 6.15. *PFA(T^*) implies that for all $\alpha < \omega_1$, $\omega_1 \rightarrow (\omega_1, \alpha)^2$.*

Corollary 6.16. *PFA(T^*) implies the non-existence of S -spaces.*

In contrast, the existence of a Suslin tree implies the existence of a compact L -space and an S -space ([8], [15, Chapter 5]). This displays contrasting pictures provided by the forcing axioms PFA(T^*) and PFA(S).

Observe that our results prove that the P -ideal dichotomy together with $\mathfrak{p} > \omega_1$ does not imply that all ω_1 -Aronszajn trees are special (or even that all ω_1 -Aronszajn trees have a stationary antichain). A similar but more restrictive result was proven earlier in [7], where it was shown that an ideal dichotomy for ω_1 -generated ideals together with $\mathfrak{p} > \omega_1$ is consistent with a non-special Aronszajn tree.

We close the paper with some topics for future research. We currently do not know whether some other notable consequences of PFA not addressed in this paper

follow from $\text{PFA}(T^*)$. These include the open coloring axiom, the mapping reflection principle, measuring, and the non-existence of weak Kurepa trees.

An important part of Todorćević's theory of $\text{PFA}(S)$ is the model $\text{PFA}(S)[S]$, which is obtained by forcing over a model of $\text{PFA}(S)$ with the coherent Suslin tree S . In contrast, $\text{PFA}(T^*)$ implies that every ω_1 -Aronszajn tree is special on cofinally many levels (see the argument from [10, Lemma 4.6, Chapter IX]). It follows that forcing with T^* collapses ω_1 . We do not know, however, whether generic extensions of models of $\text{PFA}(T^*)$ by other forcings associated with T^* could be interesting, such as the forcing for specializing T^* with finite approximations.

The theory of $\text{PFA}(T^*)$ could possibly be relevant to the issues of chain conditions and Martin's axiom which motivated $\text{PFA}(S)$. For example, it is open whether Martin's axiom follows from the statement that all ω_1 -c.c. forcings are Knaster, or from the statement that the product of ω_1 -c.c. forcings is ω_1 -c.c.

Finally, we mention a problem suggested to the author by Justin Moore and the referee: Does $\text{PFA}(T^*)$ imply that all normal Aronszajn trees with no stationary antichains are club isomorphic? This would imply that the forcing axiom $\text{PFA}(T^*)$ is independent of the choice of T^* and thus to some extent canonical.

REFERENCES

- [1] J. Baumgartner. Applications of the proper forcing axiom. In *Handbook of set-theoretic topology*, pages 913–959. North-Holland, Amsterdam, 1984.
- [2] J. Baumgartner, J. Malitz, and W. Reinhardt. Embedding trees in the rationals. *Proc. Nat. Acad. Sci.*, 67(4):1748–1753, 1970.
- [3] M. Bell. On the combinatorial principle $P(\mathfrak{c})$. *Fund. Math.*, 114(2):149–157, 1981.
- [4] K. Devlin and S. Shelah. A weak version of $\diamond$ which follows from $2^{\aleph_0} < 2^{\aleph_1}$. *Israel J. Math.*, 29(2-3):239–247, 1978.
- [5] P. Larson. An $\mathfrak{S}_{\max}$ -variation for one Souslin tree. *J. Symbolic Logic*, 64(1):81–98, 1999.
- [6] P. Larson and S. Todorćević. Katětov's problem. *Trans. Amer. Math. Soc.*, 354:1783–1791, 2002.
- [7] H. Mildenberger and L. Zdomskyy. L-spaces and the P-ideal dichotomy. *Acta Math. Hungar.*, 125(1-2):85–97, 2009.
- [8] M.E. Rudin. A normal hereditarily separable non-Lindelöf space. *Illinois J. Math.*, 16:621–626, 1972.
- [9] C. Schlindwein. Suslin's hypothesis does not imply stationary antichains. *Ann. Pure Appl. Logic*, 64(2):153–167, 1993.
- [10] S. Shelah. *Proper and Improper Forcing*. Perspectives in Mathematical Logic. Springer-Verlag, Berlin, second edition, 1998.
- [11] R. M. Solovay and S. Tennenbaum. Iterated Cohen extensions and Souslin's problem. *Ann. of Math.*, 94:201–245, 1971.
- [12] D. Talayco. Applications of cohomology to set theory. I. Hausdorff gaps. *Ann. Pure Appl. Logic*, 71(1):69–106, 1995.
- [13] S. Todorćević. Forcing with a coherent Souslin tree. Preprint.
- [14] S. Todorćević. Forcing positive partition relations. *Trans. Amer. Math. Soc.*, 280(2):703–720, 1983.
- [15] S. Todorćević. *Partition problems in topology*, volume 84 of *Contemporary Mathematics*. American Mathematical Society, Providence, RI, 1989.
- [16] S. Todorćević. A dichotomy for P-ideals of countable sets. *Fund. Math.*, 166(3):251–267, 2000.
- [17] S. Todorćević and B. Velickovic. Martin's axiom and partitions. *Compositio Math.*, 63(3):391–408, 1987.
- [18] M. Viale. A family of covering properties. *Math. Res. Lett.*, 15(2):221–238, 2008.
- [19] W.H. Woodin. *The axiom of determinacy, forcing axioms, and the nonstationary ideal. Second revised edition*. De Gruyter Series in Logic and its Applications. Walter de Gruyter GmbH & Co. KG, Berlin, Germany, 2010.

JOHN KRUEGER, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF NORTH TEXAS, 1155 UNION
CIRCLE #311430, DENTON, TX 76203
E-mail address: `jkruenger@unt.edu`