

GUESSING MODELS IMPLY THE SINGULAR CARDINAL HYPOTHESIS

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ABSTRACT. In this article we prove three main theorems: (1) guessing models are internally unbounded, (2) for any regular cardinal $\kappa \geq \omega_2$, $\text{ISP}(\kappa)$ implies that SCH holds above κ , and (3) forcing posets which have the ω_1 -approximation property also have the countable covering property. These results solve open problems of Viale [6] and Hachtman and Sinapova [2].

A major result in recent years on the consequences of forcing axioms is the theorem of M. Viale that the Proper Forcing Axiom (PFA) implies the Singular Cardinal Hypothesis (SCH). In fact, Viale showed that several strong combinatorial consequences of PFA, including the Mapping Reflection Principle (MRP) and the P -Ideal Dichotomy (PID), each imply SCH ([4], [5]).

C. Weiss [8] introduced a combinatorial principle $\text{ISP}(\kappa)$, for any regular cardinal $\kappa \geq \omega_2$, which is equivalent to κ being supercompact in the case that κ is inaccessible, but is also consistent when κ is a small successor cardinal. In particular, $\text{ISP}(\omega_2)$ (abbreviated henceforth as ISP) is a consequence of PFA, and it in turn implies many of the strong consequences of PFA, such as the failure of square principles. Later Viale and Weiss [7] provided an alternative characterization of ISP in terms of the existence of stationarily many elementary substructures which have a “guessing” property reminiscent of the approximation property in forcing theory.

In light of these developments, a natural question is whether ISP implies SCH . Viale [6] made partial progress on this question by showing that SCH follows from an apparently stronger form of ISP , namely, the existence of stationarily many guessing models which are also internally unbounded. This result raises a number of additional questions, such as whether guessing models alone imply SCH , whether guessing models are always internally unbounded, and whether the ω_1 -approximation property of forcing posets implies the countable covering property. In this article we refine the results of Viale and Weiss described above and answer all of these questions in the affirmative.

1. GUESSING AND COVERING

For the remainder of the article, N will usually denote an elementary substructure of $H(\theta)$ for some regular cardinal $\theta \geq \omega_2$, although we will not strictly require this for many of the definitions.

Date: March 2019.

2010 Mathematics Subject Classification: Primary 03E05; Secondary 03E40.

Key words and phrases. ISP , SCH , guessing model, internally unbounded, approximation property, countable covering property.

This material is based upon work supported by the National Science Foundation under Grant No. DMS-1464859.

For a set or class M , a set $x \subseteq M$ is said to be *bounded in M* if there exists $Y \in M$ such that $x \subseteq Y$.

Definition 1.1. A set N is said to be *guessing* if for any set $x \subseteq N$ which is bounded in N , if for all $a \in N \cap [N]^\omega$, $x \cap a \in N$, then there exists $E \in N$ such that $x = N \cap E$.

Definition 1.2. For any regular cardinal $\kappa \geq \omega_2$, let $ISP(\kappa)$ be the statement that for any regular cardinal $\theta \geq \kappa$, the collection of guessing sets is stationary in $P_\kappa(H(\theta))$. Let ISP be the statement $ISP(\omega_2)$.

Being *stationary* in $P_\kappa(H(\theta)) = \{a \subseteq H(\theta) : |a| < \kappa\}$ means meeting every club, where a club is any cofinal subset of $P_\kappa(H(\theta))$ closed under unions of $\subseteq$ -increasing sequences of length less than κ . The collection of all sets N such that $N \cap \kappa \in \kappa$ is club in $P_\kappa(H(\theta))$, so $ISP(\kappa)$ implies stationarily many guessing models N such that $N \cap \kappa \in \kappa$.

It is easy to prove from the definition that if N is an elementary substructure which is guessing, then for any regular uncountable cardinal $\kappa \in N$, $\sup(N \cap \kappa)$ has uncountable cofinality.

Definition 1.3. A set N is said to be *internally unbounded* if for any countable set $x \subseteq N$ which is bounded in N , there exists $y \in N \cap [N]^\omega$ such that $x \subseteq y$.

Recall that N has *countable covering* if any countable subset of N is covered by a countable set in N . Obviously, if $\sup(N \cap \omega_n)$ has cofinality ω , then N does not have this property, but under some typical assumptions, if $\sup(N \cap \omega_n)$ has uncountable cofinality then countable covering is equivalent to being internally unbounded.

Viale ([6, Remark 4.3]) asked whether it is consistent to have a guessing model which is not internally unbounded. In [1, Section 4] we showed that PFA implies the existence of stationarily many elementary substructures N of $H(\omega_2)$ of size ω_1 such that N is guessing but $\sup(N \cap \omega_2) = \omega$. Such models do not have countable covering, but they are internally unbounded according to Definition 1.3. This result solved an easy special case of Viale's question, but the next theorem provides the complete solution.

Theorem 1.4. Let $\theta \geq \omega_2$ be a regular cardinal, and suppose that N is an elementary substructure of $H(\theta)$ such that $\omega_1 \subseteq N$. If N is guessing, then N is internally unbounded.

Proof. Let $x \subseteq N$ be countably infinite and bounded in N . Fix a set $Y \in N$ such that $x \subseteq Y$. Our goal is to find a countable set y in N such that $x \subseteq y$. Observe that by elementarity, the set $[Y]^{<\omega}$ is a member of N . Fix a bijection $g : \omega \rightarrow x$, and for each n let $x_n := g[n]$. Then $x_m \subseteq x_n$ for all $m < n$, $\bigcup_n x_n = x$, and $\{x_n : n < \omega\} \subseteq [Y]^{<\omega}$.

We consider two possibilities. The first is that there exists $\mathcal{X} \in N \cap [N]^\omega$ such that

$$|\mathcal{X} \cap \{x_n : n < \omega\}| = \omega.$$

By intersecting $\mathcal{X}$ with $[Y]^{<\omega}$ if necessary, we may assume without loss of generality that $\mathcal{X} \subseteq [Y]^{<\omega}$. Since $\mathcal{X}$ is countable and its elements are finite, $y := \bigcup \mathcal{X}$ is a countable subset of Y . Also, $y \in N$ by elementarity.

We claim that $x \subseteq y$, which completes the proof in this case. Consider $a \in x$. Fix m such that $a \in x_m$. Since $\mathcal{X} \cap \{x_n : n < \omega\}$ is infinite, we can fix $n > m$ such that $x_n \in \mathcal{X}$. Then $a \in x_m \subseteq x_n \subseteq y$, so $a \in y$.

The second possibility is that for all $\mathcal{X} \in N \cap [N]^\omega$, $\mathcal{X} \cap \{x_n : n < \omega\}$ is finite. Since N is closed under finite subsets, for all such $\mathcal{X}$, $\mathcal{X} \cap \{x_n : n < \omega\}$ is a member of N . In this case we will show that x itself is a member of N , which completes the proof. Since N is guessing, we can fix $E \in N$ such that $\{x_n : n < \omega\} = N \cap E$.

Observe that E is countable. Otherwise there would exist an injection of ω_1 into E in N by elementarity. Since $\omega_1 \subseteq N$, it would follow that $N \cap E$ is uncountable. This is impossible since $N \cap E = \{x_n : n < \omega\}$, which is countable. As E is countable, $E \subseteq N$ by elementarity. So $\{x_n : n < \omega\} = N \cap E = E$. Therefore, the set $\{x_n : n < \omega\}$ is a member of N . Thus, $x = \bigcup \{x_n : n < \omega\}$ is a member of N . $\square$

Corollary 1.5. *Let $\kappa \geq \omega_2$ be a regular cardinal. Then $\text{ISP}(\kappa)$ implies that for all regular cardinals $\theta \geq \kappa$, there are stationarily many $N \in P_\kappa(H(\theta))$ such that N is guessing and internally unbounded.*

Proof. We already know that $\text{ISP}(\kappa)$ implies the existence of stationarily many $N \in P_\kappa(H(\theta))$ such that N is guessing and $N \cap \kappa \in \kappa$. By definability, $\omega_1 \in N \cap \kappa$, and it follows that $\omega_1 \subseteq N$. By Theorem 1.4, N is internally unbounded. $\square$

Viale [6, Section 7.2] proved that the existence of stationarily many internally unbounded guessing models implies SCH, but it was unknown whether guessing models alone imply SCH. This problem also appears in [2, Section 1]. By Corollary 1.5 together with Viale's result, ISP does indeed imply SCH.¹

Corollary 1.6. *ISP implies SCH.*

2. ISP AND SCH

In the previous section we showed that guessing models are internally unbounded, which combined with Viale's argument [6, Section 7.2] proves that ISP implies SCH. S. Hachtman and D. Sinapova [2] asked a more general question, which is whether for a regular cardinal $\kappa \geq \omega_2$, $\text{ISP}(\kappa)$ implies SCH above κ . In this section we solve this problem in the affirmative. We note that our proof avoids the idea of internally unbounded models entirely.

We will in fact prove something a bit stronger.

Theorem 2.1. *Let $\kappa \geq \omega_2$ be regular and assume that $\text{ISP}(\kappa)$ holds. Then either κ is supercompact, or SCH holds.*

Proposition 2.2. *Let $\kappa \geq \omega_2$ be regular and assume that $\text{ISP}(\kappa)$ holds. If $2^\omega < \kappa$, then κ is supercompact. Hence, SCH holds above κ .*

Proof. If κ is strongly inaccessible and $\text{ISP}(\kappa)$ holds, then κ is supercompact by [8, Theorem 2.10]. And if κ is supercompact, then SCH holds above κ by a well-known result of Solovay ([3, Theorem 20.8]). So it suffices to show that κ is strongly inaccessible.

¹After announcing the results of this paper, we learned that S. Hachtman had recently and independently proven that ISP implies SCH using essentially the same argument as presented in this section.

Let $\mu < \kappa$ be a cardinal and we will show that $|P(\mu)| < \kappa$. Using $\text{ISP}(\kappa)$, we can fix an elementary substructure N of $H(\kappa)$ of size less than κ such that $N \cap \kappa \in \kappa$, $N \cap \kappa$ is larger than 2^ω and μ , and N is guessing. It suffices to show that $P(\mu) \subseteq N$.

Let $x \subseteq \mu$. Then x is a subset of N which is bounded in N . Consider $a \in N \cap [N]^\omega$. Since $2^\omega < N \cap \kappa$, $P(a) \subseteq N$. In particular, $a \cap x \in N$. As N is guessing, it follows that there exists $E \in N$ such that $x = N \cap E$. By intersecting E with μ if necessary, we may assume without loss of generality that $E \subseteq \mu$. Since μ is a subset of N , so is E , and hence $x = N \cap E = E$. Thus, $x \in N$, as desired. $\square$

See [2, Theorem 2.1] for a similar argument.

Fix a regular cardinal $\kappa \geq \omega_2$ for the remainder of the section, and assume that $\text{ISP}(\kappa)$ holds. If $2^\omega < \kappa$, then κ is supercompact, and we are done. Assume that $2^\omega \geq \kappa$. We will show that SCH holds.

By a well-known theorem of Silver, the first cardinal for which SCH fails, if it exists, has cofinality ω ([3, Theorem 8.13]). Let λ be a singular cardinal of cofinality ω , and assume that SCH holds below λ . If SCH fails at λ , that means that $2^\omega < \lambda$ and $\lambda^\omega > \lambda^+$. Now $2^\omega \geq \kappa$, so $\lambda > \kappa$. Since SCH holds below λ , an easy inductive argument shows that for all cardinals $\mu < \lambda$, $\mu^\omega < \lambda$ ([3, Theorem 5.20]).

Putting it all together, assuming $\text{ISP}(\kappa)$ and $2^\omega \geq \kappa$, SCH follows from the statement: for all cardinals $\lambda > \kappa$ of cofinality ω , if $\mu^\omega < \lambda$ for all $\mu < \lambda$, then $\lambda^\omega = \lambda^+$. Our proof of this statement follows along the lines of Viale's proof [6, Section 7.2], but avoids consideration of internal unboundedness.

Lemma 2.3 ([5, Lemma 6]). *Let $\lambda > 2^\omega$ be a cardinal with cofinality ω . Then there exists a matrix*

$$\langle K(n, \beta) : n < \omega, \beta < \lambda^+ \rangle$$

of sets of size less than λ satisfying:

- (1) *for all $\beta < \lambda^+$, $\beta = \bigcup \{K(n, \beta) : n < \omega\}$;*
- (2) *for all $\beta < \lambda^+$ and $m < n < \omega$, $K(m, \beta) \subseteq K(n, \beta)$;*
- (3) *for all $\gamma < \beta < \lambda^+$ there exists $m < \omega$ such that for all $m \leq n < \omega$, $K(n, \gamma) \subseteq K(n, \beta)$;*
- (4) *for all $x \in [\lambda^+]^\omega$ there exists $\gamma < \lambda^+$ such that for all $\gamma < \beta < \lambda^+$, there exists $m < \omega$ such that for all $m \leq n < \omega$, $K(n, \beta) \cap x = K(n, \gamma) \cap x$.*

Proof. Fix an increasing sequence of uncountable cardinals $\langle \lambda_n : n < \omega \rangle$ cofinal in λ . By a straightforward argument, it is possible to fix, for each $\beta < \lambda^+$, a surjection $g_\beta : \lambda \rightarrow \beta$ satisfying that for all $\gamma < \beta$ there exists m such that for all $n \geq m$, $g_\gamma[\lambda_n] \subseteq g_\beta[\lambda_n]$.

Define $K(n, \emptyset) := \emptyset$ for all $n < \omega$. Now fix $\beta < \lambda^+$ and assume that $K(n, \gamma)$ is defined for all $n < \omega$ and $\gamma < \beta$. Define for each $n < \omega$

$$K(n, \beta) := g_\beta[\lambda_n] \cup \bigcup \{K(n, \gamma) : \gamma \in g_\beta[\lambda_n]\}.$$

This completes the definition. It is easy to prove by induction that (1), (2), and (3) hold, and each $K(n, \beta)$ has size at most λ_n .

For (4), fix $x \in [\lambda^+]^\omega$. For each $\beta < \lambda^+$, define a function $f_\beta : \omega \rightarrow P(x)$ by $f_\beta(n) := K(n, \beta) \cap x$. Observe that there are 2^ω many possibilities for such a function f_β . Since $2^\omega < \lambda$, we can fix a set $S \subseteq \lambda^+$ of size λ^+ and a function f such that for all $\beta \in S$, $f_\beta = f$. Let $\gamma := \min(S)$.

To verify that (4) holds for x , consider $\beta > \gamma$. Let $\xi := \min(S \setminus \beta)$. Using (3), fix m such that for all $n \geq m$,

$$K(n, \gamma) \subseteq K(n, \beta) \subseteq K(n, \xi).$$

In particular, $K(n, \gamma) \cap x \subseteq K(n, \beta) \cap x$. For the reverse inclusion,

$$K(n, \beta) \cap x \subseteq K(n, \xi) \cap x = f_\xi(n) = f(n) = f_\gamma(n) = K(n, \gamma) \cap x.$$

□

Lemma 2.4 ([5, Fact 9]). *Let $\lambda > 2^\omega$ be a singular cardinal with cofinality ω such that for all cardinals $\mu < \lambda$, $\mu^\omega < \lambda$. Fix $\langle K(n, \beta) : n < \omega, \beta < \lambda^+ \rangle$ as described in Lemma 2.3. Assume that there exists a set $S \subseteq \lambda^+$ of size λ^+ such that for all $x \in [S]^\omega$, there exists $n < \omega$ and $\beta < \lambda^+$ such that $x \subseteq K(n, \beta)$. Then $\lambda^\omega = \lambda^+$.*

Proof. Since S has size λ^+ , the cardinality of $[S]^\omega$ is equal to $(\lambda^+)^\omega$, which in turn equals λ^ω . So it suffices to show that $[S]^\omega$ has cardinality λ^+ . By assumption, every member of $[S]^\omega$ is a subset of $K(n, \beta)$ for some $n < \omega$ and $\beta < \lambda^+$. Thus,

$$[S]^\omega \subseteq \bigcup \{ [K(n, \beta)]^\omega : n < \omega, \beta < \lambda^+ \}.$$

Now each $K(n, \beta)$ has cardinality less than λ , so by our assumptions, $[K(n, \beta)]^\omega$ has cardinality less than λ . Thus, the union in the above inclusion has cardinality λ^+ . □

Assume that $\text{ISP}(\kappa)$ holds, and let $\lambda > \kappa$ be a singular cardinal of cofinality ω such that for all $\mu < \lambda$, $\mu^\omega < \lambda$. We will prove that $\lambda^\omega = \lambda^+$.

Fix $\mathcal{K} = \langle K(n, \beta) : n < \omega, \beta < \lambda^+ \rangle$ as described in Lemma 2.3. In order to show that $\lambda^\omega = \lambda^+$, by Lemma 2.4 it suffices to show that there exists a set $S \subseteq \lambda^+$ of size λ^+ such that for all $x \in [S]^\omega$, there exists $n < \omega$ and $\beta < \lambda^+$ such that $x \subseteq K(n, \beta)$.

Using $\text{ISP}(\kappa)$, fix an elementary substructure N of $H(\lambda^{++})$ of size less than κ such that $N \cap \kappa = \kappa$, $\mathcal{K} \in N$, and N is guessing. For each $x \in [\lambda^+]^\omega$, let $\gamma_x < \lambda^+$ be the minimal ordinal satisfying that for all $\gamma_x < \beta < \lambda^+$, there exists n such that for all $m \geq n$, $K(m, \beta) \cap x = K(m, \gamma_x) \cap x$. Observe that $\langle \gamma_x : x \in [\lambda^+]^\omega \rangle$ is a member of N by elementarity.

Consider $x \in N \cap [\lambda^+]^\omega$. Then $\gamma_x \in N \cap \lambda^+$. So there exists n such that for all $m \geq n$,

$$K(m, \sup(N \cap \lambda^+)) \cap x = K(m, \gamma_x) \cap x.$$

Since x , γ_x , and $\mathcal{K}$ are in N , $K(m, \gamma_x) \cap x$ is a member of N . Therefore,

$$K(m, \sup(N \cap \lambda^+)) \cap x \in N.$$

Now for each $x \in N \cap [\lambda^+]^\omega$, fix the smallest integer k_x satisfying that for all $m \geq k_x$, $K(m, \sup(N \cap \lambda^+)) \cap x$ is in N .

We claim that if x and y are in $N \cap [\lambda^+]^\omega$ and $x \subseteq y$, then $k_x \leq k_y$. By the minimality of k_x , it suffices to show that for all $m \geq k_y$, $K(m, \sup(N \cap \lambda^+)) \cap x \in N$. Let $m \geq k_y$. Then $K(m, \sup(N \cap \lambda^+)) \cap y \in N$. Since x is in N and $x \subseteq y$, we have that $K(m, \sup(N \cap \lambda^+)) \cap x = (K(m, \sup(N \cap \lambda^+)) \cap y) \cap x$ is in N .

Next, we claim that the collection of integers

$$A := \{k_x : x \in N \cap [\lambda^+]^\omega\}$$

is finite. Suppose for a contradiction that A is infinite. For each $n \in A$, fix $x_n \in N \cap [\lambda^+]^\omega$ such that $n = k_{x_n}$. Now define, for each $n \in A$, $y_n := \bigcup \{x_k :$

$k \in A \cap (n+1)\}$, which is in $N \cap [\lambda^+]^\omega$. Observe that if $m < n$ are in A , then $y_m \subseteq y_n$. Also, for each $n \in A$, $x_n \subseteq y_n$, and therefore by the previous paragraph, $n = k_{x_n} \leq k_{y_n}$. By thinning out the sequence $\langle y_n : n \in A \rangle$ if necessary, it is easy to find a sequence $\langle z_n : n < \omega \rangle$ of distinct sets in $N \cap [\lambda^+]^\omega$ satisfying that for all $m < n$, $z_m \subseteq z_n$ and $k_{z_m} < k_{z_n}$.

We now consider two possibilities, both of which will lead to a contradiction. First, assume that there exists a countable set $\mathcal{X} \in N$ such that

$$|\mathcal{X} \cap \{z_n : n < \omega\}| = \omega.$$

By intersecting $\mathcal{X}$ with $[\lambda^+]^\omega$ if necessary, we may assume without loss of generality that $\mathcal{X} \subseteq [\lambda^+]^\omega$. Since $\mathcal{X}$ is countable and consists of countable sets, $x^* := \bigcup \mathcal{X}$ is in $N \cap [\lambda^+]^\omega$. We claim that for all $m < \omega$, $z_m \subseteq x^*$. Indeed, given m , we can find $n \geq m$ such that $z_n \in \mathcal{X}$. Then $z_m \subseteq z_n \subseteq \bigcup \mathcal{X} = x^*$. Now for all $n < \omega$, $z_n \subseteq x^*$ implies that $k_{z_n} \leq k_{x^*}$. This is impossible, since $\{k_{z_n} : n < \omega\}$ is unbounded in ω , whereas $k_{x^*} < \omega$.

Secondly, assume that for all countable sets $\mathcal{X} \in N$, $\mathcal{X} \cap \{z_n : n < \omega\}$ is finite. Then in particular, for all countable sets $\mathcal{X} \in N$, $\mathcal{X} \cap \{z_n : n < \omega\}$ is a member of N . Also note that this assumption implies that $\{z_n : n < \omega\}$ is not in N , for otherwise we could let $\mathcal{X}$ be equal to it and get a contradiction. Since N is guessing, it follows that there exists $E \in N$ such that $\{z_n : n < \omega\} = N \cap E$. In particular, $N \cap E$ is countable. Since $\omega_1 \subseteq N$, this implies that E is countable, for otherwise by elementarity $N \cap E$ would be uncountable. Therefore, $E \subseteq N$. So $\{z_n : n < \omega\} = N \cap E = E$, and hence $\{z_n : n < \omega\}$ is a member of N , which is a contradiction.

This concludes the proof that the set $A = \{k_x : x \in N \cap [\lambda^+]^\omega\}$ is finite. Let n^* be the largest member of A . Then for all $x \in N \cap [\lambda^+]^\omega$, $k_x \leq n^*$ implies that for all $m \geq n^*$, $K(m, \sup(N \cap \lambda^+)) \cap x \in N$. It easily follows that for all $m \geq n^*$, for any countable set $Y \in N$, $K(m, \sup(N \cap \lambda^+)) \cap Y \in N$. Since N is guessing, for all $m \geq n^*$ there exists a set $E_m \in N$ such that $N \cap K(m, \sup(N \cap \lambda^+)) = N \cap E_m$. By intersecting E_m with λ^+ if necessary, we may assume without loss of generality that $E_m \subseteq \lambda^+$.

Since $\sup(N \cap \lambda^+)$ is equal to $\bigcup \{K(m, \sup(N \cap \lambda^+)) : m < \omega\}$, we have that

$$N \cap \sup(N \cap \lambda^+) = \bigcup \{N \cap K(m, \sup(N \cap \lambda^+)) : m < \omega\}.$$

As $\text{cf}(\sup(N \cap \lambda^+))$ is uncountable, there exists $m \geq n^*$ such that $N \cap K(m, \sup(N \cap \lambda^+)) = N \cap E_m$ is unbounded in $\sup(N \cap \lambda^+)$. By elementarity, it easily follows that the set $S := E_m$ is unbounded in λ^+ . To complete the proof, it suffices to show that for all $x \in [S]^\omega$, there exists $n < \omega$ and $\beta < \lambda^+$ such that $x \subseteq K(n, \beta)$. Since $S \in N$, by elementarity it suffices to show that for all $x \in N \cap [S]^\omega$, there exists $n < \omega$ and $\beta < \lambda^+$ such that $x \subseteq K(n, \beta)$. Fix $x \in N \cap [S]^\omega$. Then $x \subseteq N \cap S = N \cap E_m = N \cap K(m, \sup(N \cap \lambda^+))$. By elementarity, there exists $\beta \in N \cap \lambda^+$ such that $x \subseteq K(m, \beta)$.

3. APPROXIMATION AND COVERING

In Section 1 we saw that guessing implies internally unbounded for elementary substructures. In this section we provide analogous results concerning the approximation property implying the covering property, for models and forcing posets.

Definition 3.1. Let κ be a regular uncountable cardinal. Let $W_1 \subseteq W_2$ be transitive (sets or classes) with $\kappa \in W_1$.

- (1) The pair (W_1, W_2) is said to have the κ -approximation property provided that whenever $X \in W_2$ is a bounded subset of W_1 , if $X \cap y \in W_1$ for any set $y \in W_1$ such that $W_1 \models |y| < \kappa$, then $X \in W_1$;
- (2) The pair (W_1, W_2) is said to have the κ -covering property if whenever $X \in W_2$ is a bounded subset of W_1 , if $W_2 \models |X| < \kappa$, then there exists $Y \in W_1$ such that $W_1 \models |Y| < \kappa$ and $X \subseteq Y$.

Definition 3.2. Let κ be a regular uncountable cardinal and $\mathbb{P}$ a forcing poset. We say that $\mathbb{P}$ has the κ -approximation property if $\mathbb{P}$ forces that $(V, V^\mathbb{P})$ has the κ -approximation property, and has the κ -covering property if $\mathbb{P}$ forces that $(V, V^\mathbb{P})$ has the κ -covering property.

Theorem 3.3. Let κ be a regular uncountable cardinal and $W_1 \subseteq W_2$ be transitive models of ZFC minus power set such that $\kappa \in W_1$. Assume that for all W_2 -cardinals $\mu < \kappa$, any subset of W_1 which is a member of W_2 and has W_2 -cardinality less than μ is a member of W_1 . If (W_1, W_2) has the κ -approximation property, then it has the κ -covering property.

Proof. Let $x \in W_2$ satisfy that $W_2 \models |x| < \kappa$ and $x \subseteq Y$ for some $Y \in W_1$. We will prove that x is covered by some set in W_1 which has W_1 -cardinality less than κ . Define $\mu := |x|^{W_2}$. Since x has cardinality μ in W_2 , fix a bijection $g : \mu \rightarrow x$ in W_2 , and define for each $i < \mu$ $x_i := g[i]$. Then the sequence $\langle x_i : i < \mu \rangle$ is in W_2 , is $\subseteq$ -increasing, and has union equal to x . Moreover, each x_i has size less than μ in W_2 , hence is in W_1 by our assumptions, and has W_1 -cardinality less than μ .

We consider two possibilities. First, assume that there exists a set $\mathcal{X} \in W_1$ of W_1 -cardinality less than κ such that

$$W_2 \models |\mathcal{X} \cap \{x_i : i < \mu\}| = \mu.$$

By intersecting $\mathcal{X}$ with $([Y]^{<\mu})^{W_1}$ if necessary, we may assume without loss of generality that $\mathcal{X} \subseteq ([Y]^{<\mu})^{W_1}$. Since $\mu < \kappa$, $z := \bigcup \mathcal{X}$ is a subset of Y of W_1 -cardinality less than κ . For all $i < \mu$, there exists $j > i$ in μ such that $x_j \in \mathcal{X}$, so $x_i \subseteq x_j \subseteq z$. Hence, z is a member of W_1 of W_1 -cardinality less than κ such that $x = \bigcup \{x_i : i < \mu\}$ is a subset of z , as required.

Secondly, assume that for all $\mathcal{X} \in W_1$ of W_1 -cardinality less than κ ,

$$W_2 \models |\mathcal{X} \cap \{x_i : i < \mu\}| < \mu.$$

Since each x_i is a member of W_1 , it follows from our assumptions that $\mathcal{X} \cap \{x_i : i < \mu\}$ is a member of W_1 . Also, the set $\{x_i : i < \mu\}$ is a subset of a member of W_1 , namely the set $([Y]^{<\mu})^{W_1}$. As the pair (W_1, W_2) has the κ -approximation property, it follows that $\{x_i : i < \mu\}$ is a member of W_1 . This is impossible, since letting $\mathcal{X}$ be equal to $\{x_i : i < \mu\}$, we get a contradiction to the assumption of this case. $\square$

Corollary 3.4. Let λ be a regular cardinal and $\mathbb{P}$ a forcing poset. Assume that $\mathbb{P}$ is $<\lambda$ -distributive. If $\mathbb{P}$ has the λ^+ -approximation property, then $\mathbb{P}$ has the λ^+ -covering property.

Proof. By Theorem 3.3, it suffices to show that $\mathbb{P}$ preserves λ^+ . If not, then there exists a cofinal set $x \subseteq (\lambda^+)^V$ in $V^\mathbb{P}$ of order type at most λ . If $a \in V$ has V -cardinality less than $(\lambda^+)^V$, then $a \cap x$ is bounded in $(\lambda^+)^V$, and hence has

order type less than λ . As $\mathbb{P}$ is $< \lambda$ -distributive, $a \cap x \in V$. Since $\mathbb{P}$ has the λ^+ -approximation property, it follows that $x \in V$, which is impossible. $\square$

Observe that if κ is weakly inaccessible or the successor of a singular cardinal, then a forcing poset $\mathbb{P}$ being $< \mu$ -distributive for all cardinals $\mu < \kappa$ implies that $\mathbb{P}$ is $< \kappa$ -distributive, and hence has the κ -covering property. That is why we restricted the statement of the corollary to successors of regulars.

Corollary 3.5. *If $\mathbb{P}$ is a forcing poset which has the ω_1 -approximation property, then $\mathbb{P}$ has the ω_1 -covering property.*

This follows from the fact that $\mathbb{P}$ forces that $V^{<\omega} \cap V^{\mathbb{P}} \subseteq V$.

REFERENCES

- [1] S. Cox and J. Krueger. Namba forcing, weak approximation, and guessing. *J. Symbolic Logic*, 83(4):1539–1565, 2018.
- [2] S. Hachtman and D. Sinapova. ITP, ISP and SCH. To appear in *J. Symbolic Logic*.
- [3] T. Jech. *Set theory. The third millennium edition, revised and expanded*. Springer-Verlag, Berlin, third edition, 2003.
- [4] M. Viale. The proper forcing axiom and the singular cardinal hypothesis. *J. Symbolic Logic*, 71(2):473–479, 2006.
- [5] M. Viale. A family of covering properties. *Math. Res. Lett.*, 15(2):221–238, 2008.
- [6] M. Viale. Guessing models and generalized Laver diamond. *Ann. Pure Appl. Logic*, 163(11):1660–1678, 2012.
- [7] M. Viale and C. Weiss. On the consistency strength of the proper forcing axiom. *Adv. Math.*, 228(5):2672–2687, 2011.
- [8] C. Weiss. The combinatorial essence of supercompactness. *Ann. Pure Appl. Logic*, 163(11):1710–1717, 2012.

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