

SOME RESULTS ON NON-CLUB ISOMORPHIC ARONSZAJN TREES

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ABSTRACT. Let λ be a regular cardinal satisfying $\lambda^{<\lambda} = \lambda$ and $\diamond(S_\lambda^{\lambda^+})$. Then there exists a family of 2^{λ^+} many completely club rigid special λ^+ -Aronszajn trees which are pairwise far.

In this article we will be concerned with building Aronszajn trees which are not club isomorphic and have strong rigidity properties. This topic goes back to Gaifman-Specker [4], who proved that if λ is a regular cardinal satisfying $\lambda^{<\lambda} = \lambda$, then there exists a family of 2^{λ^+} many normal λ -complete λ^+ -Aronszajn trees which are pairwise non-isomorphic. Abraham [1] and Todorcevic [7] constructed in ZFC ω_1 -Aronszajn trees which are rigid, that is, have no automorphisms other than the identity. Later the focus shifted from isomorphisms between trees to club isomorphisms. Abraham-Shelah [2] proved that under PFA, any two normal ω_1 -Aronszajn trees are club isomorphic. Krueger [6] provided a generalization of this result to higher cardinals. Abraham-Shelah [2] also showed that the weak diamond principle on ω_1 implies the existence of a family of 2^{ω_1} many normal club rigid ω_1 -Aronszajn trees which are pairwise not club embeddable into each other.

Building off of this work, we will use the diamond principle to construct a family of pairwise non-club isomorphic Aronszajn trees. Specifically, assume that λ is a regular cardinal satisfying $\lambda^{<\lambda} = \lambda$ and the diamond principle $\diamond(S_\lambda^{\lambda^+})$ holds, where $S_\lambda^{\lambda^+} := \{\alpha < \lambda^+ : \text{cf}(\alpha) = \lambda\}$. Then there exists a family $\{\mathcal{T}_\alpha : \alpha < 2^{\lambda^+}\}$ of normal λ -complete special λ^+ -Aronszajn trees such that for each $\alpha < 2^{\lambda^+}$, the only club embedding from a downwards closed normal subtree of $\mathcal{T}_\alpha$ into $\mathcal{T}_\alpha$ is the identity, and for all $\alpha < \beta < 2^{\lambda^+}$, $\mathcal{T}_\alpha$ and $\mathcal{T}_\beta$ do not contain club isomorphic downwards closed normal subtrees. We also discuss some related results, such as obtaining a large family of Suslin trees with similar properties and generalizing the Abraham-Shelah result on weak diamond to higher cardinals.

1. BACKGROUND

We assume that the reader is familiar with the basic definitions and facts about trees, including Aronszajn and Suslin trees. Let T be a tree. We write $\text{ht}_T(x)$ for the height of a node x in the tree T , and for any ordinal δ , $T(\delta)$ is the set of nodes of T with height δ . For a set of ordinals X , let $T \restriction X$ be the set of nodes $x \in T$ such that $\text{ht}_T(x) \in X$. For a regular uncountable cardinal κ , a tree T of height

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κ is a κ -tree if its levels have size less than κ . A tree T is *normal* if it satisfies the properties that every element of T has incomparable nodes above it, different elements of T of the same limit height do not have the same set of predecessors, and every element of T has nodes above it at any larger height below the height of the tree. A *branch* of T is a maximal chain. For an infinite cardinal λ , T is λ -complete if every chain of T whose order type is less than λ has an upper bound.

For any $x \in T$, let $x \downarrow := \{y \in T : y <_T x\}$. For a limit ordinal α , let $T(\alpha) \downarrow$ denote the collection of all sets $x \downarrow$, where $x \in T(\alpha)$. In other words, $T(\alpha) \downarrow$ is the set of all cofinal branches of $T \upharpoonright \alpha$ which have upper bounds in T . A *subtree* of a tree T is any subset U of T considered as a tree with the inherited order $<_T \cap (U \times U)$. A set $X \subseteq T$ is *downwards closed* if for all $x \in X$, $\{y \in T : y <_T x\} \subseteq X$. For any set $Y \subseteq T$, the *downward closure* of Y is the set $\{z \in T : \exists y \in Y \ z \leq_T y\}$. For any $x \in T$, let T_x be the set of all nodes of T which are comparable to x .

Trees T and U of height a regular uncountable cardinal κ are *club isomorphic* if there exists a club $C \subseteq \kappa$ such that $T \upharpoonright C$ and $U \upharpoonright C$ are isomorphic, and an isomorphism between $T \upharpoonright C$ and $U \upharpoonright C$ is called a *club isomorphism* of T and U . Similarly, T is *club embeddable* into U if there exists an injective and order preserving function $f : T \upharpoonright C \rightarrow U \upharpoonright C$ for some club $C \subseteq \kappa$, and in that case f is a *club embedding*. The trees T and U are *near* if there exist downwards closed normal subtrees of T and U of height κ which are club isomorphic, and otherwise T and U are *far*. A tree T of height κ is *club rigid* if the only club isomorphism of T with itself is the identity. A tree T is *completely club rigid* if the only club isomorphism between downwards closed normal subtrees of T is the identity. A tree T being completely club rigid is equivalent to the statement that whenever $F : T' \upharpoonright C \rightarrow T \upharpoonright C$ is injective, order preserving, and height preserving, where T' is a downwards closed normal subtree of T and $C \subseteq \kappa$ is club, then F is the identity.

For a regular uncountable cardinal κ , a κ -tree is κ -Aronszajn if it has no cofinal branch, and is κ -Suslin if it has no chains or antichains of size κ . In this article we will be exclusively concerned with λ^+ -trees, where λ is a regular cardinal. A λ^+ -tree T is *special* if it is a union of λ many antichains, or equivalently, there exists a map of T into λ which is injective on chains. Every special tree is Aronszajn and not Suslin.

For a regular cardinal λ , the linear order $\mathbb{Q}_\lambda$ is defined as the set of all functions $f : \lambda \rightarrow 2$ such that the set $\{\alpha < \lambda : f(\alpha) = 1\}$ is non-empty and has size less than λ , ordered lexicographically. If $\lambda^{<\lambda} = \lambda$, then $\mathbb{Q}_\lambda$ has size λ . Two important facts about $\mathbb{Q}_\lambda$ which we will use are: between any two elements of $\mathbb{Q}_\lambda$ there exists an increasing sequence of order type λ , and any increasing sequence of $\mathbb{Q}_\lambda$ with order type less than λ has a least upper bound in $\mathbb{Q}_\lambda$ ([5, Lemma 3.4]). A tree T of height λ^+ is $\mathbb{Q}_\lambda$ -embeddable if there exists an order preserving map from T into $\mathbb{Q}_\lambda$. In particular, if $\lambda^{<\lambda} = \lambda$, then being $\mathbb{Q}_\lambda$ -embeddable implies being special.

Let κ be a regular uncountable cardinal and $S \subseteq \kappa$ stationary. The *diamond principle* $\diamond(S)$ states that there exists a sequence $\langle s_\alpha : \alpha \in S \rangle$, where each $s_\alpha \subseteq \alpha$, such that for any set $X \subseteq \kappa$ there are stationarily many $\alpha \in S$ such that $s_\alpha = X \cap \alpha$. It is common to replace subsets of κ in the above with related objects. For example, as we will use in this article, if n and $m_1, \dots, m_n$ are positive integers, then $\diamond(S)$ is equivalent to the existence of a diamond sequence which guesses a finite sequence

$X_1 \subseteq \kappa^{m_1}, \dots, X_n \subseteq \kappa^{m_n}$ instead of a single subset of κ (see the beginning of Section 3).

2. BUILDING BLOCKS OF SPECIAL TREE CONSTRUCTIONS

In this section we provide a sequence of lemmas which we will use to construct a large family of pairwise far Aronszajn trees in Section 3. For the remainder of this section assume that λ is a regular cardinal satisfying that $\lambda^{<\lambda} = \lambda$. For every ordinal $\alpha < \lambda^+$, let I_α be the interval of ordinals $[\lambda \cdot \alpha, \lambda \cdot (\alpha+1)) = \{\lambda \cdot \alpha + \delta : \delta < \lambda\}$.

Definition 2.1. Let $\alpha \leq \lambda^+$. A standard α -tree is pair (T, π) satisfying:

- (1) T is a tree of height α and $\pi : T \rightarrow \mathbb{Q}_\lambda$ is order preserving;
- (2) for all $\beta < \alpha$, $T(\beta) = I_\beta$;
- (3) every node x of T such that $\text{ht}_T(x) + 1 < \alpha$ has λ many immediate successors;
- (4) if x and y are distinct nodes of T of the same limit height then they do not have the same set of predecessors;
- (5) for all $\beta < \alpha$ of cofinality less than λ , every cofinal branch b of $T \restriction \beta$ has an upper bound y and $\pi(y) = \sup\{\pi(x) : x <_T y\}$;
- (6) for all $x \in T$, $\text{ht}_T(x) < \beta < \alpha$, and q with $\pi(x) <_{\mathbb{Q}_\lambda} q$:
 - (a) if β is either a successor ordinal or an ordinal of cofinality λ , then there exists y above x with height β such that $\pi(y) = q$;
 - (b) if β is a limit ordinal of cofinality less than λ , then there exists y above x with height β such that $\pi(y) \leq_{\mathbb{Q}_\lambda} q$.

Observe that the underlying set of any standard α -tree is equal to $\lambda \cdot \alpha$.

Definition 2.2. Let (T, π) be a standard α -tree and (U, τ) a standard β -tree, where $\alpha < \beta \leq \lambda^+$. Then (U, τ) extends (T, π) if U is an end-extension of T and $\tau \restriction T = \pi$.

The next lemma is easy to check.

Lemma 2.3. Let $\alpha \leq \lambda^+$ be a limit ordinal. Assume that $\langle (T_\beta, \pi_\beta) : \beta < \alpha \rangle$ is a sequence such that each (T_β, π_β) is a standard β -tree and for all $\beta < \gamma < \alpha$, (T_γ, π_γ) extends (T_β, π_β) . Define $T_\alpha := \bigcup \{T_\beta : \beta < \alpha\}$ and $\pi_\alpha := \bigcup \{\pi_\beta : \beta < \alpha\}$. Then (T_α, π_α) is an α -tree which extends (T_β, π_β) for all $\beta < \alpha$.

Lemma 2.4. Suppose that (T, π) is a standard α -tree, where $\alpha < \lambda^+$ is a limit ordinal, and B is a set of cofinal branches of T of size less than λ . Let $z \in T$ and $q \in \mathbb{Q}_\lambda$ with $\pi(z) <_{\mathbb{Q}_\lambda} q$. Then there exists a cofinal branch b of T satisfying that $z \in b$, for all $y \in b$, $\pi(y) <_{\mathbb{Q}_\lambda} q$, and $b \notin B$.

Proof. Fix an increasing and continuous sequence of ordinals $\langle \alpha_i : i < \text{cf}(\alpha) \rangle$ which is cofinal in α such that $\alpha_0 = \text{ht}_T(z) + 1$. Fix an increasing sequence $\langle q_i : i < \text{cf}(\alpha) \rangle$ of elements of $\mathbb{Q}_\lambda$ in between $\pi(z)$ and q .

We recursively build an increasing sequence $\langle x_i : i < \text{cf}(\alpha) \rangle$ of nodes of T above z as follows. We will maintain that x_i has height α_i and $\pi(x_i) \leq_{\mathbb{Q}_\lambda} q_i$. To define x_0 , consider the immediate successors of z . By Definition 2.1(6a) and the fact that there are λ many elements of $\mathbb{Q}_\lambda$ between $\pi(z)$ and q_0 , it follows that there are λ many immediate successors y of z such that $\pi(y) <_{\mathbb{Q}_\lambda} q_0$. Since $|B| < \lambda$, we can choose x_0 as such a y which does not belong to any branch in B .

If x_i is defined for some $i < \text{cf}(\alpha)$, apply Definition 2.1(6) to choose x_{i+1} above x_i of height α_{i+1} such that $\pi(x_{i+1}) \leq_{\mathbb{Q}_\lambda} q_{i+1}$. If x_j is defined for all $j < \delta$, where $\delta < \text{cf}(\alpha)$ is a limit ordinal, then since $\text{cf}(\delta) < \lambda$, we can let x_δ be the unique node of T with height α_δ which is above x_j for all $j < \delta$. Then $\pi(x_\delta) = \sup\{\pi(x_j) : j < \delta\} \leq \sup\{q_j : j < \delta\} \leq q_\delta$. This completes the construction of the sequence. Let b be the downward closure of $\{x_i : i < \text{cf}(\alpha)\}$. $\square$

Lemma 2.5. *Suppose that $\alpha < \lambda^+$ is either a successor ordinal or a limit ordinal of cofinality less than λ . Let (T, π) be a standard α -tree. Then there exists a standard $(\alpha + 1)$ -tree (T^+, π^+) which extends (T, π) .*

Proof. First, assume that $\alpha = \beta + 1$ is a successor ordinal, so that $T(\beta)$ is the top level of T . Partition I_α into λ many subsets of size λ . Define T^+ so that the map sending a node of $T(\beta)$ to the set of immediate successors of that node is a bijection between $T(\beta)$ and the partition. Define π^+ which extends π so that for each $x \in T(\beta)$, π^+ restricted to the set of immediate successors of x in T^+ is a bijection between the set of such successors and the set $\{q \in \mathbb{Q}_\lambda : \pi(x) <_{\mathbb{Q}_\lambda} q\}$.

Secondly, assume that α is a limit ordinal of cofinality less than λ . Since $\lambda^{<\lambda} = \lambda$, T has exactly λ many cofinal branches. Define T^+ which end-extends T by adding a top level consisting of $I(\alpha)$, where we put exactly one node above each cofinal branch of T . For each $z \in T^+(\alpha)$ above a branch b , define $\pi^+(z) := \sup\{\pi(y) : y \in b\}$. Most of the requirements of (T^+, π^+) being an $(\alpha + 1)$ -tree are obvious, but let us check 6b of Definition 2.1 carefully. Let $x \in T$ and $q \in \mathbb{Q}_\lambda$ with $\pi(x) <_{\mathbb{Q}_\lambda} q$. Applying Lemma 2.4 with $B = \emptyset$, fix a cofinal branch b of T such that $x \in b$ and for all $x' \in b$, $\pi(x') <_{\mathbb{Q}_\lambda} q$. Then the upper bound y of b on level α of T^+ is above x and satisfies that $\pi(y) = \sup\{\pi(x') : x' \in b\} \leq_{\mathbb{Q}_\lambda} q$. $\square$

Notation 2.6. *Let $\alpha < \lambda^+$. If (T, π) is a standard α -tree, define $X_{T, \pi} := \{(x, q) : x \in T, \pi(x) <_{\mathbb{Q}_\lambda} q\}$.*

Note that since $\lambda^{<\lambda} = \lambda$, $X_{T, \pi}$ has size λ .

Definition 2.7. *Let (T, π) be a standard α -tree, where $\alpha \in S_\lambda^{\lambda^+}$. For a set B of cofinal branches of T , a $\mathbb{Q}_\lambda$ -tethering of B is any bijection $h : X_{T, \pi} \rightarrow B$ satisfying that for all $(x, q) \in X_{T, \pi}$, $x \in h(x, q)$ and for all $y \in h(x, q)$, $\pi(y) <_{\mathbb{Q}_\lambda} q$. If there exists a $\mathbb{Q}_\lambda$ -tethering for B , then B is $\mathbb{Q}_\lambda$ -tethered.*

Definition 2.8. *Let $\alpha \leq \lambda^+$ and suppose that (T, π) is a standard α -tree. A downwards closed subtree T' of T is said to be avoidable if for all $x \in T$ and $q \in \mathbb{Q}_\lambda$ with $\pi(x) <_{\mathbb{Q}_\lambda} q$, there exists some $y \in T \setminus T'$ above x such that $\pi(y) <_{\mathbb{Q}_\lambda} q$.*

Lemma 2.9. *Let (T, π) be a standard α -tree, where $\alpha \in S_\lambda^{\lambda^+}$, and suppose that T' is a downwards closed subtree of T which is avoidable. Then there exist disjoint $\mathbb{Q}_\lambda$ -tethered sets of cofinal branches B_0 and B_1 of T satisfying that every branch in B_0 or B_1 contains a node which is not in T' .*

Proof. Let $\langle (x_i, q_i) : i < \lambda \rangle$ enumerate $X_{T, \pi}$. We will define $B_0 = \{b_{0,i} : i < \lambda\}$ and $B_1 = \{b_{1,i} : i < \lambda\}$ by recursion.

Let $i < \lambda$ and assume that $b_{0,j}$ and $b_{1,j}$ are defined for all $j < i$. Consider (x_i, q_i) . Then $\pi(x_i) <_{\mathbb{Q}_\lambda} q_i$. Since T' is avoidable, we can fix z above x_i in $T \setminus T'$ such that $\pi(z) <_{\mathbb{Q}_\lambda} q_i$. Applying Lemma 2.4, fix a cofinal branch $b_{0,i}$ of T satisfying that $z \in b_{0,i}$, for all $y \in b_{0,i}$, $\pi(y) <_{\mathbb{Q}_\lambda} q_i$, and $b_{0,i}$ is not a member of $\{b_{n,j} : j < i, n < 2\}$.

Applying Lemma 2.4 again, fix a cofinal branch $b_{1,i}$ of T satisfying that $z \in b_{1,i}$, for all $y \in b_{1,i}$, $\pi(y) <_{\mathbb{Q}_\lambda} q_i$, and $b_{1,i}$ is not a member of $\{b_{n,j} : j < i, n < 2\} \cup \{b_{0,i}\}$.

This completes the construction. The maps given by $h_0(x_i, q_i) := b_{0,i}$ and $h_1(x_i, q_i) := b_{1,i}$ for all $i < \lambda$ are $\mathbb{Q}_\lambda$ -tetherings of B_0 and B_1 respectively. $\square$

Lemma 2.10. *Let (T, π) be a standard α -tree, where $\alpha \in S_\lambda^{\lambda^+}$, and T' a downwards closed subtree of T . Suppose that $f : T' \restriction c \rightarrow T \restriction c$ is injective, order preserving, and height preserving, where $c \subseteq \alpha$ is a club. Assume that $x \in T' \restriction c$, $f(x) \neq x$, and T'_x is not avoidable. Fix $x^* \in T'_x$ and $q \in \mathbb{Q}_\lambda$ with $\pi(x^*) <_{\mathbb{Q}_\lambda} q$ such that for all y above x^* in T , if $\pi(y) <_{\mathbb{Q}_\lambda} q$ then $y \in T'_x$. Fix q^* with $\pi(x^*) <_{\mathbb{Q}_\lambda} q^* <_{\mathbb{Q}_\lambda} q$. Then there exists a set B of cofinal branches of T and a $\mathbb{Q}_\lambda$ -tethering h of B such that $b := h(x^*, q^*)$ satisfies that $b \subseteq T'_x$, but the downward closure of $f[b \cap (T \restriction c)]$ is not in B .*

Proof. Let $\langle (x_i, q_i) : i < \lambda \rangle$ be an enumeration of $X_{T, \pi}$ such that $x_0 = x^*$ and $q_0 = q^*$. We define $B = \{b_i : i < \lambda\}$ by recursion.

Begin by choosing b_0 to be some cofinal branch of T which contains x_0 and satisfies that for all $y \in b_0$, $\pi(y) <_{\mathbb{Q}_\lambda} q_0$; this is possible by Lemma 2.4. Observe that $b_0 \subseteq T'_x$. Let b_f be the downward closure in T of $f[b_0 \cap (T \restriction c)]$. Note that $b_f \neq b_0$ since $x \in b_0$, $f(x) \in b_f$, and b_0 cannot contain two distinct elements from the same level of T .

Let $0 < i < \lambda$ and suppose that b_j is defined for all $j < i$. Applying Lemma 2.4, fix a cofinal branch b_i of T satisfying that $x_i \in b_i$, for all $y \in b_i$, $\pi(y) <_{\mathbb{Q}_\lambda} q_i$, and b_i is not a member of the set $\{b_j : j < i\} \cup \{b_f\}$. This completes the construction of B . Now define $h(x_i, q_i) := b_i$ for all $i < \lambda$. $\square$

Lemma 2.11. *Suppose that (T, π) is a standard α -tree, where $\alpha \in S_\lambda^{\lambda^+}$, and B is a set of cofinal branches of (T, π) and h a $\mathbb{Q}_\lambda$ -tethering of B . Then there exists a standard $(\alpha + 1)$ -tree (T^+, π^+) which extends (T, π) satisfying that $T^+(\alpha) \downarrow = B$ and for all $(x, q) \in X_{T, \pi}$, if z is the upper bound of $h(x, q)$ in T^+ then $\pi^+(z) = q$.*

Proof. Extend T to T^+ by letting the top level of T^+ be I_α and placing one node above each branch in B . If z is an upper bound of $b = h(x, q)$, then for all $y \in b$, $\pi(y) <_{\mathbb{Q}_\lambda} q$. Thus, it makes sense to define $\pi^+(z) := q$. Now easily (T^+, π^+) is as required. $\square$

Lemma 2.12. *Suppose that (T, π) is a standard α -tree, where $\alpha \in S_\lambda^{\lambda^+}$, and T' is a downwards closed subtree of T which is avoidable. Then there exists a standard $(\alpha + 1)$ -tree (T^+, π^+) which extends (T, π) such that every member of $T^+(\alpha) \downarrow$ contains an element which is not in T' .*

Proof. Immediate from Lemma 2.9, just using one of the two sets of branches mentioned there, and Lemma 2.11. $\square$

Lemma 2.13. *Suppose that (T, π) is a standard α -tree, where $\alpha \in S_\lambda^{\lambda^+}$. Then there exist standard $(\alpha + 1)$ -trees (T^0, π^0) and (T^1, π^1) which extend (T, π) and satisfy that $T^0(\alpha) \downarrow$ and $T^1(\alpha) \downarrow$ are disjoint.*

Proof. Immediate from Lemma 2.9, letting $T' = \emptyset$, and Lemma 2.11. $\square$

Lemma 2.14. *Let (T, π) be a standard α -tree, where $\alpha \in S_\lambda^{\lambda^+}$, and T' a downwards closed subtree of T . Suppose that $f : T' \restriction c \rightarrow T \restriction c$ is injective, order preserving, and height preserving, where $c \subseteq \alpha$ is a club. Assume that $x \in T' \restriction c$, $f(x) \neq x$, and*

T'_x is not avoidable. Fix $x^* \in T'_x$ and $q \in \mathbb{Q}_\lambda$ with $\pi(x^*) <_{\mathbb{Q}_\lambda} q$ such that for all y above x^* in T , if $\pi(y) <_{\mathbb{Q}_\lambda} q$ then $y \in T'_x$. Then there exists a standard $(\alpha+1)$ -tree (T^+, π^+) which extends (T, π) , there exists $b \in T^+(\alpha) \downarrow$ for which x and x^* are in b and $b \subseteq T'$, and there exists $z \in T^+(\alpha)$ which is an upper bound of b such that $\pi^+(z) <_{\mathbb{Q}_\lambda} q$, but the downward closure of $f[b \cap (T \restriction c)]$ is not in $T^+(\alpha) \downarrow$.

Proof. Immediate from Lemmas 2.10 and 2.11. $\square$

Lemma 2.15. Suppose that (T, π) and (U, τ) are standard α -trees, where $\alpha \in S_\lambda^{\lambda^+}$, $c \subseteq \alpha$ is a club, T' is a downwards closed subtree of T which is not avoidable, and $f : T' \restriction c \rightarrow U \restriction c$ is injective and order preserving. Let $x \in T$ and $q \in \mathbb{Q}_\lambda$ be such that $\pi(x) <_{\mathbb{Q}_\lambda} q$ and for all y above x in T , if $\pi(y) <_{\mathbb{Q}_\lambda} q$ then $y \in T'$.

Then there exist standard $(\alpha+1)$ -trees (T^+, π^+) and (U^+, τ^+) which extend (T, π) and (U, τ) respectively, there exists $b \in T^+(\alpha) \downarrow$ which contains x , and there exists $z \in T^+(\alpha)$ which is an upper bound of b such that $\pi^+(z) <_{\mathbb{Q}_\lambda} q$, satisfying that $b \subseteq T'$ but $f[b \cap (T \restriction c)]$ does not have an upper bound in U^+ .

Proof. Using Lemma 2.13, fix a standard $(\alpha+1)$ -tree (T^+, π^+) which extends (T, π) . Using Lemma 2.13 again, fix standard $(\alpha+1)$ -trees (U^0, τ^0) and (U^1, τ^1) which extend (U, τ) such that the sets $U^0(\alpha) \downarrow$ and $U^1(\alpha) \downarrow$ are disjoint.

As (T^+, π^+) is a standard $(\alpha+1)$ -tree, we can fix $z \in T^+(\alpha)$ above x such that $\pi(z) <_{\mathbb{Q}_\lambda} q$. Let $b := \{y \in T : y <_T z\}$. Note that for all $y \in b$ above x , $\pi(y) <_{\mathbb{Q}_\lambda} \pi(z) = q$, and hence $y \in T'$ by the choice of x and q . Since T' is downwards closed, $b \subseteq T'$.

Let b_f be the downward closure of the chain $f[b \cap (T \restriction c)]$ in U . Then b_f is a cofinal branch of U . Since the sets $U^0(\alpha) \downarrow$ and $U^1(\alpha) \downarrow$ are disjoint, b_f does not belong to one of them. Fix $j < 2$ such that $b_f \notin U^j(\alpha) \downarrow$. Then b_f has no upper bound in $U^j(\alpha)$. Define (U^+, π^+) as (U^j, π^j) . Then $f[b \cap (T \restriction c)]$ does not have an upper bound in U^+ . $\square$

3. A PAIRWISE FAR FAMILY FROM DIAMOND

We now use the results of the preceding section to prove our main theorem.

Theorem 3.1. Let λ be a regular cardinal such that $\lambda^{<\lambda} = \lambda$. Assume $\diamond(S_\lambda^{\lambda^+})$. Then there exists a pairwise far family of 2^{λ^+} many normal λ -complete special λ^+ -Aronszajn trees which are completely club rigid.

Using $\diamond(S_\lambda^{\lambda^+})$, fix a sequence $\langle (x_\alpha, s_\alpha, t_\alpha, f_\alpha) : \alpha \in S_\lambda^{\lambda^+} \rangle$, where each $x_\alpha \subseteq \alpha$ and $s_\alpha, t_\alpha, f_\alpha \subseteq \alpha^2$, such that for any sets $X \subseteq \lambda^+$ and $G, H, F \subseteq (\lambda^+)^2$, there are stationarily many $\alpha \in S_\lambda^{\lambda^+}$ such that $X \cap \alpha = x_\alpha$, $G \cap \alpha^2 = s_\alpha$, $H \cap \alpha^2 = t_\alpha$, and $F \cap \alpha^2 = f_\alpha$.

We will recursively construct a family $\{(T_s, \pi_s) : s \in {}^{\lambda^+}2\}$ satisfying that each (T_s, π_s) is a standard $\text{dom}(s)$ -tree and if $s \subseteq t$ then (T_t, π_t) extends (T_s, π_s) . After this construction is complete, we will define a large subset of ${}^{\lambda^+}2$ which will index the required pairwise far family.

As the base case, let $T_\emptyset := \emptyset$ and $\pi_\emptyset := \emptyset$. For each $s \in {}^12$, let T_s be the tree of height one whose single level equals I_0 and let π_s be any bijection from λ onto $\mathbb{Q}_\lambda$.

Suppose that $\alpha \leq \lambda^+$ is a limit ordinal and (T_s, π_s) is defined as required for all $s \in {}^{<\alpha}2$. For any $s \in {}^\alpha 2$, define $T_s := \bigcup \{T_{s \restriction \gamma} : \gamma < \alpha\}$ and $\pi_s := \bigcup \{\pi_{s \restriction \gamma} : \gamma < \alpha\}$. The validity of this step follows from Lemma 2.3.

Now assume that $\alpha + 1$ is a successor ordinal and (T_s, π_s) is defined as required for all $s \in {}^\alpha 2$. We split the definition into two cases. In the first case, assume that α is either a successor ordinal or a limit ordinal of cofinality less than λ . Fix $s \in {}^\alpha 2$, and we will define (T_t, π_t) for $t \in \{s \hat{\ } 0, s \hat{\ } 1\}$. Using Lemma 2.5, fix an $(\alpha + 1)$ -tree (T_s^+, π_s^+) which extends (T_s, π_s) . Now define $(T_{s \hat{\ } 0}, \pi_{s \hat{\ } 0})$ and $(T_{s \hat{\ } 1}, \pi_{s \hat{\ } 1})$ as (T_s^+, π_s^+) .

In the second case, assume that $\alpha \in S_\lambda^{\lambda^+}$. Let us define $T_{s \hat{\ } 0}$ and $T_{s \hat{\ } 1}$ for all $s \in {}^\alpha 2$ except for a few exceptional functions s which we will discuss momentarily. For all s other than the possible exceptions, apply Lemma 2.13 to fix an $(\alpha + 1)$ -tree (T_s^+, π_s^+) which extends (T_s, π_s) . Now define $(T_{s \hat{\ } 0}, \pi_{s \hat{\ } 0})$ and $(T_{s \hat{\ } 1}, \pi_{s \hat{\ } 1})$ as (T_s^+, π_s^+) .

Now we discuss the possible exceptions by considering the objects s_α , t_α , and f_α . We assume that $\alpha = \lambda \cdot \alpha$; if not, then there are no exceptions and we are done. There are four mutually exclusive cases we will consider. If none of these cases occurs, then there are no exceptions and we are done.

Case 1: The objects s_α and t_α are in ${}^\alpha 2$, $s_\alpha = t_\alpha$, and there exists a downwards closed subtree T' of T_{s_α} and a club $c \subseteq \alpha$ such that $f_\alpha : T' \restriction c \rightarrow T_{t_\alpha} \restriction c$ is injective, order preserving, height preserving, and not the identity. Note that T' is the downward closure of $T' \restriction c$ in T_{s_α} and hence it is uniquely determined from f_α . Let $s := s_\alpha = t_\alpha$ and $f := f_\alpha$.

Subcase 1a: There exists $x \in T' \restriction c$ such that $f(x) \neq x$ and T'_x is avoidable. Fix such an x . Since T'_x is avoidable and downwards closed, applying Lemma 2.12 we can fix a standard $(\alpha + 1)$ -tree (T_s^+, π_s^+) which extends (T_s, π_s) such that every member of $T_s^+(\alpha) \downarrow$ contains an element which is not in T'_x . Now define $(T_{s \hat{\ } 0}, \pi_{s \hat{\ } 0})$ and $(T_{s \hat{\ } 1}, \pi_{s \hat{\ } 1})$ as (T_s^+, π_s^+) .

Subcase 1b: For all $x \in T' \restriction c$ with $f(x) \neq x$, T'_x is not avoidable. Since f is not the identity, we can fix $x \in T_s \restriction c$ such that $f(x) \neq x$. Fix $x^* \in T'_x$ and $q \in \mathbb{Q}_\lambda$ with $\pi(x^*) <_{\mathbb{Q}_\lambda} q$ such that for all y above x^* in T , if $\pi(y) <_{\mathbb{Q}_\lambda} q$ then $y \in T'_x$. By Lemma 2.14 there exists a standard $(\alpha + 1)$ -tree (T_s^+, π_s^+) such that for some $b \in T_s^+(\alpha) \downarrow$, x and x^* are in b , $b \subseteq T'$, the upper bound z of b in T_s^+ satisfies that $\pi(z) <_{\mathbb{Q}_\lambda} q$, and the downward closure of $f[b \cap (T \restriction c)]$ is not in $T_s^+(\alpha) \downarrow$. Now define $(T_{s \hat{\ } 0}, \pi_{s \hat{\ } 0})$ and $(T_{s \hat{\ } 1}, \pi_{s \hat{\ } 1})$ as (T_s^+, π_s^+) .

Case 2: The objects s_α and t_α are in ${}^\alpha 2$, $s_\alpha \neq t_\alpha$, and there exists a downwards closed subtree T' of T_{s_α} and a club $c \subseteq \alpha$ such that $f_\alpha : T' \restriction c \rightarrow T_{t_\alpha} \restriction c$ is injective and order preserving. Note again that T' is determined from f_α . Let $s := s_\alpha$, $t := t_\alpha$, and $f := f_\alpha$.

Subcase 2a: T' is avoidable. Since T' is avoidable and downwards closed, applying Lemma 2.12 there exists a standard $(\alpha + 1)$ -tree (T_s^+, π_s^+) which extends (T_s, π_s) such that every member of $T_s^+(\alpha) \downarrow$ contains an element which is not in T' . By Lemma 2.13, fix a standard $(\alpha + 1)$ -tree (T_t^+, π_t^+) which extends (T_t, π_t) . Now define $(T_{s \hat{\ } 0}, \pi_{s \hat{\ } 0})$ and $(T_{s \hat{\ } 1}, \pi_{s \hat{\ } 1})$ as (T_s^+, π_s^+) and $(T_{t \hat{\ } 0}, \pi_{t \hat{\ } 0})$ and $(T_{t \hat{\ } 1}, \pi_{t \hat{\ } 1})$ as (T_t^+, π_t^+) .

Subcase 2b: T' is not avoidable. Since T' is not avoidable, fix $x \in T_s$ and $q \in \mathbb{Q}_\lambda$ such that $\pi_s(x) <_{\mathbb{Q}_\lambda} q$ and for all y above x in T_s with $\pi_s(y) <_{\mathbb{Q}_\lambda} q$, $y \in T'$.

Applying Lemma 2.15, fix standard $(\alpha + 1)$ -trees (T_s^+, π_s^+) and (T_t^+, π_t^+) which extend (T_s, π_s) and (T_t, π_t) respectively, $b \in T_s^+(\alpha) \downarrow$ which contains x , and $z \in T_s^+(\alpha)$ which is an upper bound of b with $\pi_s^+(z) <_{\mathbb{Q}_\lambda} q$, such that $b \subseteq T'$ but $f[b \cap (T \restriction c)]$ does not have an upper bound in T_t^+ . Now define $(T_{s \smallfrown 0}, \pi_{s \smallfrown 0})$ and $(T_{s \smallfrown 1}, \pi_{s \smallfrown 1})$ as (T_s^+, π_s^+) and $(T_t \smallfrown 0, \pi_t \smallfrown 0)$ and $(T_t \smallfrown 1, \pi_t \smallfrown 1)$ as (T_t^+, π_t^+) .

This completes the construction. Note that for all $s \in {}^{\lambda^+}2$, T_s is a normal λ -complete special λ^+ -Aronszajn tree.

We now define a large subcollection of $\{T_s : s \in {}^{\lambda^+}2\}$ which will witness Theorem 3.1.

Definition 3.2. For each $X \subseteq \lambda^+$, define a function $h_X : \lambda^+ \rightarrow 2$ by letting $h_X(\alpha) = 1$ iff $X \cap \alpha = x_\alpha$.

Observe that if X and Y are distinct subsets of λ^+ , then for any $\alpha < \lambda^+$ greater than the least member of the symmetric difference of X and Y , if $h_X(\alpha) = 1$ then $h_Y(\alpha) = 0$. Also there exists an α such that $h_X(\alpha) = 1$ by the diamond property.

Notation 3.3. For every set $X \subseteq \lambda^+$, let $\mathcal{T}_X := T_{h_X}$ and $\pi_X := \pi_{h_X}$.

We complete the proof by showing that for every set $X \subseteq \lambda^+$, $\mathcal{T}_X$ is completely club rigid, and for all distinct subsets X and Y of λ^+ , $\mathcal{T}_X$ and $\mathcal{T}_Y$ are far.

Let $X \subseteq \lambda^+$. Suppose for a contradiction that there exists a downwards closed normal subtree $\mathcal{T}'_X$ of $\mathcal{T}_X$, a club $C \subseteq \lambda^+$, and an injective, order preserving, and height preserving function $F : \mathcal{T}'_X \restriction C \rightarrow \mathcal{T}_X \restriction C$ which is not the identity.

By an elementary argument, we can find a club $D \subseteq \lambda^+$ satisfying that for all $\alpha \in D$:

- $\lambda \cdot \alpha = \alpha$;
- $C \cap \alpha$ is club in α ;
- $F \restriction \alpha = F \cap \alpha^2$;
- $F \restriction \alpha$ is not the identity;
- for all $x \in \mathcal{T}_X \restriction \alpha$ and $q \in \mathbb{Q}_\lambda$, if there exists some y above x in $\mathcal{T}_X \setminus \mathcal{T}'_X$ such that $\pi_X(y) <_{\mathbb{Q}_\lambda} q$, then there exists such a y in $\mathcal{T}_X \restriction \alpha$.

Note that to obtain the last statement, we use the fact that $\lambda^{<\lambda} = \lambda$ and so $\mathbb{Q}_\lambda$ has size λ .

By the diamond property, fix a stationary set $S \subseteq S_\lambda^{\lambda^+}$ satisfying that for all $\alpha \in S$:

- $x_\alpha = X \cap \alpha$;
- $s_\alpha = h_X \cap \alpha^2$;
- $t_\alpha = h_X \cap \alpha^2$;
- $f_\alpha = F \cap \alpha^2$.

Observe that since h_X maps into 2, $h_X \cap \alpha^2 = h_X \restriction \alpha$.

Fix $\alpha \in S \cap D$ and we will get a contradiction. Note that we are in Case 1 of the construction as witnessed by $T' := \mathcal{T}'_X \restriction \alpha$ and $c := C \cap \alpha$. Let $s := s_\alpha = t_\alpha = h_X \restriction \alpha$ and $f := f_\alpha$. Since $X \cap \alpha = x_\alpha$, $h_X(\alpha) = 1$. Hence, $\mathcal{T}_X \restriction (\alpha + 1) = T_{h_X \restriction (\alpha + 1)} = T_{s \smallfrown 1}$. Also, since $\lambda \cdot \alpha = \alpha$, the underlying set of T_s is α , so $f = F \restriction \alpha = F \restriction T_s$.

We claim that for all $x \in T' \restriction c$ with $f(x) \neq x$, T'_x is not avoidable. Suppose for a contradiction that this is not the case. Then we are in Subcase 1a of the construction. By definition, for some $x \in T' \restriction c$ such that $f(x) \neq x$ and T'_x is avoidable, we have that every member of $T_{s \smallfrown 1}(\alpha) \downarrow$ contains an element which is

not in T'_x . Since $\mathcal{T}'_X$ is normal, we can find $z \in \mathcal{T}'_X$ of height α above x . Then $b := \{y : y <_{\mathcal{T}_X} z\}$ is in $\mathcal{T}_X(\alpha) \downarrow = T_{s \wedge 1}(\alpha) \downarrow$. But since $\mathcal{T}'_X$ is downwards closed and contains z , $b \subseteq T'_x$, which is a contradiction.

So indeed for all $x \in T' \upharpoonright c$ with $f(x) \neq x$, T'_x is not avoidable. Hence, we are in Subcase 1b of the construction. By definition, (a) there exists $x \in T_s \upharpoonright c$ such that $f(x) \neq x$, (b) there exists $x^* \in T'_x$ and $q \in \mathbb{Q}_\lambda$ with $\pi(x^*) <_{\mathbb{Q}_\lambda} q$ such that for all y above x^* in T_s , if $\pi(y) <_{\mathbb{Q}_\lambda} q$ then $y \in T'_x$, and (c) there exists $b \in T_{s \wedge 1}(\alpha) \downarrow$ and an upper bound z of b in $T_{s \wedge 1}(\alpha)$ such that $\pi_{s \wedge 1}(z) <_{\mathbb{Q}_\lambda} q$, the nodes x and x^* are in b , $b \subseteq T'$, but the downward closure of $f[b \cap (T \upharpoonright c)]$ is not in $T_{s \wedge 1}(\alpha) \downarrow$. In particular, the downward closure of $f[b \cap (T \upharpoonright c)]$ does not have an upper bound in $\mathcal{T}_X$.

We claim that $z \in \mathcal{T}'_X$. If not, then $z \in \mathcal{T}_X \setminus \mathcal{T}'_X$ and $\pi_X(z) = \pi_{s \wedge 1}(z) <_{\mathbb{Q}_\lambda} q$. As $\alpha \in D$, by the last property in the description of D it follows that there is some y above x^* in $(\mathcal{T}_X \upharpoonright \alpha) \setminus \mathcal{T}'_X = T_s \setminus T'$ such that $\pi_X(y) = \pi_s(y) <_{\mathbb{Q}_\lambda} q$. But this contradicts the choice of x^* and q .

Since $F \upharpoonright \alpha = f$, $F(z)$ is an upper bound of the downward closure of $f[b \cap (T \upharpoonright c)]$ in $\mathcal{T}_X$, which is a contradiction.

Now let X and Y be distinct subsets of λ^+ , and we will prove that $\mathcal{T}_X$ and $\mathcal{T}_Y$ are far. Suppose for a contradiction that there exist downwards closed normal subtrees $\mathcal{T}'_X$ and $\mathcal{T}'_Y$ of $\mathcal{T}_X$ and $\mathcal{T}_Y$ respectively, a club $C \subseteq \lambda^+$, and a club isomorphism $F : \mathcal{T}'_X \upharpoonright C \rightarrow \mathcal{T}'_Y \upharpoonright C$.

By an elementary argument, we can find a club $D \subseteq \lambda^+$ satisfying that for all $\alpha \in D$:

- $\lambda \cdot \alpha = \alpha$;
- $C \cap \alpha$ is club in α ;
- $F \upharpoonright \alpha = F \cap \alpha^2$;
- α is greater than the least member of the symmetric difference of X and Y ;
- $h_X \upharpoonright \alpha \neq h_Y \upharpoonright \alpha$;
- for all $x \in \mathcal{T}_X \upharpoonright \alpha$ and $q \in \mathbb{Q}_\lambda$, if there exists some y above x in $\mathcal{T}_X \setminus \mathcal{T}'_X$ such that $\pi_X(y) <_{\mathbb{Q}_\lambda} q$, then there exists such a y in $\mathcal{T}_X \upharpoonright \alpha$.

Note that to obtain the last statement, we use the fact that $\lambda^{<\lambda} = \lambda$ and so $\mathbb{Q}_\lambda$ has size λ .

By the diamond property, fix a stationary set $S \subseteq S_\lambda^{\lambda^+}$ satisfying that for all $\alpha \in S$:

- $x_\alpha = X \cap \alpha$;
- $s_\alpha = h_X \cap \alpha^2$;
- $t_\alpha = h_Y \cap \alpha^2$;
- $f_\alpha = F \cap \alpha^2$.

Observe that since h_X and h_Y map into 2, $h_X \cap \alpha^2 = h_X \upharpoonright \alpha$ and $h_Y \cap \alpha^2 = h_Y \upharpoonright \alpha$.

Fix $\alpha \in S \cap D$ and we will get a contradiction. Since $X \cap \alpha = x_\alpha$, $h_X(\alpha) = 1$. As α is greater than the least member of the symmetric difference of X and Y , it follows that $h_Y(\alpha) = 0$. Note that we are in Case 2 of the construction as witnessed by $T' := \mathcal{T}'_X \upharpoonright \alpha$ and $c := C \cap \alpha$. Let $f := f_\alpha$, $s := s_\alpha = h_X \upharpoonright \alpha$, and $t := t_\alpha = h_Y \upharpoonright \alpha$. Then $\mathcal{T}_X \upharpoonright (\alpha + 1) = \mathcal{T}_{h_X \upharpoonright (\alpha + 1)} = \mathcal{T}_{s \wedge 1}$ and $\mathcal{T}_Y \upharpoonright (\alpha + 1) = \mathcal{T}_{h_Y \upharpoonright (\alpha + 1)} = \mathcal{T}_{t \wedge 0}$. Also, since $\lambda \cdot \alpha = \alpha$, the underlying set of the tree T_s is α . So $f = F \upharpoonright \alpha = F \upharpoonright T_s$.

We claim that T' is not avoidable. Suppose for a contradiction that T' is avoidable. Then we are in Subcase 2a. By definition, every member of $T_{s \wedge 1}(\alpha) \downarrow$ contains an element which is not in T' . Fix a node z of $\mathcal{T}'_X$ of height α and let $b := \{y \in \mathcal{T}_X : y <_{\mathcal{T}_X} z\}$. Then b has an upper bound in $\mathcal{T}_X(\alpha) = T_{s \wedge 1}(\alpha)$. By assumption, b contains an element which is not in T' . But $z \in \mathcal{T}'_X$ and $\mathcal{T}'_X$ is downwards closed, so $b \subseteq T'$ which is a contradiction.

So indeed, T' is not avoidable and we are in Subcase 2b. By the definition of $T_{s \wedge 1}$ and $T_{t \wedge 0}$ in this case, (a) there exists $x \in T_s$ and $q \in \mathbb{Q}_\lambda$ such that $\pi_s(x) <_{\mathbb{Q}_\lambda} q$ and for all y above x in T_s with $\pi_s(y) <_{\mathbb{Q}_\lambda} q$, $y \in T'$, and (b) there exists $b \in T_{s \wedge 1}(\alpha) \downarrow$ which contains x , has some upper bound $z \in T_{s \wedge 1}(\alpha)$ such that $\pi_{s \wedge 1}(z) <_{\mathbb{Q}_\lambda} q$, $b \subseteq T'$, and $f[b \cap (T_s \upharpoonright c)]$ does not have an upper bound in $T_{t \wedge 0}$.

We claim that $z \in \mathcal{T}'_X$. If not, then $z \in \mathcal{T}_X \setminus \mathcal{T}'_X$ and $\pi_X(z) = \pi_{s \wedge 1}(z) <_{\mathbb{Q}_\lambda} q$. As $\alpha \in D$, by the last property in the description of D it follows that there is some y above x in $(\mathcal{T}_X \upharpoonright \alpha) \setminus \mathcal{T}'_X = T_s \setminus T'$ such that $\pi_X(y) = \pi_s(y) <_{\mathbb{Q}_\lambda} q$. But this contradicts the choice of x and q .

Now z is an upper bound of b in $T_{s \wedge 1}$. As $\alpha \in C$, $z \in \mathcal{T}'_X \upharpoonright C$. Since $F \upharpoonright \alpha = f$, $F(z)$ is an upper bound of $f[b \cap (T_s \upharpoonright c)]$ in $\mathcal{T}_Y(\alpha) = T_{t \wedge 0}(\alpha)$. So $f[b \cap (T_s \upharpoonright c)]$ has an upper bound in $T_{t \wedge 0}$, which contradicts the choice of b .

4. ADDITIONAL RESULTS

In this section we briefly discuss two additional results which are related to Theorem 3.1.

There is an alternative version of Theorem 3.1 which involves Suslin trees in place of special Aronszajn trees.

Theorem 4.1 ([3]). *Let λ be a regular cardinal such that $\lambda^{<\lambda} = \lambda$. Assume $\diamond(S_\lambda^{\lambda^+})$. Then there exists a pairwise far family of 2^{λ^+} many normal λ -complete λ^+ -Suslin trees which are completely club rigid.*

This theorem can be proven by a modification of the proof of Theorem 3.1. Let us point out the main changes which are needed and refer the interested reader to [3] for the details. First, the lemmas of Section 2 can be adjusted to the Suslin tree setting with the following differences. Any references in the lemmas to order preserving maps of trees into $\mathbb{Q}_\lambda$ are removed. The idea of a $\mathbb{Q}_\lambda$ -tethered set of branches is replaced by a *normal* set of branches B , which means for all $x \in T$ there exists $b \in B$ with $x \in b$. We add a lemma which asserts the existence of a normal set of branches such that each branch in the collection meets a given maximal antichain of T . The idea of an avoidable subtree is replaced with the simpler property of a subtree T' satisfying that for all $x \in T$, there exists y above x in $T \setminus T'$.

We then construct a family of Suslin trees with the following changes to Section 3. The diamond sequence is expanded to include a parameter y_α for guessing maximal antichains. For $\alpha \in S_\lambda^{\lambda^+}$, we adjust the case division as follows. Add a new case in which y_α is a maximal antichain of T_{s_α} , and then extend T_{s_α} so that every branch in $T_{s_\alpha \wedge 0}(\alpha) \downarrow$ and $T_{s_\alpha \wedge 1}(\alpha) \downarrow$ meets y_α . This case will allow for the sealing off of potential antichains as in the classical construction of a Suslin tree from diamond. When y_α is not a maximal antichain of T_{s_α} , we use Subcases 1b and 2b as before with the obvious modifications. Subcases 1a and 2a are no longer necessary by the following lemma.

Lemma 4.2. *Let κ be a regular uncountable cardinal and suppose that T is a κ -Suslin tree. Then for any downwards closed subtree T' of T of size κ , there exists $x \in T'$ such that $T_x \subseteq T'$.*

Proof. Suppose for a contradiction that T' is a counterexample to the lemma. We build a sequence $\langle x_i, y_i : i < \kappa \rangle$ by recursion. Let $i < \kappa$ and suppose that for all $j < i$, x_j and y_j are defined. Choose some $x_i \in T'$ with height greater than the heights of x_j and y_j for all $j < i$. Since T_{x_i} is not subset of T' and T' is downwards closed, we can find y_i above x_i in $T \setminus T'$. Now check that $\{y_i : i < \kappa\}$ is an antichain. $\square$

Now we discuss the second result. Abraham-Shelah [2] proved that assuming $2^\omega < 2^{\omega_1}$, there exists a family of 2^{ω_1} many normal special ω_1 -Aronszajn trees which are pairwise non-club embeddable into each other. It is not known whether this theorem generalizes to cardinals greater than ω_1 . For example, it is an open question whether GCH implies the existence of two countably complete normal special ω_2 -Aronszajn trees which are not club isomorphic ([8]).

On the other hand, $2^\omega < 2^{\omega_1}$ is equivalent to the weak diamond principle on ω_1 , and weak diamond can be relativized to stationary sets. In order to provide a generalization of the Abraham-Shelah result to a cardinal $\lambda > \omega$ in place of ω , it is natural to make the stronger assumption that the weak diamond principle holds on the critical cofinality, rather than assuming just $2^\lambda < 2^{\lambda^+}$. Thus, the next theorem appears to be the appropriate generalization of their result to higher cardinals.

Theorem 4.3. *Let λ be a regular cardinal such that $\lambda^{<\lambda} = \lambda$ and assume that the weak diamond principle holds on $S_\lambda^{\lambda^+}$. Then there exists a family of 2^{λ^+} many λ -complete normal club rigid special λ^+ -Aronszajn trees which are pairwise not club embeddable into each other.*

The proof of this theorem is essentially the same as the proof of the case $\lambda = \omega$ from [2], taking into account the standard methods for building higher Aronszajn trees. We refer the interested reader to [2] or [3] for the details.

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