

A STRONG KUREPA TREE

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ABSTRACT. We prove that it is consistent that there exists a Kurepa tree T such that ${}^{\omega_1}2$ is a continuous image of the topological space $[T]$ consisting of all cofinal branches of T with respect to the cone topologies. This result solves an open problem due to Bergfalk, Chodounský, Guzmán, and Hrušák. We also prove that any Kurepa tree with the above property contains an Aronszajn subtree.

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1. INTRODUCTION

In this article we are concerned with the topological space consisting of the set of all cofinal branches $[T]$ of a tree T with height ω_1 with the cone topology. More specifically, a basic open set in this space is any set of the form $\{g \in [T] : x \in g\}$ for some $x \in T$. Recall that a Kurepa tree is an ω_1 -tree (that is, a tree with height ω_1 and countable levels) such that T has at least ω_2 -many cofinal branches. For any Kurepa tree T , there exists a binary downwards closed subtree U of ${}^{<\omega_1}2 = \{f : \exists \alpha < \omega_1 (f : \alpha \rightarrow 2)\}$ such that $[T]$ and $[U]$ are homeomorphic. So $[T]$ can be thought of as a subspace of the generalized Baire space ${}^{\omega_1}2$, which is the set of all functions from ω_1 into 2.

In recent work in generalized descriptive set theory, Bergfalk, Chodounský, Guzmán, and Hrušák introduced the idea of a *strong Kurepa tree*, which is a Kurepa tree T such that ${}^{\omega_1}2$ is a continuous image of $[T]$ with respect to the cone topologies ([BCGH25]). This definition arose out of the following context. Consider the statement that there exists a closed subset of the product space ${}^{\omega_1}2 \times {}^{\omega_1}2$ which is universal for the closed subsets of ${}^{\omega_1}2$, but its universality

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is not absolute with respect to any σ -closed forcing which adds a new subset of ω_1 . They formulated the idea of a strong Kurepa tree as a natural kind of object whose existence implies this statement. Later they were able to derive the statement as a consequence of a somewhat weaker hypothesis which follows from $\diamond^+$. This left open the question of whether the existence of a strong Kurepa tree is consistent. The main result of the article solves this problem affirmatively.

Main Theorem. *Assume CH. Let $1 \leq \kappa \leq 2^{\omega_1}$ be a cardinal. Then there exists a forcing poset which is σ -closed, ω_2 -c.c., and adds a Kurepa tree T and a surjective κ -to-one continuous function $F : [T] \rightarrow {}^{\omega_1}2$.*

We also prove that any strong Kurepa tree contains an Aronszajn subtree.¹ Since there can consistently exist Kurepa trees which do not contain Aronszajn subtrees, not every Kurepa tree is necessarily strong ([Dev74]).

2. A NON-STRONG KUREPA TREE

We begin by proving that the standard generic Kurepa tree obtained by forcing with countable conditions is not strong. This result introduces some key definitions and ideas which we use later and serves as a warm-up for what follows. Throughout the whole article we use the basic methods due to Jech for forcing an ω_1 -tree with countable initial segments ([Jec67], [Jec72]).

For any ordinal γ , let ${}^\gamma 2$ denote the set of all functions from γ to $2 = \{0, 1\}$, and let $\leq^\gamma 2 = \bigcup \{ {}^\beta 2 : \beta < \gamma \}$.

Definition 2.1. *A standard binary countable tree is a countable set $t \subseteq {}^{<\omega_1} 2$ satisfying that for some $\delta_t < \omega_1$:*

- (1) *t is a countable subset of $\leq^{\delta_t} 2$;*
- (2) *if $x \in t$ then for all $\beta < \text{dom}(x)$, $x \restriction \beta \in t$;*
- (3) *for all $x \in t \cap {}^{<\delta_t} 2$, $x \restriction 0$ and $x \restriction 1$ are in t ;*
- (4) *for all $x \in t$, there exists some $y \in t \cap {}^{\delta_t} 2$ such that $x \subseteq y$.*

If t and u are standard binary countable trees, then u end-extends t if $\delta_t \leq \delta_u$ and $u \cap \leq^{\delta_t} 2 = t$.

For a standard binary countable tree t , we write $\text{top}(t)$ for $t \cap {}^{\delta_t} 2$. Note that if u end-extends t and $u \neq t$, then $\delta_t < \delta_u$.

Lemma 2.2. *Suppose that $\langle t_n : n < \omega \rangle$ is a sequence of distinct standard binary countable trees such that for each $n < \omega$, t_{n+1} is an end-extension of t_n . Let $t' = \bigcup \{ t_n : n < \omega \}$ and $\delta = \sup \{ \delta_{t_n} : n < \omega \}$. Assume that B is a countable collection of cofinal branches of t' such that every member of t' belongs to some member of B . Then $t = t' \cup B$ is a standard binary countable tree, $\delta_t = \delta$, $\text{top}(t) = B$, and t end-extends t_n for all $n < \omega$.*

The proof is straightforward.

Definition 2.3 ([Ste66]). *Define $\mathbb{P}^*$ as the forcing poset consisting of conditions p satisfying:*

- (1) *p is a function whose domain is a countable subset of ω_2 ;*
- (2) *$0 \in \text{dom}(p)$ and $p(0)$ is a standard binary countable tree;*
- (3) *for all $\alpha \in \text{dom}(p) \setminus \{0\}$, $p(\alpha) \in \text{top}(p(0))$.*

¹This fact was also observed by Bergfalk, Chodounský, Guzmán, and Hrušák independently.

Let $q \leq p$ if $\text{dom}(p) \subseteq \text{dom}(q)$, $q(0)$ end-extends $p(0)$, and for all $\alpha \in \text{dom}(p) \setminus \{0\}$, $p(\alpha) \subseteq q(\alpha)$.

For notational simplicity, whenever $p \in \mathbb{P}^*$ and $\alpha < \omega_2$, we occasionally write $p(\alpha)$ even when we do not know if $\alpha \in \text{dom}(p)$, and let $p(\alpha)$ denote the empty-set in the case that it is not. For any $p \in \mathbb{P}^*$, we write $\delta_{p(0)}$ as δ_p .

Definition 2.4. Let $\dot{T}^*$ be a $\mathbb{P}^*$ -name for

$$\bigcup \{p(0) : p \in \dot{G}_{\mathbb{P}^*}\},$$

and for each $0 < \alpha < \omega_2$, let $\dot{b}_\alpha$ be a $\mathbb{P}^*$ -name for

$$\bigcup \{p(\alpha) : p \in \dot{G}_{\mathbb{P}^*}\},$$

where $\dot{G}_{\mathbb{P}^*}$ is the canonical name for a generic filter on $\mathbb{P}^*$.

The following summarizes the main facts about $\mathbb{P}^*$, which we state without proof.

Theorem 2.5. The forcing poset $\mathbb{P}^*$ is σ -closed, forces that $\dot{T}^*$ is a normal binary ω_1 -tree, and forces that $\langle \dot{b}_\alpha : 0 < \alpha < \omega_2 \rangle$ is a sequence of distinct cofinal branches of $\dot{T}^*$. If CH holds, then $\mathbb{P}^*$ is ω_2 -c.c. and forces that $\dot{T}^*$ is a Kurepa tree.

The proofs of the next two lemmas are straightforward.

Lemma 2.6. Suppose that $p \in \mathbb{P}^*$. Then there is $q \leq p$ such that for all $x \in p(0)$ there exists some $\alpha \in \text{dom}(q) \setminus \{0\}$ such that $x \subseteq q(\alpha)$.

Lemma 2.7. Suppose that $\langle p_n : n < \omega \rangle$ is a decreasing sequence of conditions in $\mathbb{P}^*$ such that $\delta_{p_n} < \delta_{p_{n+1}}$ for all $n < \omega$. Let $t' = \bigcup \{p_n(0) : n < \omega\}$, $\delta = \sup \{\delta_{p_n} : n < \omega\}$, and $X = \bigcup \{\text{dom}(p_n) : n < \omega\}$. For each $\alpha \in X \setminus \{0\}$, let $b_\alpha = \bigcup \{p_n(\alpha) : n < \omega\}$. Assume that B is a countable set of cofinal branches of t' such that $\{b_\alpha : \alpha \in X\} \subseteq B$ and every element of t' is in some member of B . Define q with domain X so that $q(0) = t' \cup B$ and for all $\alpha \in X \setminus \{0\}$, $q(\alpha) = b_\alpha$. Then $q \in \mathbb{P}^*$, $\delta_q = \delta$, $\text{top}(q(0)) = B$, and $q \leq p_n$ for all $n < \omega$.

The next lemma is the basis for showing that the generic Kurepa tree is not strong.

Lemma 2.8. Suppose that T is a strong Kurepa tree. Then there exists a function $H : T \rightarrow {}^{<\omega_1}2$ satisfying:

- (1) $x <_T y$ implies $H(x) \subseteq H(y)$;
- (2) for all $s \in {}^{<\omega_1}2$, there exists some $x \in T$ such that $s \subseteq H(x)$;
- (3) for any cofinal branch b of T and for all $\zeta < \omega_1$, there exists some $y \in b$ such that $\zeta \leq \text{dom}(H(y))$.

Proof. Fix a surjective continuous function $F : [T] \rightarrow {}^{\omega_1}2$. For each $x \in T$, define

$$H(x) = \bigcap \{F(b) : \exists b \in [T] (x \in b)\}.$$

In other words, $H(x)$ is the largest function which is an initial segment of $F(b)$ for all $b \in [T]$. (1) is obvious. For (2), let $s \in {}^{<\omega_1}2$. Pick some $g \in {}^{\omega_1}2$ such that $s \subseteq g$. Since F is surjective, fix $b \in [T]$ such that $F(b) = g$. By the continuity of F , there exists some $y \in b$ such that for all $c \in [T]$, if $y \in c$ then $s \subseteq F(c)$. Then $s \subseteq H(y)$. (3) Let b be a cofinal branch of T and let $\zeta < \omega_1$. By the continuity of F , fix $y \in b$ such that for all $c \in [T]$, if $y \in c$ then $F(c) \restriction \zeta = F(b) \restriction \zeta$. It follows that $F(b) \restriction \zeta \subseteq H(y)$, so $\zeta \leq \text{dom}(H(y))$. $\square$

Theorem 2.9. *The forcing $\mathbb{P}^*$ forces that $\dot{T}^*$ is not strong.*

Proof. Suppose for a contradiction that $p \in \mathbb{P}^*$ and p forces that $\dot{T}^*$ is a strong Kurepa tree. Fix a $\mathbb{P}^*$ -name $\dot{H}$ satisfying the properties of Lemma 2.8. We define by induction a descending sequence $\langle p_n : n < \omega \rangle$ of conditions in $\mathbb{P}^*$, an increasing sequence of ordinals $\langle \delta_n : n < \omega \rangle$, and a sequence $\langle s(n, \alpha) : n < \omega, \alpha \in \text{dom}(p_n) \setminus \{0\} \rangle$ as follows.

Let $p_0 = p$ and let $\delta_0 = \delta_p$. Now let $n < \omega$ and assume that p_n and δ_n are defined. For each $\alpha \in \text{dom}(p_n) \setminus \{0\}$, apply property (3) of Lemma 2.8 to fix a name $\dot{x}(n, \alpha)$ for a member of $\dot{b}_\alpha$ above $p_n(\alpha)$ such that $\delta_n \leq \text{dom}(\dot{H}(\dot{x}(n, \alpha)))$. By a straightforward argument making use of Lemma 2.6 and the fact that $\mathbb{P}^*$ does not add reals, we can find p_{n+1} , δ_{n+1} , and $\langle s(n, \alpha) : \alpha \in \text{dom}(p_n) \setminus \{0\} \rangle$ satisfying:

- (1) $p_{n+1} \leq p_n$;
- (2) $\delta_n < \delta_{n+1} = \delta_{p_{n+1}}$;
- (3) for all $x \in p_n(0)$ there exists some $\beta \in \text{dom}(p_{n+1})$ such that $x \subseteq p_{n+1}(\beta)$;
- (4) for all $\alpha \in \text{dom}(p_n) \setminus \{0\}$, p_{n+1} forces that $\dot{x}(n, \alpha) \subseteq p_{n+1}(\alpha)$ and $\dot{H}(\dot{x}(n, \alpha)) \restriction \delta_n = s(n, \alpha)$.

This completes the induction. Let $t' = \bigcup \{p_n(0) : n < \omega\}$, $\delta = \sup\{\delta_n : n < \omega\}$, and $X = \bigcup \{\text{dom}(p_n) : n < \omega\}$. For each $\alpha \in X \setminus \{0\}$, define $b_\alpha = \bigcup \{p_n(\alpha) : n < \omega\}$ and $s_\alpha = \bigcup \{s(n, \alpha) : n < \omega, \alpha \in \text{dom}(p_n)\}$. By property (3), every member of t' is contained in b_α for some $\alpha \in X \setminus \{0\}$. By property (4), $s_\alpha \in {}^\delta 2$. Define $B = \{b_\alpha : \alpha \in X \setminus \{0\}\}$ and $t = t' \cup B$.

Let q be the function with domain X such that $q(0) = t$ and for all $\alpha \in X \setminus \{0\}$, $q(\alpha) = b_\alpha$. By Lemma 2.7, $q \in \mathbb{P}^*$, $\delta_q = \delta$, $\text{top}(q(0)) = B$, and $q \leq p_n$ for all $n < \omega$. By property (3) above and property (1) of Lemma 2.8, q forces that for all $\alpha \in X \setminus \{0\}$, $s_\alpha \subseteq \dot{H}(b_\alpha)$.

Since B is countable and ${}^\delta 2$ is uncountable, fix some $s \in {}^\delta 2$ such that for all $\alpha \in X \setminus \{0\}$, $s \neq s_\alpha$. By properties (1) and (2) of Lemma 2.8, we can fix $r \leq q$ and $x \in r(0)$ such that $\text{dom}(x) \geq \delta$ and r forces that $s \subseteq \dot{H}(x)$. Since $r(0)$ end-extends $q(0)$ and $\text{top}(q(0)) = B$, there exists some $\alpha \in X \setminus \{0\}$ such that $x \restriction \delta = b_\alpha$. Then r forces that $s_\alpha \subseteq \dot{H}(b_\alpha) \subseteq \dot{H}(x)$. So r forces that both s_α and s are subsets of the function $\dot{H}(x)$. But s_α and s both have domain δ , so they must be equal, which contradicts the choice of s . $\square$

Next we show that any strong Kurepa tree contains an Aronszajn subtree. This actually gives a second (less direct) proof of Theorem 2.9, since $\mathbb{P}^*$ forces that $\dot{T}^*$ does not contain an Aronszajn subtree.

Proposition 2.10. *Suppose that T is an ω_1 -tree and there exists a function $H : T \rightarrow {}^{<\omega_1} 2$ satisfying:*

- (1) $x <_T y$ implies $H(x) \subseteq H(y)$;
- (2) for all $s \in {}^{<\omega_1} 2$, there exists some $x \in T$ such that $s \subseteq H(x)$;
- (3) for any cofinal branch b of T and for all $\zeta < \omega_1$, there exists some $y \in b$ such that $\zeta \leq \text{dom}(H(y))$.

Then T contains an Aronszajn subtree.

Proof. We begin with the following claim.

Claim: There exists some $x \in T$ such that $H(x) \in {}^{<\omega_1} 2$ and for all $\text{ht}_T(x) < \zeta < \omega_1$, there is some $y \in T_\zeta$ such that $x <_T y$ and $H(x) = H(y)$.

Proof: Suppose for a contradiction that for all $x \in T$, if $H(x) \in {}^{<\omega_1}2$ then there exists some $\text{ht}_T(x) < \zeta_x < \omega_1$ such that for all $y \in T_{\zeta_x}$, if $x <_T y$ then $H(x) \subsetneq H(y)$. Since the levels of T are countable, we can find a club $C \subseteq \omega_1$ such that for all $\delta \in C$, if $x \in T \restriction \delta$ and $H(x) \in 2^{<\omega_1}$, then $\zeta_x < \delta$. Fix some $\theta \in C$ such that $\text{ot}(C \cap \theta) = \theta$. Now consider any $x \in T_\theta$. We prove that $\text{dom}(H(x)) \geq \theta$. First, assume that $H(x) \in 2^{<\omega_1}$. Then $\{\text{dom}(H(x \restriction \gamma)) : \gamma \in C \cap \theta\}$ is strictly increasing and hence has order type $\text{ot}(C \cap \theta) = \theta$. It follows that $\theta \leq \text{dom}(H(x))$. Secondly, assume that $H(x) \in 2^{\omega_1}$. Then $\text{dom}(H(x)) = \omega_1 > \theta$.

Define

$$Z = \{H(x) \restriction \theta : x \in T_\theta\} \cup \{H(y) \restriction \theta : y \in T \restriction \theta, \text{dom}(H(y)) \geq \theta\}.$$

By the previous paragraph, $Z \subseteq {}^\theta 2$. As Z is countable and ${}^\theta 2$ is uncountable, we can find some $s \in {}^\theta 2$ which is not in Z . By property (2), fix $y \in T$ such that $s \subseteq H(y)$. In particular, $\text{dom}(H(y)) \geq \theta$. By the definition of Z and the fact that $s \notin Z$, $y \notin T \restriction \theta$. So $\text{ht}_T(y) \geq \theta$. Then $H(y \restriction \theta) \restriction \theta$ is in Z and is a subset of $H(y)$, and therefore is $\subseteq$ -comparable with s . But both $H(y \restriction \theta) \restriction \theta$ and s have domain θ , so they are equal, contradicting that $s \notin Z$. This completes the proof of the claim.

Now fix x as in the claim. Let U be the set of $y \in T$ such that either $y \leq x$, or $x \leq y$ and $H(y) = H(x)$. By property (1), U is downwards closed, and U is uncountable by the choice of x . Now every $y \in U$ satisfies that $\text{dom}(H(y)) \leq \text{dom}(H(x))$. So by property (3), U does not contain a cofinal branch. $\square$

Corollary 2.11. *Any strong Kurepa tree contains an Aronszjan subtree.*

Proof. By Lemma 2.8 and Proposition 2.10. $\square$

3. THE MAIN IDEAS

We now turn towards proving the consistency of the existence of a strong Kurepa tree. In this section, we describe the basic ideas on which the consistency proof is based. In particular, we use a function H similar to that described in Lemma 2.8 but with stronger properties.

Definition 3.1. *Define a relation $\mathcal{R}$ by letting $\mathcal{R}(T, H)$ hold if:*

- (1) T is a normal binary ω_1 -tree which is a downwards closed subtree of ${}^{<\omega_1}2$;
- (2) $H : T \rightarrow {}^{<\omega_1}2$ is a function satisfying:
 - (a) $H(\emptyset) = \emptyset$;
 - (b) $x <_T y$ implies $H(x) \subseteq H(y)$;
 - (c) whenever $z \in T$ has limit height, then $H(z) = \bigcup \{H(y) : y <_T z\}$;
- (3) for all $x \in T$, for all $\zeta < \omega_1$, and for all $s \in {}^{<\omega_1}2$, if $H(x) \subseteq s$, then there exists some $z \in T$ such that $x \leq_T z$, $\text{dom}(z) \geq \zeta$, and $H(z) = s$.

The following lemma is immediate.

Lemma 3.2. *Assume that $\mathcal{R}(T, H)$ holds. Then any forcing poset which does not add reals forces that $\mathcal{R}(T, H)$ holds.*

Definition 3.3. *Define a relation $\mathcal{R}^*$ by letting $\mathcal{R}^*(T, H, B, F)$ hold if:*

- (1) $\mathcal{R}(T, H)$ holds;
- (2) $B \subseteq [T]$;

(3) $F : B \rightarrow {}^{\omega_1}2$ is a function satisfying that for all $b \in B$,

$$F(b) = \bigcup \{H(b \upharpoonright \gamma) : \gamma < \omega_1\}.$$

Lemma 3.4. *Suppose that $\mathcal{R}^*(T, H, B, F)$ holds. Then $F : B \rightarrow {}^{\omega_1}2$ is continuous.*

Proof. Let $b \in B$ and we show that F is continuous at b . So consider an open set $O \subseteq {}^{\omega_1}2$ such that $F(b) \in O$. By the definition of the cone topology, find some $\beta < \omega_1$ such that for all $h \in {}^{\omega_1}2$, if $h \upharpoonright \beta = F(b) \upharpoonright \beta$, then $h \in O$. Since $F(b) = \bigcup \{H(b \upharpoonright \gamma) : \gamma < \omega_1\}$, there exists some $\gamma < \omega_1$ such that $F(b) \upharpoonright \beta \subseteq H(b \upharpoonright \gamma)$.

Let U be the open subset of B in the cone topology consisting of all $c \in B$ such that $c \upharpoonright \gamma = b \upharpoonright \gamma$. Obviously, $b \in U$. We claim that $F[U] \subseteq O$. So let $c \in U$ and we show that $F(c) \in O$. Then $c \upharpoonright \gamma = b \upharpoonright \gamma$. So

$$F(b) \upharpoonright \beta \subseteq H(b \upharpoonright \gamma) = H(c \upharpoonright \gamma) \subseteq F(c),$$

and hence $F(c) \upharpoonright \beta = F(b) \upharpoonright \beta$. By the choice of β , $F(c) \in O$. $\square$

Corollary 3.5. *Assume that $\mathcal{R}^*(T, H, [T], F)$ holds and the range of F is equal to ${}^{\omega_1}2$. Then ${}^{\omega_1}2$ is a continuous image of $[T]$.*

In Section 5, we prove that assuming CH there exists a σ -closed, ω_2 -c.c. forcing which adds a Kurepa tree T and functions H and F such that $\mathcal{R}^*(T, H, [T], F)$ holds, and the range of F equals ${}^{\omega_1}2$. By Corollary 3.5, this forcing shows the consistency of the existence of a strong Kurepa tree.

4. THE FIRST STEP

Unlike the standard forcing for adding a Kurepa tree discussed in the previous section which is a product forcing, our construction of a strong Kurepa tree involves a forcing iteration. The reason we need to use a forcing iteration instead of a product forcing is because all functions in ${}^{\omega_1}2$ in the generic extension need to be handled in order for the continuous function witnessing strongness to be surjective. In this section, we describe the first step of this forcing iteration. This step adds an ω_1 -tree T and a function $H : T \rightarrow {}^{<\omega_1}2$ such that $\mathcal{R}(T, H)$ holds.

Definition 4.1. *Let $\mathbb{P}$ be the forcing poset consisting of conditions which are pairs (t, h) satisfying:*

- (1) t is a standard binary countable tree;
- (2) $h : t \rightarrow {}^{<\omega_1}2$ is a function such that:
 - (a) $h(\emptyset) = \emptyset$;
 - (b) for all $x, y \in t$, if $x \subseteq y$ then $h(x) \subseteq h(y)$;
 - (c) whenever $z \in t$ has limit height, then $h(z) = \bigcup \{h(y) : y \in t, y \subseteq z\}$;
- (3) for all $x \in t$, there exists some $y \in \text{top}(t)$ such that $h(x) = h(y)$.

The ordering on $\mathbb{P}$ is given by letting $(u, k) \leq (t, h)$ if:

- (4) u end-extends t ;
- (5) $h \subseteq k$.

For any $p \in \mathbb{P}$, we write p as (t_p, h_p) and δ_{t_p} as δ_p . Observe that if $q \leq p$ and $q \neq p$, then $\delta_p < \delta_q$.

Note that $\mathbb{P}$ has size 2^ω , and hence is $(2^\omega)^+$ -c.c.

Lemma 4.2. *Let $p \in \mathbb{P}$. Then there exists $q \leq p$ such that $\delta_q = \delta_p + 1$ and for all $x \in t_q \cap {}^{<\delta_q}2$, there exist distinct z_0 and z_1 in $\text{top}(t_q)$ such that $y \subseteq z_0$, $y \subseteq z_1$, and $h_q(z_0) = h_q(z_1) = h_q(x)$.*

Proof. Define $t_q = t_p \cup \{z \smallfrown i : z \in \text{top}(t_p), i < 2\}$. Define $h_q : t_q \rightarrow {}^{<\omega_1}2$ by letting $h_q \upharpoonright t_p = h_p$ and $h_q(z \smallfrown i) = h_p(z)$ for all $z \in \text{top}(t_p)$ and $i < 2$. It is easy to check that $q = (t_q, h_q)$ is as required. $\square$

Lemma 4.3. *Suppose that $\langle (t_n, h_n) : n < \omega \rangle$ is a decreasing sequence of distinct conditions in $\mathbb{P}$. Let $t' = \bigcup \{t_n : n < \omega\}$ and $h' = \bigcup \{h_n : n < \omega\}$. Then for all $x \in t'$, there exists a cofinal branch b of t' such that $x \subseteq b$ and for all $y \in t'$ with $x \subseteq y \subseteq b$, $h'(y) = h'(x)$.*

Proof. Observe that the function h' has the property that for all $y, z \in t'$, if $y \subseteq z$ then $h'(y) \subseteq h'(z)$. Let $x \in t'$. Using property (3) of Definition 4.1, it is possible to construct a sequence $\langle y_n : n < \omega \rangle$ of elements of t' above x which is cofinal in t' so that for all $n < \omega$, $h'(x) = h'(y_n)$. Let $b = \bigcup \{y_n : n < \omega\}$. $\square$

Lemma 4.4. *Suppose that $\langle (t_n, h_n) : n < \omega \rangle$ is a decreasing sequence of distinct conditions in $\mathbb{P}$. Let $t' = \bigcup \{t_n : n < \omega\}$, $h' = \bigcup \{h_n : n < \omega\}$, and $\delta = \sup \{\delta_{t_n} : n < \omega\}$. Suppose that B is a countable collection of cofinal branches of t' such that for all $x \in t'$, there exists some $b \in B$ such that $x \subseteq b$ and for all $y \in t'$ with $x \subseteq y \subseteq b$, $h'(y) = h'(x)$. Define $t = t' \cup B$. Define $h : t \rightarrow {}^{<\omega_1}2$ so that $h \upharpoonright t' = h'$ and for all $b \in B$, $h(b) = \bigcup \{h'(y) : y \in t', y \subseteq b\}$. Then (t, h) is in $\mathbb{P}$, $\delta_t = \delta$, $\text{top}(t) = B$, and $(t, h) \leq (t_n, h_n)$ for all $n < \omega$.*

The proof is routine.

Lemma 4.5. *The forcing $\mathbb{P}$ is σ -closed.*

Proof. Let $\langle (t_n, h_n) : n < \omega \rangle$ be a decreasing sequence of conditions in $\mathbb{P}$. Define t' and h' as in Lemma 4.4. For each $x \in t'$, apply Lemma 4.3 to fix a cofinal branch b_x of t' such that for all $y \in t'$ with $x \subseteq y \subseteq b_x$, $h'(x) = h'(y)$. Let $B = \{b_x : x \in t'\}$. Now apply Lemma 4.4. $\square$

Lemma 4.6. *For any $p \in \mathbb{P}$ and for any $\zeta < \omega_1$, there exists some $q \leq p$ such that $\delta_q \geq \zeta$.*

Proof. This can be easily proven by induction on ζ using Lemma 4.2 at successor stages and Lemma 4.5 at limit stages. $\square$

Definition 4.7. *For any generic filter G on $\mathbb{P}$, let $T_G = \bigcup \{t : \exists h ((t, h) \in G)\}$ and $H_G = \bigcup \{h : \exists t ((t, h) \in G)\}$. Let $\dot{T}$ and $\dot{H}$ be $\mathbb{P}$ -names for these objects.*

Lemma 4.8. *Let G be a generic filter on $\mathbb{P}$. Then in $V[G]$:*

- (1) T_G is a normal binary ω_1 -tree which is a downwards closed subtree of ${}^{<\omega_1}2$;
- (2) $H_G : T_G \rightarrow {}^{<\omega_1}2$ is a function satisfying:
 - (a) $H(\emptyset) = \emptyset$;
 - (b) $x <_{T_G} y$ implies $H(x) \subseteq H(y)$;
 - (c) if $z \in T_G$ has limit height, then $H_G(z) = \bigcup \{H_G(y) : y <_{T_G} z\}$.

Proof. The height of T_G is ω_1 by Lemma 4.6. The rest of the properties stated in the lemma follow from the fact that if $(t, h) \in G$, then $T_G \upharpoonright (\delta_t + 1) = t$ and $H_G \upharpoonright t = h$. $\square$

Lemma 4.9. *Let G be a generic filter on $\mathbb{P}$. Then for all $x \in T_G$, for all $\zeta < \omega_1$, and for all $s \in {}^{<\omega_1}2$, if $H_G(x) \subseteq s$, then there exists some $z \in T_G$ such that $x \leq_{T_G} z$, $\text{ht}_T(z) \geq \zeta$, and $H_G(z) = s$.*

Proof. We prove the lemma by using a density argument in V . Let $(t, h) \in \mathbb{P}$, let $\zeta < \omega_1$, and suppose that (t, h) forces that $\dot{H}(\dot{x}) \subseteq \dot{s} \in {}^{<\omega_1}2$. Since $\mathbb{P}$ is σ -closed, by extending further if necessary we may assume that $\delta_t \geq \zeta$ and for some $x \in t$ and for some $s \in {}^{<\omega_1}2$, (t, h) forces that $\dot{x} = x$, $\dot{s} = s$, and $\dot{H}(\dot{x}) = h(x) \subseteq s$. Fix some $y \in \text{top}(t)$ such that $x \subseteq y$ and $h(x) = h(y)$. Then $\text{dom}(y) \geq \zeta$.

Define

$$u = t \cup \{w \frown i : w \in \text{top}(t), i < 2\}.$$

Define $k : u \rightarrow {}^{<\omega_1}2$ by letting $k \upharpoonright t = h$ and for all $w \in \text{top}(t)$ and for all $i < 2$:

- if $w \neq y$ then $k(w \frown i) = h(w)$;
- if $w = y$ and $i = 0$ then $k(w \frown i) = h(w)$;
- if $w = y$ and $i = 1$ then $k(w \frown i) = s$.

Let $z = y \frown 1$. It is easy to check that (u, k) is a condition which extends (t, h) and forces that $\dot{x} \leq_{\dot{T}} z$ and $\dot{H}(z) = k(z) = s$. $\square$

Corollary 4.10. *The forcing poset $\mathbb{P}$ forces that $\mathcal{R}(\dot{T}, \dot{H})$ holds.*

Proof. By Lemmas 4.8 and 4.9. $\square$

5. THE GENERAL STEP

We now describe the kind of forcings which we use after the first step in the forcing iteration of the proof of the main theorem.

Definition 5.1. *Assume that $\mathcal{R}(T, H)$ holds and $g \in {}^{\omega_1}2$. Define $\mathbb{Q}(T, H, g)$ to be the forcing poset whose conditions are elements $x \in T$ such that $H(x) \subseteq g$, ordered by $y \leq x$ if $x \leq_T y$.*

Lemma 5.2. *Assume that $\mathcal{R}(T, H)$ holds and $g \in {}^{\omega_1}2$. Then:*

- (1) $\emptyset \in \mathbb{Q}(T, H, g)$;
- (2) *for all $x \in \mathbb{Q}(T, H, g)$ and for all $\zeta < \omega_1$, there exists some $z \in \mathbb{Q}(T, H, g)$ such that $x \leq_T z$ and $\text{ht}_T(z) \geq \zeta$.*

Proof. The first statement follows from the fact that $H(\emptyset) = \emptyset$. For the second statement, consider $x \in \mathbb{Q}(T, H, g)$ and $\zeta < \omega_1$. Then $H(x) \subseteq g$. By Definition 3.1(3) applied to $s = H(x)$, there exists some $z \in T$ such that $x \leq_T z$, $\text{ht}_T(z) \geq \zeta$, and $H(z) = H(x)$. Then $H(z) \subseteq g$ so $z \in \mathbb{Q}(T, H, g)$. $\square$

Definition 5.3. *Assume that $\mathcal{R}(T, H)$ holds and $g \in {}^{\omega_1}2$. Let $\dot{d}(T, H, g)$ be a $\mathbb{Q}(T, H, g)$ -name for the union of the generic filter.*

Lemma 5.4. *Assume that $\mathcal{R}(T, H)$ holds and $g \in {}^{\omega_1}2$. Let G be a generic filter on $\mathbb{Q}(T, H, g)$. Let $d = \dot{d}(T, H, g)^G$. Then d is a cofinal branch of T and $\bigcup \{H(d \upharpoonright \beta) : \beta < \omega_1\} = g$.*

Proof. The fact that d is a cofinal branch of T follows easily from Lemma 5.2(2). Note that $G = \{d \upharpoonright \beta : \beta < \omega_1\}$. Therefore, $\bigcup \{H(d \upharpoonright \beta) : \beta < \omega_1\}$ is a subset of g by the definition of $\mathbb{Q}(T, H, g)$.

Conversely, let $\gamma < \omega_1^V$ and we show that $g \restriction \gamma \subseteq H(z)$ for some $z \in G$. We give a density argument in V . So let $x \in \mathbb{Q}(T, H, g)$, and we find some $z \in \mathbb{Q}(T, H, g)$ such that $x \leq_T z$ and $g \restriction \gamma \subseteq H(z)$. Since $x \in \mathbb{Q}(T, H, g)$, $H(x) = g \restriction \text{dom}(H(x))$. If $\gamma \leq \text{dom}(H(x))$, then $g \restriction \gamma \subseteq H(x)$ and we are done. Otherwise, $\text{dom}(H(x)) < \gamma$ so $H(x) \subseteq g \restriction \gamma$. Applying Definition 3.1(3) to x and $g \restriction \gamma$, there exists some $z \in T$ such that $x \leq_T z$ and $H(z) = g \restriction \gamma$. Then z is as required. $\square$

Lemma 5.5. *Assume that $\mathcal{R}(T, H)$ holds and $g \in {}^{\omega_1}2$. Suppose that $\langle x_n : n < \omega \rangle$ is a descending sequence of conditions in $\mathbb{Q}(T, H, g)$ such that $y = \bigcup \{x_n : n < \omega\}$ is in T . Then $y \in \mathbb{Q}(T, H, g)$ and for all $n < \omega$, $y \leq x_n$.*

Proof. For all $n < \omega$, $H(x_n) \subseteq g$. By Definition 3.1(2c), $H(y) = \bigcup \{H(x_n) : n < \omega\} \subseteq g$. So $y \in \mathbb{Q}(T, H, g)$. $\square$

6. THE FORCING ITERATION

We now develop the forcing iteration which we use in the next section to prove the main theorem.

Definition 6.1. *A countable support forcing iteration $\langle \mathbb{P}_i, \dot{\mathbb{Q}}_j : i \leq \theta, j < \theta \rangle$ is suitable if $\theta \geq 1$ and there exists a sequence $\langle \dot{g}_i : 0 < i < \theta \rangle$ such that the following are satisfied:*

- (1) $\dot{\mathbb{Q}}_0$ is a $\mathbb{P}_0$ -name for $\mathbb{P}$;
- (2) for all $0 < i < \theta$, $\mathbb{P}_i$ does not add reals;
- (3) for all $0 < i < \theta$, $\mathbb{P}_i$ forces that $\dot{g}_i \in {}^{\omega_1}2$ and $\dot{\mathbb{Q}}_i = \mathbb{Q}(\dot{T}, \dot{H}, \dot{g}_i)$.

Note that we are not assuming as a matter of definition that $\mathbb{P}_\theta$ does not add reals. The fact that it does not, which we prove later, allows us to define suitable forcing iterations by recursion.

The main result of this section is the following theorem.

Theorem 6.2. *Suppose that $\langle \mathbb{P}_i, \dot{\mathbb{Q}}_j : i \leq \theta, j < \theta \rangle$ is a countable support forcing iteration which is suitable. Then:*

- (1) $\mathbb{P}_\theta$ contains a σ -closed dense subset D_θ , and hence does not add reals;
- (2) $|D_\theta| = |\max\{2, \theta\}|^\omega$;
- (3) $\mathbb{P}_\theta$ is $(2^\omega)^+$ -c.c.

For the rest of the section, fix a suitable forcing iteration $\langle \mathbb{P}_i, \dot{\mathbb{Q}}_j : i \leq \theta, j < \theta \rangle$ and a sequence $\langle \dot{g}_i : 0 < i < \theta \rangle$ as in Definition 6.1.

Lemma 6.3. *Let $i \leq \theta$. If $p \in \mathbb{P}_i$ and $\gamma < i$, then there exists $q \leq p$ such that $\gamma \in \text{dom}(p)$.*

Proof. If $\gamma \in \text{dom}(p)$ then we are done, so assume not. Define a function q so that $\text{dom}(q) = \text{dom}(p) \cup \{\gamma\}$, $q \restriction \text{dom}(p) = p$, and $q(\gamma)$ is a $\mathbb{P}_\gamma$ -name for the empty-set. $\square$

Definition 6.4. *For each $0 < i \leq \theta$, let D_i be the set of $p \in \mathbb{P}_i$ such that there exists a function p' satisfying the following:*

- (1) $\text{dom}(p) = \text{dom}(p')$;
- (2) $0 \in \text{dom}(p')$;
- (3) $p'(0) = (t_p, h_p) \in \mathbb{P}$;
- (4) for all $\alpha \in \text{dom}(p) \setminus \{0\}$, $p'(\alpha)$ is in $\text{top}(t_p)$;

(5) for all $\alpha \in \text{dom}(p)$, $p \restriction \alpha \Vdash_\alpha p(\alpha) = p'(\alpha)$.

Note that the function p' above is unique. We adopt the same notational convention about functions p' for $p \in D_i$ as described in the comment after Definition 2.3 for conditions in $\mathbb{P}^*$. For any $p \in D_i$, we write δ_{t_p} as δ_p .

The next two lemmas are easy.

Lemma 6.5. *The set D_1 has size 2^ω and for all $1 < \tau \leq \theta$, D_τ has size $|\tau|^\omega$.*

Lemma 6.6. *For all $0 < i < j < \theta$, $D_i \subseteq D_j$, $D_i = \{p \restriction i : p \in D_j\}$, and if $p \in D_j$ as witnessed by p' , then $p \restriction i \in D_i$ as witnessed by $p' \restriction i$.*

Lemma 6.7. *Let $i \leq \theta$. Suppose that $\langle p_n : n < \omega \rangle$ is a descending sequence of conditions in D_i with corresponding witnesses $\langle p'_n : n < \omega \rangle$ such that $\delta_{p_n} < \delta_{p_{n+1}}$ for all $n < \omega$. Let $t' = \bigcup \{t_{p_n} : n < \omega\}$, $h' = \bigcup \{h_{p_n} : n < \omega\}$, $\delta = \sup \{\delta_{p_n} : n < \omega\}$, and $X = \bigcup \{\text{dom}(p_n) : n < \omega\}$. Suppose that B is any countable set of cofinal branches of t' satisfying that (a) for all $x \in t'$ there exists $b \in B$ such that $x \subseteq b$ and for all $y \in t'$ with $x \subseteq y \subseteq b$, $h'(x) = h'(y)$, and (b) for all $\alpha \in X$, $b_\alpha = \bigcup \{p'_n(\alpha) : n < \omega\}$ is in B . Then there exists some $q \in D_i$ such that $\delta_q = \delta$, $\text{top}(t_q) = B$, and $q \leq p_n$ for all $n < \omega$.*

Proof. Define t , h , q' , and q as follows:

- $t = t' \cup B$;
- h has domain t , $h \restriction t' = h'$, and for all $b \in B$, $h(b) = \bigcup \{h'(b \restriction \gamma) : \gamma < \delta\}$;
- q' has domain X , $q'(0) = (t, h)$, and for all $\alpha \in X \setminus \{0\}$, $q'(\alpha) = b_\alpha$;
- q has domain X , and for all $\alpha \in X$, $q(\alpha)$ is a $\mathbb{P}_\alpha$ -name for $q'(\alpha)$.

Using Lemmas 4.4 and 5.5, it is straightforward to check that q is as required. $\square$

Lemma 6.8. *For all $0 < i \leq \theta$, D_i is σ -closed.*

Proof. Let $\langle p_n : n < \omega \rangle$ be a descending sequence of conditions in D_i with corresponding witnesses $\langle p'_n : n < \omega \rangle$.

Case 1: There exists an infinite set $A \subseteq \omega$ such that for all $m < n$ in A , $\delta_{p_n} < \delta_{p_m}$. Then we are done by Lemmas 4.3 and 6.7.

Case 2: There exists some $m < \omega$ such that for all $m \leq n < \omega$, $p'_n(0) = p'_m(0)$. Let $X = \bigcup \{\text{dom}(p_n) : n < \omega\}$. For each $\alpha \in X \setminus \{0\}$, let $b_\alpha = p'_n(\alpha)$ for some (any) n such that $m \leq n < \omega$ and $\alpha \in \text{dom}(p_n)$. Define q' with domain X so that $q'(0) = p'_m(0)$ and $q'(\alpha) = b_\alpha$ for all $\alpha \in X \setminus \{0\}$. Define q with domain X so that for all $\alpha \in X$, $q(\alpha)$ is a $\mathbb{P}_\alpha$ -name for $q'(\alpha)$. It is routine to check that $q \in D_i$ as witnessed by q' and $q \leq p_n$ for all $n < \omega$. $\square$

Lemma 6.9. *For all $0 < i \leq \theta$, D_i is a dense subset of $\mathbb{P}_i$.*

Proof. We prove the statement by induction on $i \leq \theta$. The case that $i = 1$ is trivial, so assume that $i > 1$.

Case 1: i is a successor ordinal. Let $i = i_0 + 1$. By the inductive hypothesis, D_{i_0} is dense in $\mathbb{P}_{i_0}$. Let $p \in \mathbb{P}_i$. By extending further if necessary, we may assume that $i_0 \in \text{dom}(p)$.

Define by induction sequences $\langle v_n : n < \omega \rangle$, $\langle v'_n : n < \omega \rangle$, $\langle \delta_n : n < \omega \rangle$, and $\langle x_n : n < \omega \rangle$ as follows. Extend $p \restriction i_0$ to some v_0 in D_{i_0} which decides $p(i_0)$ as some x_0 , where $\delta_0 = \delta_{v_0} > \text{dom}(x_0)$. Let v'_0 be the witness that $v_0 \in D_{i_0}$. Now let $n < \omega$ and suppose we have defined v_n , v'_n , δ_n , and x_n , where $v_n \in D_{i_0}$ with witness v'_n , $\delta_n = \delta_{v_n}$, and $\text{dom}(x_n) \subseteq \delta_n$.

Fix a $\mathbb{P}_{i_0}$ -name $\dot{y}$ for an extension of x_n in $\mathbb{Q}(\dot{T}, \dot{H}, \dot{g}_{i_0})$ with $\text{dom}(\dot{y}) = \delta_n$. Fix $v_{n+1} \leq v_n$ in D_{i_0} and some x_{n+1} such that v_{n+1} forces that $\dot{y} = x_{n+1}$, where $\delta_{n+1} = \delta_{v_{n+1}} > \delta_n$. Let v'_{n+1} be the witness that $v_{n+1} \in D_{i_0}$.

Let $t' = \bigcup \{t_{v_n} : n < \omega\}$, $h' = \bigcup \{h_{v_n} : n < \omega\}$, $\delta = \sup\{\delta_n : n < \omega\}$, and $X = \bigcup \{\text{dom}(v_n) : n < \omega\}$. For each $x \in t'$, apply Lemma 4.3 to fix a cofinal branch b_x of t' such that $x \subseteq b_x$ and for all $y \in t'$ with $x \subseteq y \subseteq b_x$, $h'(x) = h'(y)$. For each $\alpha \in X \setminus \{0\}$ define $b_\alpha = \bigcup \{v'_n(\alpha) : n < \omega\}$. Define $b = \bigcup \{x_n : n < \omega\}$. Finally, define $B = \{b_x : x \in t'\} \cup \{b_\alpha : \alpha \in X \setminus \{0\}\} \cup \{b\}$. By Lemma 6.7 applied to D_{i_0} , fix some $v \in D_{i_0}$ such that $t_v = t' \cup B$, $\delta_v = \delta$, and $v \leq v_n$ for all $n < \omega$. Now let $q = v \restriction \check{b}$. It is routine to check that $q \in D_i$ and $q \leq p$.

Case 2: i is a limit ordinal. By the inductive hypothesis, for all $j < i$, D_j is dense in $\mathbb{P}_j$. First, assume that i has uncountable cofinality. Consider $p \in \mathbb{P}_i$. Then for some $\zeta < i$, $p \in \mathbb{P}_\zeta$. Applying the inductive hypothesis, fix $q \leq_\zeta p$ in D_ζ . Then $q \leq_i p$ and $q \in D_i$.

Secondly, assume that i has cofinality ω . Fix an increasing sequence $\langle i_n : n < \omega \rangle$ of ordinals which is cofinal in i . Let $p \in \mathbb{P}_i$. Define by induction a descending sequence $\langle v_n : n < \omega \rangle$ of conditions in D_i as follows. Let $v_0 \leq p \restriction i_0$ be in D_{i_0} . Now let $n < \omega$ and suppose that v_n is defined, is in D_{i_n} , and is below $p \restriction i_n$. Then $v_n \cup p \restriction [i_n, i_{n+1})$ is in $\mathbb{P}_{i_{n+1}}$, so we can find some $v_{n+1} \in D_{i_{n+1}}$ which extends it. This completes the construction. By Lemma 6.8, there exists some $q \in D_i$ such that $q \leq v_n$ for all $n < \omega$. Then easily $q \leq p$. $\square$

Lemma 6.10. *The forcing $\mathbb{P}_\theta$ is $(2^\omega)^+$ -c.c.*

Proof. Since D_θ is dense in $\mathbb{P}_\theta$, it suffices to prove that D_θ is $(2^\omega)^+$ -c.c. Note that for any $p \in D_\theta$ and $\alpha \in \text{dom}(p)$, there are 2^ω -many possibilities for $p'(\alpha)$. Using this fact, the chain condition follows by a standard application of the Δ -system lemma applied to the domains of conditions. $\square$

Theorem 6.2 now follows from Lemmas 6.5, 6.8, 6.9, and 6.10.

Definition 6.11. *For each $0 < i < \theta$, let $\dot{d}_i$ be a $\mathbb{P}_\theta$ -name for $\bigcup G_{\dot{\mathbb{Q}}_i}$, where $G_{\dot{\mathbb{Q}}_i}$ is a $\mathbb{P}_\theta$ -name for the generic filter on $\dot{\mathbb{Q}}_i$ induced by the generic filter on $\mathbb{P}_\theta$. Let $\dot{B}_\theta$ be a $\mathbb{P}_\theta$ -name for the collection $\{\dot{d}_i : 0 < i < \theta\}$.*

Proposition 6.12. *The forcing $\mathbb{P}_\theta$ forces that $[\dot{T}] = \dot{B}_\theta$.*

Proof. By Lemma 5.4, $\mathbb{P}_\theta$ forces that $\dot{B}_\theta \subseteq [\dot{T}]$. Suppose for a contradiction that $p \in \mathbb{P}_\theta$ and p forces that $\dot{b}$ is a cofinal branch of $\dot{T}$ which is not in $\dot{B}_\theta$. Define by induction a decreasing sequence of conditions $\langle p_n : n < \omega \rangle$ in D_θ , an increasing sequence of ordinals $\langle \delta_n : n < \omega \rangle$, and a sequence $\langle b_n : n < \omega \rangle$ as follows. Let p_0 be some extension of p in D_θ and define $\delta_0 = \delta_{p_0}$.

Let $n < \omega$ and assume that p_n and δ_n are defined. Find a countable ordinal $\zeta > \delta_n$ and some $p_n^0 \leq p_n$ such that p_n^0 forces that for all $\alpha \in \text{dom}(p_n)$, $\dot{b} \restriction \zeta \neq \dot{d}_\alpha \restriction \zeta$. Using the fact that $\mathbb{P}_\theta$ does not add reals, find $p_n^1 \leq p_n^0$ in D_θ and some b_n such that p_n^1 forces that $\dot{b} \restriction \zeta = b_n$ and $\delta_{p_n^1} \geq \zeta$. Now extend p_n^1 to p_n^2 so that $p_n^2(0)$ satisfies the property described in Lemma 4.2. Finally, let p_{n+1} be an extension of p_n^2 in D_θ and let $\delta_{n+1} = \delta_{p_{n+1}}$. This completes the induction.

Define the following objects:

- $t' = \bigcup \{t_{p_n} : n < \omega\}$;

- $h' = \bigcup \{h_{p_n} : n < \omega\}$;
- $\delta = \sup \{\delta_n : n < \omega\}$;
- $X = \bigcup \{\text{dom}(p_n) : n < \omega\}$;
- $b^* = \bigcup \{b_n : n < \omega\}$;
- for all $\alpha \in X \setminus \{0\}$, $b_\alpha = \bigcup \{p'_n(\alpha) : n < \omega\}$.

Consider $x \in t'$. Let $n < \omega$ be least such that $x \in t_{p_n}$. By the choice of p_n^2 , we can find $x_1 \in t_{p_{n+1}}$ such that $x \subseteq x_1$, $h'(x_1) = h'(x)$, and $x_1 \not\subseteq b^*$. Now apply Lemma 4.3 to find a cofinal branch b_x of t' satisfying that $x_1 \subseteq b_x$ and for all $y \in t'$ with $x_1 \subseteq y \subseteq b_x$, $h'(y) = h'(x_1)$. Note that $b_x \neq b^*$ and for all $y \in t'$ with $x \subseteq y \subseteq b_x$, $h'(y) = h'(x)$. Consider $\alpha \in X \setminus \{0\}$. Let $n < \omega$ be least such that $\alpha \in \text{dom}(p_n)$. Then by the choice of p_n^0 and p_n^1 , $p'_{n+1}(\alpha) \restriction \delta_{n+1} \neq b^* \restriction \delta_{n+1}$. Consequently, for all $\alpha \in X \setminus \{0\}$, $b_\alpha \neq b^*$.

Let $B = \{b_x : x \in t'\} \cup \{b_\alpha : \alpha \in X \setminus \{0\}\}$. Observe that $b^* \notin B$. Apply Lemma 6.7 to find $q \in D_\theta$ such that $\delta_q = \delta$, $\text{top}(t_q) = B$, and $q \leq p_n$ for all $n < \omega$. Then $q \Vdash \dot{b} \restriction \delta = b^* \notin B = \text{top}(t_q) = \dot{T}_\delta$, which is a contradiction to the fact that q forces that $\dot{b}$ is cofinal in $\dot{T}$. $\square$

7. THE MAIN THEOREM

We are now prepared to prove the main theorem of the article.

Theorem 7.1. *Assume CH. Let $1 \leq \kappa \leq 2^{\omega_1}$ be a cardinal. Then there exists a forcing poset which is σ -closed, ω_2 -c.c., and forces that there exists a Kurepa tree T , a function $H : T \rightarrow {}^{<\omega_1}2$, and a function $F : [T] \rightarrow {}^{\omega_1}2$ such that F is κ -to-one and $\mathcal{R}^*(T, H, [T], F)$ holds.*

Proof. We define by recursion a countable support forcing iteration $\langle \mathbb{P}_i, \dot{\mathbb{Q}}_j : i \leq 2^{\omega_1}, j < 2^{\omega_1} \rangle$ as follows. Our inductive hypothesis is that for all $\alpha \leq 2^{\omega_1}$, $\langle \mathbb{P}_i, \dot{\mathbb{Q}}_j : i \leq \alpha, j < \alpha \rangle$ is suitable and $\mathbb{P}_\alpha$ does not add reals. Let $\mathbb{P}_0$ be the trivial forcing, let $\dot{\mathbb{Q}}_0$ be a $\mathbb{P}_0$ -name for $\mathbb{P}$, and let $\mathbb{P}_1 = \mathbb{P}_0 * \dot{\mathbb{Q}}_0$.

Successor stage: Let $1 \leq \alpha < 2^{\omega_1}$ and assume that $\langle \mathbb{P}_i, \dot{\mathbb{Q}}_j : i \leq \alpha, j < \alpha \rangle$ is defined as required. By Lemma 3.2 applied to the quotient forcing $\mathbb{P}_\alpha / \dot{G}_{\mathbb{P}_1}$ in $V^{\mathbb{P}_1}$, $\mathbb{P}_\alpha$ forces that $\mathcal{R}(\dot{T}, \dot{H})$ holds. Using a booking device which we describe in more detail later in the proof, we fix a nice D_α -name $\dot{g}_\alpha$ for a function in ${}^{\omega_1}2$. Let $\dot{\mathbb{Q}}_\alpha$ be a $\mathbb{P}_\alpha$ -name for $\mathbb{Q}(\dot{T}, \dot{H}, \dot{g}_\alpha)$ and let $\mathbb{P}_{\alpha+1} = \mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha$. Clearly, $\langle \mathbb{P}_i, \dot{\mathbb{Q}}_j : i \leq \alpha + 1, j < \alpha + 1 \rangle$ is suitable, and $\mathbb{P}_{\alpha+1}$ does not add reals by Theorem 6.2.

Limit stage: Let $\delta \leq 2^{\omega_1}$ be a limit ordinal and suppose that for all $i < \delta$, $\mathbb{P}_i$ and $\dot{\mathbb{Q}}_i$ are defined such that for all $\alpha < \delta$, $\langle \mathbb{P}_i, \dot{\mathbb{Q}}_j : i \leq \alpha, j < \alpha \rangle$ is suitable and $\mathbb{P}_\alpha$ does not add reals. Define $\mathbb{P}_\delta$ as the countable support limit of $\langle \mathbb{P}_i : i < \delta \rangle$. It is easy to check that $\langle \mathbb{P}_i, \dot{\mathbb{Q}}_j : i \leq \delta, j < \delta \rangle$ is suitable, and $\mathbb{P}_\delta$ does not add reals by Theorem 6.2.

This completes the construction. Let $\mathbb{P} = \mathbb{P}_{2^{\omega_1}}$. By Theorem 6.2 and CH, $D_{2^{\omega_1}}$ has size 2^{ω_1} and $\mathbb{P}$ is ω_2 -c.c. So by standard methods we can arrange our bookkeeping device so that every member of ${}^{\omega_1}2$ in $V^{\mathbb{P}}$ is named by $\dot{g}_\alpha$ for exactly κ -many ordinals $\alpha < 2^{\omega_1}$. Let $\dot{F}$ be a $\mathbb{P}$ -name for the function with domain $\dot{B}_{2^{\omega_1}}$ such that for all $0 < i < 2^{\omega_1}$, $\dot{F}(\dot{d}_i) = \dot{g}_i$. Note that $\mathbb{P}$ forces that $\dot{F}$ is κ -to-one. By Lemma 5.4, $\mathbb{P}$ forces that for all $0 < i < 2^{\omega_1}$, $\dot{F}(\dot{d}_i) = \bigcup \{\dot{H}(\dot{d}_i \restriction \gamma) : \gamma < \omega_1\}$. Since $\mathbb{P}$ does not add reals, applying Lemma 3.2 to the quotient forcing $\mathbb{P} / \dot{G}_{\mathbb{P}_1}$ in $V^{\mathbb{P}_1}$, $\mathbb{P}$ forces that $\mathcal{R}(\dot{T}, \dot{H})$ holds. By Proposition 6.12, $\mathbb{P}$ forces that $[T] = \dot{B}_{2^{\omega_1}}$. So $\mathbb{P}$ forces that $\mathcal{R}^*(\dot{T}, \dot{H}, [\dot{T}], \dot{F})$ holds. $\square$

Corollary 7.2. *Assume CH. Let $1 \leq \kappa \leq 2^{\omega_1}$ be a cardinal. Then there exists a forcing poset which does not add reals, is ω_2 -c.c., and adds a normal binary Kurepa tree T and a continuous κ -to-one surjective function $F : [T] \rightarrow {}^{\omega_1}2$.*

We note that the forcing of Theorem 7.1 preserves CH because it does not add reals, preserves all cardinals because it is ω_2 -c.c., and by an easy counting argument, does not change the value of 2^{ω_1} . In particular, the existence of a strong Kurepa tree is consistent with arbitrarily large values of 2^{ω_1} .

8. REMARKS AND OPEN PROBLEMS

The Kurepa trees we considered in this article are mostly downwards closed binary subtrees of $2^{<\omega_1}$. This focus does not provide any limitations for the following reason.

Lemma 8.1. *Suppose that T is a Kurepa tree. Then there exists a Kurepa tree U which is a binary downwards closed subtree of ${}^{<\omega_1}2$ such that $[T]$ and $[U]$ are homeomorphic.*

Proof. Modify T in several steps as follows. Add a root to T to obtain T_0 . Insert new limit levels to T_0 to obtain T_1 which is Hausdorff. Between successive elements of T_1 add a copy of ${}^{<\omega}2$ preserving Hausdorffness to obtain T_2 . It is easy to show that each of these steps preserves the homeomorphism type of the space of cofinal branches. Now T_2 is rooted, binary, and Hausdorff. By a straightforward argument, any tree with these three properties is isomorphic to a binary downwards closed subtree of ${}^{<\omega_1}2$. $\square$

The existence of a strong Kurepa tree implies CH. Indeed, suppose that T is a Kurepa tree and $F : [T] \rightarrow {}^{\omega_1}2$ is continuous and surjective. Then T has size ω_1 , and therefore the cones of T provide a base for $[T]$ with size ω_1 . Assuming that CH is false, let $\langle s_i : i < \omega_2 \rangle$ enumerate distinct functions in ${}^{\omega_2}2$, and for each $i < \omega_2$ let U_i be the open set $\{g \in {}^{\omega_1}2 : s_i \subseteq g\}$. Then for any $i < j < \omega_2$, $U_i \cap U_j = \emptyset$, and hence $F^{-1}(U_i) \cap F^{-1}(U_j) = \emptyset$. Since $[T]$ has a base of size ω_1 and F is continuous, there must exist some $i < \omega_2$ such that $F^{-1}(U_i)$ is empty. So F is not surjective. In particular, generic Kurepa trees obtained by standard c.c.c. forcing posets assuming $\square_{\omega_1}$ are not strong ([Vel92], [Tod07]).

A natural question is whether the main theorem of the article can be improved to get a Kurepa tree T such that $[T]$ is homeomorphic to ${}^{\omega_1}2$. It cannot by the following theorem of Lücke and Schlicht ([LS20]).

Theorem 8.2. *Assume CH. Let T be a Kurepa tree which is a downwards closed subtree of ${}^{\omega_1}\omega_1$. Then $[T]$ is not the continuous image of ${}^{\omega_1}\omega_1$.*

Corollary 8.3. *Let T be a Kurepa tree. Then $[T]$ is not homeomorphic to ${}^{\omega_1}2$.*

Proof. Suppose for a contradiction that $[T]$ is homeomorphic to ${}^{\omega_1}2$. Then T is a strong Kurepa tree. By the remarks above, CH holds. But ${}^{\omega_1}2$ and ${}^{\omega_1}\omega_1$ are homeomorphic ([HN73]). So $[T]$ is homeomorphic to ${}^{\omega_1}\omega_1$, which contradicts Theorem 8.2. $\square$

In the absence of CH, there are still restrictions on the space of all cofinal branches of a Kurepa tree being the continuous image of ${}^{\omega_1}2$. Recall that a subspace Y of a topological space X is a *retract* of X if there exists a continuous function $F : X \rightarrow Y$ such that $F(y) = y$ for all $y \in Y$.

Theorem 8.4. *Suppose that T is a Kurepa tree which is a downwards closed subtree of ${}^{<\omega_1}2$. Then $[T]$ is not a retract of ${}^{\omega_1}2$.*

This theorem is essentially due to Lücke and Schlicht, who proved that for any infinite cardinal μ , if T is a μ^+ -Kurepa tree (see the definition given below) which is a downwards closed subtree of ${}^{<\mu^+}(\mu^+)$ such that $[T]$ has no isolated points, then $[T]$ is not a retract of $\mu^+(\mu^+)$ ([LS20]). They also showed that if $V = L$, then for any cardinal μ with uncountable cofinality, there exists a μ^+ -Kurepa tree T such that $[T]$ is a retract of $\mu^+(\mu^+)$. It turns out, however, in the special case that $\mu = \omega$, the argument of their first result can be adjusted to eliminate the assumption of not having isolated points ([Kru25]).

Since the existence of a strong Kurepa tree implies CH whereas the existence of a Kurepa tree does not, it is consistent that there exists a Kurepa tree and there does not exist a strong Kurepa tree.

Question 8.5. *Is it consistent with CH that there exists a Kurepa tree but there does not exist a strong Kurepa tree?*

In the other extreme, we ask the following.

Question 8.6. *Is it consistent that CH holds, there exists a Kurepa tree, and every Kurepa tree is a strong Kurepa tree?*

Note that by Corollary 2.11, any model which gives an affirmative answer to Question 8.6 satisfies that there exists a Kurepa tree and every Kurepa tree contains an Aronszajn subtree. A model with this property was constructed by Komjáth ([Kom87]).

We briefly consider generalizations of the ideas of this article to higher cardinals. Recall that for any regular cardinal $\kappa \geq \omega_1$, a κ -tree is a tree with height κ and levels of size less than κ . A κ -tree is a κ -Aronszajn tree if it has no cofinal branch. The κ -tree property is the statement that there does not exist a κ -Aronszajn tree. For a successor cardinal κ , a κ -Kurepa tree is a κ -tree which has at least κ^+ -many cofinal branches. We say that T is a strong κ -Kurepa tree if T is a κ -Kurepa tree and ${}^\kappa 2$ is a continuous image of $[T]$ with respect to the cone topologies.

As the reader can easily check, the arguments of Section 2 generalize to show the following.

Theorem 8.7. *Let κ be a successor cardinal and suppose that T is a strong κ -Kurepa tree. Then T contains a downwards closed subtree which is a κ -Aronszajn subtree.*

Corollary 8.8. *Suppose that $\kappa \geq \omega_2$ is a successor cardinal and the κ -tree property holds. Then there does not exist a strong κ -Kurepa tree.*

The proof of the main theorem can be easily adjusted to prove a more general theorem.

Theorem 8.9. *Suppose that μ is a regular cardinal, $\kappa = \mu^+$, $\mu^{<\mu} = \mu$, and $2^\mu = \kappa$. Then there exists a κ -closed, κ^+ -c.c. forcing which adds a strong κ -Kurepa tree.*

On the other hand, we do not know the answer to the following.

Question 8.10. *Is it consistent that for some singular cardinal λ , there exists a strong λ^+ -Kurepa tree T ?*

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