A VIRTUAL FIVE ELEMENT BASIS FOR THE UNCOUNTABLE LINEAR ORDERS

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ABSTRACT. We prove that for every Aronzsajn line A and every Countryman line C, there is a proper forcing extension in which A contains an isomorphic copy of either C or its converse C^*. As a corollary, we obtain answers to several related questions asked by the second author in the literature: if there is an inaccessible cardinal, then there is a proper forcing extension in which the uncountable linear orders have a five element basis; BPFA implies the existence of a five element basis for the uncountable linear orders; BPFA is equiconsistent with the conjunction of BPFA and Aronszajn tree saturation. These results are derived from new preservation results concerning subtrees of Aronszajn trees, proper forcings, and countable support iterations, generalizing work of Miyamoto, Abraham, and Shelah.

1. INTRODUCTION

In [16], Shelah proved the existence an uncountable linear order C such that $C \times C$ is a union of countably many chains in the coordinate-wise partial order, answering a question of R. Countryman [5]. Such orders are now known as *Countryman lines* and, as we will see momentarily, are fundamental objects within the class of linear orders. Countryman had already observed that such linear orders are necessarily *Aronszajn*—they have no uncountable separable suborders and they do not contain a copy of ω_1 or ω_1^*. Galvin observed that if C is a Countryman line, no uncountable linear order can embed into both C and C^*. Clearly, any uncountable suborder of a Countryman line is Countryman.

Thus, Shelah's result demonstrated that any basis for the Aronszajn lines must contain at least two elements: a Countryman line C and its converse C^*. Shelah conjectured [16, 9A] that it was consistent that any Aronszajn line contains a Countryman suborder. Subsequent work of Abraham and Shelah [1], when combined with prior work of Baumgartner [4], established the following reduction:

Theorem 1.1 ([1]). *Assume PFA. The following are equivalent:*

(1) Shelah's conjecture.
(2) If $X \subseteq \mathbb{R}$ has cardinality ω_1 and C is a Countryman line, then any uncountable linear order contains an isomorphic copy of X, ω_1, ω_1^*, C, or C^*.
(3) There is an Aronszajn tree T such that $\mathsf{CAT}(T)$: whenever $K \subseteq T$, there is an uncountable antichain $X \subseteq T$ such that $\wedge(X)$ is contained in K or $T \setminus K$.

PFA denotes the *Proper Forcing Axiom*, a powerful and versatile axiomatic assumption which strengthens Martin's Axiom and the negation of the Continuum Hypothesis; see [15, 22]. In fact, the bounded form of PFA (denoted BPFA) introduced by Goldstern and Shelah [6] suffices to prove the equivalence in Theorem 1.1. BPFA asserts that if a Σ_1-sentence with parameters in $\mathscr{P}(\omega_1)$ holds in a proper forcing extension, it is true. Unlike PFA, which has considerable large cardinal strength (see, e.g., [21]), BPFA is equiconsistent with the existence of a reflecting cardinal [6, 25], a hypothesis which is weaker in consistency strength than the existence of a Mahlo cardinal.

Shelah's conjecture was eventually confirmed by the second author:

Theorem 1.2 ([14])**.** *Assume PFA. Any Aronszajn line contains a Countryman suborder.*

Corollary 1.3. *Assume PFA and let $X \subseteq \mathbb{R}$ have cardinality ω_1 and C be any Countryman line. Any uncountable linear order contains an isomorphic copy of X, ω_1, ω_1^*, C, or C^*.*

The proof of Theorem 1.2 had two unusual related features. First, it required PFA just to prove that certain posets needed in the argument were proper. In particular, even though instances of Shelah's conjecture are equivalent to Σ_1-sentences with parameters in $\mathscr{P}(\omega_1)$, the proof of [14] did not yield that Bounded Proper Forcing Axiom (BPFA) implies Shelah's conjecture. Second, the proof relied on consequences of PFA which have large cardinal strength. Unlike MA_{ω_1}, whose consistency can be established relative to ZFC, PFA has considerable large cardinal strength (see [21]). Subsequent work of König, Larson, Veličković, and the second author [9] reduced the large cardinal strength of Shelah's conjecture to something less than a Mahlo cardinal. Still, this work left open the possibility that the consistency strength of Shelah's conjecture was stronger than that of BPFA, which is equiconsistent with the existence of a reflecting cardinal [6, 25].

In the present paper, we remedy these shortcomings.

Theorem 1.4. *If A is any Aronszajn line and C is any Countryman line, then there is a proper forcing extension in which A contains an isomorphic copy of C or C^*.*

This yields a number of corollaries, which answer questions in the literature.

Corollary 1.5. *Let C be any Countryman line and $X \subseteq \mathbb{R}$ have cardinality ω_1. If L is any uncountable linear order, there is a proper forcing extension in which L contains an isomorphic copy of X, ω_1, ω_1^*, C, or C^*.*

Corollary 1.6. *Assume BPFA and let $X \subseteq \mathbb{R}$ have cardinality ω_1 and C be any Countryman line. Any uncountable linear order contains an isomorphic copy of X, ω_1, ω_1^*, C, or C^*.*

Corollary 1.7. *The following are equiconsistent:*

(1) The conjunction of: PFA for posets of cardinality at most ω_1, "Any two ω_1-dense sets of reals are isomorphic," Shelah's conjecture, and φ.
(2) There is an inaccessible cardinal.

In last corollary, φ is a combinatorial consequence of PFA which was introduced in [9], where it was shown that the conjunction of BPFA and φ implies Shelah's conjecture.

Central to the proof of our main result is a new forcing iteration preservation lemma for ω_1-trees. This lemma says that for any countable support iteration $(\mathbb{P}_i, \dot{\mathbb{Q}}_i : i \in \beta)$ of proper forcings, where β is a limit ordinal, and for any ω_1-tree T in the ground model, every subtree of T which appears in $V^{\mathbb{P}_\beta}$ contains a subtree which lies in $V^{\mathbb{P}_i}$ for some $i \in \beta$. We also use this iteration lemma to prove that the statement φ is equiconsistent with an inaccessible cardinal, which answers a question of [9].

Terminology and notation: A tree T is an ω_1*-tree* if it has height ω_1 and countable levels. Level α of T is denoted by T_α, and $T \upharpoonright \beta = \bigcup_{\alpha<\beta} T_\alpha$. If $x \in T$ and β is less than the height of x, we write $x \upharpoonright \beta$ for the unique member of T with height β. By a *subtree* of an ω_1-tree we mean a subset which is uncountable and downwards closed. Whenever $T_0, \ldots, T_{n-1}$ are ω_1-trees, we write $T_0 \otimes \cdots \otimes T_{n-1}$ to denote the product tree consisting of n-tuples in $T_0 \times \cdots \times T_{n-1}$ whose elements all have the same height, ordered componentwise. If all of these trees are equal to the same tree T, this product tree is denoted by $T^{\otimes n}$.

An ω_1-tree T is *coherent* if it is a downwards closed subtree of ${}^{<\omega_1}2$ satisfying that for all $s, t \in T$, the set of $\alpha \in \operatorname{dom}(s) \cap \operatorname{dom}(t)$ such that $s(\alpha) \neq s(\beta)$ is finite. For any $s, t \in T$, let $s \wedge t$ be the greatest lower bound of s and t in T. For an antichain $X \subseteq T$, let $\wedge(X) = \{\wedge(s,t) : s, t \in X\}$. For incomparable $s, t \in T$, let $\Delta(s,t)$ be the least $\alpha \in \operatorname{dom}(s) \cap \operatorname{dom}(t)$ such that $s(\alpha) \neq t(\alpha)$. Note that $\Delta(s,t)$ is the height of the meet $s \wedge t$. For a set $Z \subseteq T$ and $t \in T$, let $\Delta(Z,t)$ be the set of all ordinals of the form $\Delta(s,t)$, where $s \in Z$ is incomparable with t.

2. Saturation of ω_1-trees, φ, and ψ

In this section, we review the concept of saturation for ω_1-trees and the related statements φ and ψ. We also briefly discuss the ideas of a subtree base and subtree preservation which are due to Baumgartner [3].

Definition 2.1 ([9])**.** Let T be an ω_1-tree.

1. Subtrees U and W of T are *almost disjoint* if $U \cap W$ is countable.
2. An *antichain of subtrees of* T is a pairwise almost disjoint family of subtrees of T.
3. A family of subtrees $\mathcal{F}$ of T is *predense* if any subtree of T has uncountable intersection with some member of $\mathcal{F}$.
4. For a given family of subtrees $\mathcal{F}$ of T, $\mathcal{F}^{\perp}$ is defined as the set of all subtrees of T which are almost disjoint with every member of $\mathcal{F}$.

Note that if $\mathcal{F}$ is a family of subtrees of an ω_1-tree T, then $\mathcal{F} \cup \mathcal{F}^{\perp}$ is always predense, and $\mathcal{F}$ itself is predense iff $\mathcal{F}^{\perp} = \varnothing$.

Definition 2.2 ([9])**.** An ω_1-tree T is *saturated* if every antichain of subtrees of T has cardinality less than ω_2.

Definition 2.3 ([9])**.** Let T be an ω_1-tree and let $\mathcal{F}$ be a family of subtrees of T. Define $\varphi(T, \mathcal{F})$ as the statement that there exists a sequence $(U_\xi : \xi \in \omega_1)$ of subtrees of T and a club $C \subseteq \omega_1$ such that for all $\alpha \in C$ and for any $t \in T_\alpha$, either:

1. there exists $\xi < \alpha$ such that $t \in U_\xi$ and $U_\xi \in \mathcal{F}^{\perp}$, or
2. there exists $\beta < \alpha$ such that for all $\nu \in (\beta, \alpha) \cap C$, there is $\xi < \nu$ such that $t \restriction \nu \in U_\xi$ and $U_\xi \in \mathcal{F}$.

Define $\psi(T, \mathcal{F})$ as the same statement except with alternative (1) omitted.

When T is understood from context, we write $\varphi(\mathcal{F})$ and $\psi(\mathcal{F})$ to abbreviate $\varphi(T, \mathcal{F})$ and $\psi(T, \mathcal{F})$. Observe that if $\mathcal{F}$ is predense, then $\varphi(\mathcal{F})$ and $\psi(\mathcal{F})$ are equivalent. Also note that $\varphi(\mathcal{F})$ implies $\psi(\mathcal{F} \cup \mathcal{F}^{\perp})$. It is easily seen that $\varphi(\mathcal{F})$ and $\psi(\mathcal{F})$ are upwards absolute with respect to any ω_1-preserving forcing extension.

Lemma 2.4 ([9, Lemma 2.2])**.** *Let T be an ω_1-tree and let $\mathcal{F}$ a family of subtrees of T. If $\psi(\mathcal{F})$ holds, then $\mathcal{F}$ is predense.*

The next two lemmas are implicit in [9].

Lemma 2.5. *Suppose that T is an ω_1-tree and for every predense family $\mathcal{F}$ of subtrees of T, $\psi(\mathcal{F})$ holds. Then T is saturated.*

Proof. If T is not saturated, then there exists an antichain $\mathcal{F}$ of subtrees of T with cardinality at least ω_2, which we may assume is maximal and hence predense. Let $(U_\xi : \xi \in \omega_1)$ and $C \subseteq \omega_1$ witness that $\psi(\mathcal{F})$ holds. Define $\mathcal{F}^* = \{U_\xi : \xi \in \omega_1\}$. Note that $(U_\xi : \xi \in \omega_1)$ and C also witness that $\psi(\mathcal{F}^*)$ holds. By Lemma 2.4, $\mathcal{F}^*$ is predense. Since $\mathcal{F}$ is an antichain, it must be the case that $\mathcal{F}^* = \mathcal{F}$ which is a contradiction. □

Lemma 2.6. *Suppose that T is a saturated ω_1-tree and for any family $\mathcal{F}$ of subtrees of T of size at most ω_1, $\varphi(\mathcal{F})$ holds. Then for every family $\mathcal{F}$ of subtrees of T, $\varphi(\mathcal{F})$ holds.*

Proof. It suffices to show that whenever $\mathcal{F}$ is a family of subtrees of T, there exists a subfamily $\mathcal{F}^* \subseteq \mathcal{F}$ of size at most ω_1 such that $\mathcal{F}^\perp = (\mathcal{F}^*)^\perp$. For then any sequence and club which witness that $\varphi(\mathcal{F}^*)$ holds also witnesses that $\varphi(\mathcal{F})$ holds. If not, then we can prove that T is not saturated by recursively constructing an almost disjoint sequence $(U_\alpha : \alpha \in \omega_2)$ of subtrees of T, each contained in some member of $\mathcal{F}$, as follows. Assume that $\alpha \in \omega_2$ and for all $\beta < \alpha$, U_β is defined and contained in some $W_\beta \in \mathcal{F}$. By assumption, $\{W_\beta : \beta < \alpha\}^\perp \neq \mathcal{F}^\perp$, so we can find a subtree U of T which is almost disjoint from W_β for all $\beta < \alpha$, but for some $W \in \mathcal{F}$, $W \cap U$ is uncountable. Let $U_\alpha = W \cap U$ and $W_\alpha = W$. □

Definition 2.7 ([9])**.** *Aronszajn tree saturation* is the statement that every Aronszajn tree is saturated.

Definition 2.8 ([9])**.** Let φ denote the statement that $\varphi(T, \mathcal{F})$ holds for every Aronszajn tree T and any family of subtrees $\mathcal{F}$ of T. Define ψ similarly.

By Lemma 2.5, φ implies Aronszajn tree saturation.

Remark 2.9. In [9], φ and ψ were only defined for Aronszajn trees. The generalization for ω_1-trees is equivalent, however: if T is an ω_1-tree and U is an Aronszajn tree, then $T \otimes U$ is Aronszajn, and if $\varphi(T \otimes U, \mathcal{G})$ holds for any family of subtrees $\mathcal{G}$ of $T \otimes U$, then $\varphi(T, \mathcal{F})$ holds for any family of subtrees $\mathcal{F}$ of T.

We now briefly discuss the idea of a subtree base which was introduced in [3].

Definition 2.10 ([3])**.** Let T be an ω_1-tree. A family $\mathcal{F}$ of subtrees of T is a *subtree base* if every subtree of T contains a member of $\mathcal{F}$.

Definition 2.11 ([3])**.** An ω_1-tree has a *small subtree base* if it has a subtree base of size at most ω_1.

Note that if an ω_1-tree has a small subtree base, then it is saturated. Baumgartner and Todorčević [3] proved that if there exists a Kurepa tree, then there is a special Aronszajn tree with no small subtree base. In fact, if T is a special Aronszajn tree and U is a Kurepa tree, then $T \otimes U$ is a special non-saturated Aronszajn tree. On the other hand, Stejskalová and the first author proved that the existence of a non-saturated Aronszajn tree does not imply that there exists a Kurepa tree ([10, 11]).

Baumgartner [3] proved that the Lévy collapse which collapses an inaccessible cardinal to become ω_2 forces that every ω_1-tree has a small subtree base. The proof is based on the fact that countably closed forcings satisfy the property described in the next definition.

Definition 2.12. For an ω_1-tree T, a forcing $\mathbb{P}$ *preserves subtrees of T* if $\mathbb{P}$ forces that every subtree of T in $V^{\mathbb{P}}$ contains a subtree which lies in V. A forcing $\mathbb{P}$ *preserves subtrees* if for every ω_1-tree T, $\mathbb{P}$ preserves subtrees of T.

The proof of the next lemma is easy.

Lemma 2.13. *Let T be an ω_1-tree. Suppose that $\mathbb{P}$ preserves subtrees of T. If $\mathcal{F}$ is a predense family of subtrees of T, then $\mathbb{P}$ forces that $\mathcal{F}$ is predense.*

3. Forcing φ and ψ

It was shown in [9] that instances of φ and ψ can be forced by adding reflecting sequences for open stationary set mappings. We review this method and prove a new result that the standard forcing for adding a reflecting sequence with countable conditions preserves subtrees.

Definition 3.1 ([13])**.** Let X be an uncountable set and let M be a countable elementary submodel of $H(\theta)$, where θ is a regular cardinal and $[X]^\omega \in M$. Let $S \subseteq [X]^\omega$.

(1) S is *M-stationary* if for any club $E \subseteq [X]^\omega$ in M, $M \cap S \cap E \neq \varnothing$.
(2) S is *open* if for all $N \in S$ there exists a finite set $x \subseteq N$ such that
$$\{P \in [X]^\omega : x \subseteq P \subseteq N\} \subseteq S.$$

Definition 3.2 ([13])**.** A function Σ is an *open stationary set mapping* if there exists an uncountable set X_Σ and a regular cardinal θ_Σ such that $\mathrm{dom}(\Sigma)$ is a club subset of $[H(\theta_\Sigma)]^\omega$ consisting of elementary submodels of $H(\theta_\Sigma)$ with $[X_\Sigma]^\omega$ as an member, and for all $M \in \mathrm{dom}(\Sigma)$, $\Sigma(M)$ is open and M-stationary.

Definition 3.3 ([13])**.** Let Σ be any open stationary set mapping. Define $\mathbb{P}_\Sigma$ as the forcing poset consisting of conditions, ordered by reverse extension, which are $\in$-increasing and continuous sequences $(N_i : i \leq \alpha)$ of elements of $\mathrm{dom}(\Sigma)$, for some $\alpha \in \omega_1$, such that for all $0 < \beta \leq \alpha$ there exists $\xi < \beta$ such that for all $\gamma \in (\xi, \beta)$, $N_\gamma \cap X_\Sigma \in \Sigma(N_\beta)$.

Theorem 3.4 ([13])**.** *Let Σ be an open stationary set mapping. Then $\mathbb{P}_\Sigma$ is proper and the union of any generic filter on $\mathbb{P}_\Sigma$ is a reflecting sequence for Σ, by which we mean an $\in$-increasing and continuous sequence $(N_\alpha : \alpha \in \omega_1)$ of elements of $\mathrm{dom}(\Sigma)$ such that for all non-zero $\alpha \in \omega_1$, there exists $\beta < \alpha$ such that for all $\gamma \in (\beta, \alpha)$, $N_\gamma \cap X_\Sigma \in \Sigma(N_\alpha)$.*

Theorem 3.5. *Let Σ be an open stationary set mapping. Then $\mathbb{P}_\Sigma$ preserves subtrees.*

Proof. Let $X = X_\Sigma$ and $\theta = \theta_\Sigma$. Fix an ω_1-tree T and a $\mathbb{P}$-name $\dot{U}$ for a subtree of T. It suffices to prove that for every $p \in \mathbb{P}_\Sigma$, there exists $r \leq p$ such that the set $\{x \in T : r \Vdash x \in \dot{U}\}$ is uncountable. Suppose for a contradiction that p is a counter-example. Fix a large enough regular cardinal λ and a countable elementary submodel M of $H(\lambda)$ such that $[X]^\omega$, θ, Σ, $\mathbb{P}_\Sigma$, $H(|\mathbb{P}_\Sigma|^+)$, p, T, and $\dot{U}$ are members of M. Let $M' = M \cap H(\theta)$. Enumerate $T_{M \cap \omega_1}$ as $(t_n : n \in \omega)$ and let $(D_n : n \in \omega)$ enumerate all dense open subsets of $\mathbb{P}_\Sigma$ which lie in M.

We construct by recursion a descending sequence of conditions $(p_n : n \in \omega)$ in $M \cap \mathbb{P}$ as follows. Let $p_0 = p$. Now let $n \in \omega$ and assume that we have defined p_n. Define E_n as the set of all $N \in [X]^\omega$ such that for some countable $N^* \prec H(|\mathbb{P}_\Sigma|^+)$ which contains as elements Σ, $\mathbb{P}_\Sigma$, p_n, and D_n, N is equal to $N^* \cap X$. Then $E_n \in M$ and E_n is a club. Since $\Sigma(M')$ is open and M'-stationary, we can fix $N_n \in M' \cap \Sigma(M') \cap E_n$ and $z_n \in [N_n]^{<\omega}$ such that $\{P \in [X]^\omega : z_n \subseteq P \subseteq N_n\} \subseteq \Sigma(M')$. Let $N_n^* \in M$ witness that $N_n \in E_n$. By the elementarity of N_n^*, fix $K_n \in N_n^* \cap \mathrm{dom}(\Sigma)$ such that $z_n \subseteq K_n$. Define $q_n = p_n \cup \{(\mathrm{dom}(p_n), K_n)\}$, which is in $\mathbb{P}_\Sigma \cap N_n^*$ and extends p_n. By assumption and the elementarity of N_n^*, there exists $\alpha_n < N_n^* \cap \omega_1$ such that for all $x \in T_{\alpha_n}$, q_n does not force that $x \in \dot{U}$. In particular, we can find $p_{n+1} \leq q_n$ in $N_n^* \cap D_n$ which forces that $t_n \restriction \alpha_n \notin \dot{U}$. Since $\dot{U}$ is forced to be downwards closed, p_{n+1} forces that $t_n \notin \dot{U}$.

This completes the recursion. Define

$$q = \bigcup_n p_n \cup \{(M \cap \omega_1, M')\}.$$

Note that for all $\gamma \in [\mathrm{dom}(p_0), M \cap \omega_1)$, there exists n such that

$$z_n \subseteq K_n \cap X \subseteq p_{n+1}(\gamma) \cap X = q(\gamma) \cap X \subseteq N_n,$$

and hence $q(\gamma) \cap X \in \Sigma(M')$. It easily follows that $q \in \mathbb{P}_\Sigma$ and $q \leq p_n$ for all n. By construction, q forces that $t_n \notin \dot{U}$ for all n. Hence, $q \Vdash \dot{U} \subseteq \check{T} \restriction (M \cap \omega_1)$, which contradicts that $\dot{U}$ is forced to be a subtree. $\square$

In [9], a method for forcing $\varphi(\mathcal{F})$ using the poset of Definition 3.3 was introduced and is based on the following lemma.

Lemma 3.6 ([9, Lemma 3.6])**.** *Let T be an ω_1-tree and let $\mathcal{F}$ be a family of subtrees of T. Let E_0 be the club subset of $[H(\omega_2)]^\omega$ consisting of all elementary submodels of $H(\omega_2)$ which contain T as a member. Let $\kappa \geq 2^{\omega_1 +}$ be regular and let M be a countable elementary submodel of $H(\kappa)$ with T and $\mathcal{F}$ members of M. Consider any $t \in T_{M \cap \omega_1}$ and suppose that there does not exist a set $A \in M \cap \mathcal{F}^\perp$ such that $t \in A$. Let Σ_M be the set of all $P \in [H(\omega_2)]^\omega$ such that, if $P \in E_0$, then there exists some $U \in P \cap \mathcal{F}$ such that $t \restriction (P \cap \omega_1) \in U$. Then Σ_M is open and M-stationary.*

We force $\varphi(\mathcal{F})$ by decomposing T as a two-dimensional matrix and then force witnesses to $\varphi(\mathcal{F})$ along each column in a forcing iteration of length ω. The same construction also forces $\psi(\mathcal{F})$ in the case that $\mathcal{F}$ is predense.

Proposition 3.7. *Let T be an ω_1-tree and let $\mathcal{F}$ be a family of subtrees of T. Then there exists a countable support forcing iteration $(\mathbb{P}_n, \dot{\mathbb{Q}}_n : n \in \omega)$ satisfying:*

(1) for each $n \in \omega$, $\mathbb{P}_n$ forces that $\dot{\mathbb{Q}}_n = \mathbb{P}_{\dot{\Sigma}_n}$ for some open stationary set mapping $\dot{\Sigma}_n$;
(2) $\mathbb{P}_\omega$ is proper;
(3) $\mathbb{P}_\omega$ forces $\varphi(\mathcal{F})$;
(4) if $\mathcal{F}$ is predense, then $\mathbb{P}_\omega$ forces $\psi(\mathcal{F})$.

Proof. We start by describing the open stationary set mappings used in the factors of the forcing iteration. Suppose that $\{t_\alpha : \alpha \in \omega_1\}$ is a set of elements of T such that each t_α has height α. Define Σ as follows. Let $X_\Sigma = H(\omega_2)$ and $\theta_\Sigma = 2^{\omega_1 +}$. Define E_0 as the club set of all countable elementary submodels of $H(\omega_2)$ which contain T as a member. Let $\mathrm{dom}(\Sigma)$ be the set of all countable elementary submodels of $H(\theta_\Sigma)$ which contain T and $\mathcal{F}$ as members. Consider $M \in \mathrm{dom}(\Sigma)$. If there exists some $A \in M \cap \mathcal{F}^\perp$ such that $t_{M \cap \omega_1} \in A$, then let $\Sigma(M)$ be equal to $[H(\omega_2)]^\omega$. Otherwise, let $\Sigma(M)$ be the set of all $P \in [H(\omega_2)]^\omega$ such that, if $P \in E_0$, then there exists some $U \in P \cap \mathcal{F}$ such that $t \restriction (P \cap \omega_1) \in U$. Then $\Sigma(M)$ is open and M-stationary by Lemma 3.6. Now it is straightforward to use a reflecting sequence for Σ given by a generic filter on $\mathbb{P}_\Sigma$ to define a sequence $(U_\xi : \xi \in \omega_1)$ which witnesses the definition of $\varphi(\mathcal{F})$ restricted to elements of $\{t_\alpha : \alpha \in \omega_1\}$.

Now working in the ground model, decompose the tree T as $\{t(\alpha, n) : \alpha \in \omega_1,\ n \in \omega\}$, where each $t(\alpha, n)$ has height α. Define a countable support forcing iteration $(\mathbb{P}_n,\ \dot{\mathbb{Q}}_n : n \in \omega)$ so that for each n, $\mathbb{P}_n$ forces that $\dot{\mathbb{Q}}_n$ equals $\mathbb{P}_{\dot{\Sigma}_n}$, where $\dot{\Sigma}_n$ is a $\mathbb{P}_n$-name for the open stationary set mapping described in the previous paragraph for the set $\{t(\alpha, n) : \alpha \in \omega_1\}$. In any generic extension by $\mathbb{P}_\omega$, it is easy to combine the ω-many sequences obtained at each stage of the forcing iteration into a single sequence which witnesses that $\varphi(\mathcal{F})$ holds. In the case that $\mathcal{F}$ is predense, Theorem 3.5 implies that each $\mathbb{P}_n$ preserves subtrees, and therefore by Lemma 2.13 preserves the fact that $\mathcal{F}$ is predense. So each $\mathbb{P}_n$ forces that $\mathcal{F}^\perp$ is empty, and consequently the sequence obtained at stage $n+1$ contains only members of $\mathcal{F}$. Hence, the sequence which witnesses $\varphi(\mathcal{F})$ in $V^{\mathbb{P}_\omega}$ only contains members of $\mathcal{F}$ and therefore witnesses that $\psi(\mathcal{F})$ holds. $\square$

We note that a different method for forcing $\psi(\mathcal{F})$ was described in [9], as Theorem 3.5 was not known at the time.

4. An Iteration Lemma for Trees

In this section, we prove a forcing iteration preservation lemma concerning subtrees of ω_1-trees which is an essential tool for the main theorems of the article. Roughly speaking, we show that for an ω_1-tree T and a countable support forcing iteration of proper forcings of limit length, any subtree of T which lies in a generic extension by the iteration contains a subtree from an intermediate model by an earlier stage of the iteration.

Let us clarify notation. Suppose that $(\mathbb{P}_i, \dot{\mathbb{Q}}_i : i \in \beta)$ is a countable support forcing iteration. We let $\mathbb{P}_\beta$ denote the countable support limit of this iteration in the case that β is a limit ordinal, and $\mathbb{P}_\gamma * \dot{\mathbb{Q}}_\gamma$ in the case that $\beta = \gamma + 1$ is a successor ordinal. Let $\alpha \in \beta + 1$. Then $\leq_\alpha$ denotes the order on $\mathbb{P}_\alpha$, $\dot{G}_\alpha$ is the canonical $\mathbb{P}_\alpha$-name for the generic filter, and $\Vdash_\alpha$ abbreviates the forcing relation $\Vdash_{\mathbb{P}_\alpha}$. We also write $\Vdash_\alpha^W$ for the $\mathbb{P}_\alpha$-forcing relation relativized to a model W.

Our iteration lemma can be thought of as a natural extension to the theory of trees of the following landmark preservation theorem of Shelah. This theorem has provided a blueprint for a large number of countable support (and revised countable support) iteration preservation results across many different areas of set theory.

Theorem 4.1 ([17, Chapter III, Theorem 3.2])**.** *Suppose that $(\mathbb{P}_i, \dot{\mathbb{Q}}_i : i \in \beta)$ is a countable support forcing iteration of proper forcings. Then $\mathbb{P}_\beta$ is proper.*

One of several possible proofs of Shelah's theorem is based on the following lemma, which we will make use of (see Lemma 31.17 of [8]).

Lemma 4.2. *Let $(\mathbb{P}_i, \dot{\mathbb{Q}}_i : i \in \beta)$ be a countable support forcing iteration of proper forcings. Let λ be a large enough regular cardinal and let N be a countable elementary submodel of $H(\lambda)$ containing the forcing iteration as a member. Let $\alpha \in N \cap \beta$. Suppose that $\dot{p}$ is a $\mathbb{P}_\alpha$-name for a condition in $\mathbb{P}_\beta$, $q \in \mathbb{P}_\alpha$, and $q \Vdash_\alpha \dot{p} \in N$. Also, assume that q is $(N, \mathbb{P}_\alpha)$-generic and $q \Vdash_\alpha \dot{p} \restriction \alpha \in \dot{G}_\alpha$. Then there exists $q^+ \in \mathbb{P}_\beta$ such that $q^+ \restriction \alpha = q$, q^+ is $(N, \mathbb{P}_\beta)$-generic, and $q^+ \Vdash_\beta \dot{p} \in \dot{G}_\beta$.*

Lemma 4.3. *Let T be an ω_1-tree. Suppose that $(\mathbb{P}_i, \dot{\mathbb{Q}}_i : i \in \beta)$ is a countable support forcing iteration such that $\mathbb{P}_\beta$ preserves ω_1, $p_0 \in \mathbb{P}_\beta$, and D is a dense open subset of $\mathbb{P}_\beta$. Assume that p_0 forces in $\mathbb{P}_\beta$ that $\dot{U}$ is a downwards closed subset of $\check{T}$ such that for all $\alpha < \beta$, $\dot{U}$ contains no subtree which lies in $V^{\mathbb{P}_\alpha}$. Let $\alpha < \beta$. Then $\mathbb{P}_\alpha$ forces that, if $p_0 \restriction \alpha \in \dot{G}_\alpha$, then there exists some $\gamma \in \omega_1$ such that for all $u \in T_\gamma$, there exists some $p_{1,u} \leq_\beta p_0$ in D such that $p_{1,u} \restriction \alpha \in \dot{G}_\alpha$ and $p_{1,u} \Vdash_\beta^V u \notin \dot{U}$.*

Proof. Let $r \in \mathbb{P}_\alpha$. By extending r further if necessary, we may assume without loss of generality that r decides whether or not $p_0 \restriction \alpha \in \dot{G}_\alpha$, and in the case that it does, that $r \leq_\alpha p_0 \restriction \alpha$. If r forces that $p_0 \restriction \alpha \notin \dot{G}_\alpha$, then we are done. So assume that $r \leq_\alpha p_0 \restriction \alpha$. Extend $r \cup p_0 \restriction [\alpha, \beta)$ to some condition s which is in D. Then $s \leq_\beta p_0$. We claim that $s \restriction \alpha$ forces in $\mathbb{P}_\alpha$ that there exists $\gamma \in \omega_1$ such that for all $u \in T_\gamma$, there exists $p_{1,u} \leq_\beta p_0$ in D such that $p_{1,u} \restriction \alpha \in \dot{G}_\alpha$ and $p_{1,u} \Vdash_\beta^V u \notin \dot{U}$. To prove this claim, let G_α be a generic filter for $\mathbb{P}_\alpha$ containing $s \restriction \alpha$, and we work in $V[G_\alpha]$. Let $\mathbb{P}_\beta / G_\alpha$ be the suborder of $\mathbb{P}_\beta$ consisting of all $q \in \mathbb{P}_\beta$ such that $q \restriction \alpha \in G_\alpha$. Recall that $\mathbb{P}_\beta$ is forcing equivalent to $\mathbb{P}_\alpha * (\mathbb{P}_\beta / \dot{G}_\alpha)$. Since $s \restriction \alpha \in G_\alpha$, $s \in \mathbb{P}_\beta / G_\alpha$.

In $V[G_\alpha]$, let $U_0 = \{u \in T : s \Vdash_{\mathbb{P}_\beta / G_\alpha} u \in \dot{U}\}$. Note that $U_0 \in V[G_\alpha]$ and s forces in $\mathbb{P}_\beta / G_\alpha$ that U_0 is a downwards closed subset of $\dot{U}$. Hence, by our assumption on p_0 and $\dot{U}$, U_0 is countable. Find $\gamma \in \omega_1$ such that $U_0 \subseteq T \restriction \gamma$. Consider $u \in T_\gamma$. Then $u \notin U_0$, which means that there exists some $s_u \leq_{\mathbb{P}_\beta / G_\alpha} s$ which forces that $u \notin \dot{U}$. Then $s_u \leq_\beta s$. Since $s_u \restriction \alpha \in G_\alpha$, we can find $t_u \leq_\alpha s_u \restriction \alpha$ in G_α which forces (in $\mathbb{P}_\alpha$ over V) that s_u forces (in $\mathbb{P}_\beta / \dot{G}_\alpha$ over $V[\dot{G}_\alpha]$) that $u \notin \dot{U}$. Let $p_{1,u} = t_u \cup s_u \restriction [\alpha, \beta)$. Then $p_{1,u} \leq_\beta p_0$, $p_{1,u} \restriction \alpha = t_u \in G_\alpha$, $p_{1,u} \in D$, and $p_{1,u} \Vdash_\beta^V u \notin \dot{U}$. □

Lemma 4.4 (Iteration Lemma for Trees)**.** *Let T be an ω_1-tree. Suppose that $(\mathbb{P}_i, \dot{\mathbb{Q}}_i : i \in \beta)$ is a countable support forcing iteration of proper forcings, where β is a limit ordinal. Then $\mathbb{P}_\beta$ forces that for any subtree U of T lying in $V^{\mathbb{P}_\beta}$, there exists some $\alpha < \beta$ and a subtree W in $V^{\mathbb{P}_\alpha}$ such that $W \subseteq U$.*

Proof. Let $\dot{U}$ be a $\mathbb{P}_\beta$-name for a downwards closed subset of T. It suffices to show that whenever $p \in \mathbb{P}_\beta$ forces that for all $\alpha < \beta$, $\dot{U}$ does not contain a subtree which lies in $V^{\mathbb{P}_\alpha}$, then there exists $q \leq p$ which forces that $\dot{U}$ is countable. Without loss of generality, assume that each $\mathbb{P}_i$ is separative, for if not then we can find an equivalent forcing iteration for which this is true.

Fix a large enough regular cardinal λ and let $N \prec H(\lambda)$ be countable which contains the above parameters. Let $\delta = N \cap \omega_1$. Let $(D_n : n \in \omega)$ be an enumeration of all of the dense open subsets of $\mathbb{P}_\beta$ which lie in N and let $(a_n : n \in \omega)$ be an enumeration of T_δ. Fix an increasing sequence $(\alpha_n : n \in \omega)$ of ordinals in $N \cap \beta$ cofinal in $\sup(N \cap \beta)$ with $\alpha_0 = 0$.

We define by recursion two sequences $(q_i : i \in \omega)$ and $(\dot{p}_i : i \in \omega)$ satisfying that for all i:

(1) $\dot{p}_i$ is a $\mathbb{P}_{\alpha_i}$-name for a condition in $\mathbb{P}_\beta$, and $\dot{p}_0$ is a $\mathbb{P}_0$-name for p;
(2) q_i is an $(N, \mathbb{P}_{\alpha_i})$-generic condition and $q_i \Vdash_{\alpha_i} \dot{p}_i \in N \wedge \dot{p}_i \restriction \alpha_i \in \dot{G}_{\alpha_i}$;
(3) $q_{i+1} \restriction \alpha_i = q_i$, and q_{i+1} forces in $\mathbb{P}_{\alpha_{i+1}}$ that:
   (a) $\dot{p}_{i+1} \leq_\beta \dot{p}_i$;
   (b) $\dot{p}_{i+1} \in D_i$;
   (c) $\dot{p}_{i+1} \Vdash_\beta^V a_i \notin \dot{U}$.

Begin by letting q_0 be the empty condition in $\mathbb{P}_0$ and letting $\dot{p}_0$ be a $\mathbb{P}_0$-name for p. Now let $i \in \omega$ and assume that $\dot{p}_i$ and q_i are defined as required. Applying Lemma 4.2 to $\dot{p}_i \restriction \alpha_{i+1}$ and q_i, find $q_{i+1} \in \mathbb{P}_{\alpha_{i+1}}$ such that $q_{i+1} \restriction \alpha_i = q_i$, q_{i+1} is $(N, \mathbb{P}_{\alpha_{i+1}})$-generic, and $q_{i+1} \Vdash_{\alpha_{i+1}} \dot{p}_i \restriction \alpha_{i+1} \in \dot{G}_{\alpha_{i+1}}$.

To define $\dot{p}_{i+1}$, consider a generic filter $G_{\alpha_{i+1}}$ which contains q_{i+1}. Since q_{i+1} is $(N, \mathbb{P}_{\alpha_{i+1}})$-generic, $N[G_{\alpha_{i+1}}] \cap V = N$. Let $G_{\alpha_i} = G_{\alpha_{i+1}} \cap \alpha_i$. Let $p_i = \dot{p}_i^{G_{\alpha_i}}$, which is in N. So $p_i \restriction \alpha_{i+1} \in G_{\alpha_{i+1}}$. Also, $p_i \leq_\beta p$, so p_i forces (in $\mathbb{P}_\beta$ over V) that $\dot{U}$ is a downwards closed subset of T such that for all $\alpha < \beta$, $\dot{U}$ has no subtree which lies in $V^{\mathbb{P}_\alpha}$.

Applying Lemma 4.3 to p_i, D_i, and α_{i+1}, we conclude that there exists some $\gamma \in \omega_1$ such that for all $u \in T_\gamma$, there is some $p_{1,u} \leq_\beta p_i$ in D_i such that $p_{1,u} \restriction \alpha_{i+1} \in G_{\alpha_{i+1}}$ and $p_{1,u} \Vdash_\beta^V u \notin \dot{U}$. Since p_i and D_i are in $N[G_{\alpha_{i+1}}]$, we may assume that $\gamma < N[G_{\alpha_{i+1}}] \cap \omega_1 = \delta$. Consider $u = a_i \restriction \gamma$, which is in $N[G_{\alpha_{i+1}}]$. By elementarity, we can find $p_{i+1} \in N[G_{\alpha_{i+1}}]$ such that $p_{i+1} \leq_\beta p_i$ is in D_i, $p_{i+1} \restriction \alpha_{i+1} \in G_{\alpha_{i+1}}$, and $p_{i+1} \Vdash^V_\beta a_i \restriction \gamma \notin \dot{U}$. As $\dot{U}$ is forced to be downwards closed, it follows that $p_{i+1} \Vdash^V_\beta a_i \notin \dot{U}$. Finally, since $N[G_{\alpha_{i+1}}] \cap V = N$, $p_{i+1} \in N$.

Since $G_{\alpha_{i+1}}$ was an arbitrary generic filter on $\mathbb{P}_{\alpha_{i+1}}$ containing q_{i+1}, we can fix a $\mathbb{P}_{\alpha_{i+1}}$-name $\dot{p}_{i+1}$ for a member of $\mathbb{P}_\beta$ which q_{i+1} forces has the properties described above. Now it is routine to check that q_{i+1} and $\dot{p}_{i+1}$ are as required.

This completes the construction. Let $q = \bigcup_n q_n$, which is in $\mathbb{P}_\beta$. We will argue that $q \leq p$ and q forces that $\dot{U}$ is a subset of N, and hence is countable. Let G_β be a generic filter which contains q and let $U = \dot{U}^{G_\beta}$. For all $i \in \omega$, let $G_{\alpha_i} = G_\beta \cap \mathbb{P}_{\alpha_i}$ and $p_i = \dot{p}_i^{G_{\alpha_i}}$. Since $q \restriction \alpha_i$ forces that $\dot{p}_i \restriction \alpha_i \in \dot{G}_{\alpha_i}$ and $\mathbb{P}_{\alpha_i}$ is separative, it follows that $q \restriction \alpha_i \leq_{\alpha_i} p_i \restriction \alpha_i$. Hence, for all $i < j$, $q \restriction \alpha_j \leq_{\alpha_j} p_j \restriction \alpha_j \leq_{\alpha_j} p_i \restriction \alpha_j$. Since $\mathrm{dom}(q) \subseteq \sup_n \alpha_n$, it follows that $q \leq_\beta p_j$ for all j. In particular, $q \leq p_0 = p$. Also, $q \in G_\beta$ implies that $p_i \in G_\beta$ for all i, and by the choice of p_{i+1}, $a_i \notin U$. So $U \subseteq T \restriction \delta \subseteq N$. As G_β was arbitrary, q forces all of the above statements. So $q \leq p$ and $q \Vdash_\beta \dot{U} \subseteq N$. □

5. Some Consequences of the Iteration Lemma

In this section, we derive some consequences of Lemma 4.4. Further applications appear in subsequent sections.

Corollary 5.1. *Let T be an ω_1-tree. Suppose that $(\mathbb{P}_i, \dot{\mathbb{Q}}_i : i \in \beta)$ is a countable support forcing iteration of proper forcing such that for all $i \in \beta$, $\mathbb{P}_i$ forces that $\dot{\mathbb{Q}}_i$ preserves subtrees of T. Then $\mathbb{P}_\beta$ preserves subtrees of T.*

Proof. By induction on β, using the assumption of the corollary at successor stages and Lemma 4.4 at limit stages. □

Combining Proposition 3.7 with Corollary 5.1, we see that the forcing of Proposition 3.7 which adds a φ-sequence preserves subtrees.

Corollary 5.2. *Let T be an ω_1-tree and let $\mathcal{F}$ be a family of subtrees of T. Then there exists a forcing $\mathbb{P}$ which is proper, preserves subtrees, forces $\varphi(\mathcal{F})$, and if $\mathcal{F}$ is predense, forces $\psi(\mathcal{F})$.*

Proof. Immediate from Theorem 3.5, Proposition 3.7, and Corollary 5.1. □

Lemma 5.3. *Let $\{(T_i, \mathcal{F}_i) : i \in \delta\}$ be such that for each $i \in \delta$, T_i is an ω_1-tree and $\mathcal{F}_i$ is a predense family of subtrees of T_i. Then there exists a proper forcing which preserves subtrees and forces that for all $i \in \delta$, $\psi(T_i, \mathcal{F}_i)$ holds.*

Proof. Define a countable support forcing iteration $(\mathbb{P}_i, \dot{\mathbb{Q}}_i : i \in \delta)$ as follows. Arrange that each $\mathbb{P}_i$ forces that $\dot{\mathbb{Q}}_i$ preserves subtrees, so by Corollary 5.1 each $\mathbb{P}_i$ preserves subtrees. Assuming that $\mathbb{P}_i$ is defined, let $\dot{\mathbb{Q}}_i$ be a $\mathbb{P}_i$-name for the forcing of Corollary 5.2 for T_i and $\mathcal{F}_i$. By Lemma 2.13, $\mathbb{P}_i$ forces that $\mathcal{F}_i$ is predense. Therefore, $\mathbb{P}_i * \dot{\mathbb{Q}}_i$ forces $\psi(T_i, \mathcal{F}_i)$. Since ψ is upwards absolute, $\mathbb{P}_\delta$ is as required. □

We now revisit some classic forcing iteration preservation theorems for trees, and show that they can be derived as corollaries of Lemma 4.4. Silver [20] demonstrated the consistency of the non-existence of a Kurepa tree from an inaccessible cardinal using the fact that countably closed forcings do not add new branches to ω_1-trees. Shelah proved that for an ω_1-tree T, the property of a forcing poset that it is proper and does not add new cofinal branches of T is preserved under countable support forcing iterations (see [18, Chapter III, Theorem 8.5]). This theorem generalizes Silver's method to a broader context by allowing for the construction of models with no Kurepa trees using iterated forcing.

Note that any cofinal branch of an ω_1-tree T is a subtree of T.

Corollary 5.4. *Suppose that T is an ω_1-tree and $(\mathbb{P}_i, \dot{\mathbb{Q}}_i : i \in \beta)$ is a countable support forcing iteration of proper forcings such that for all $i \in \beta$, $\mathbb{P}_i$ forces that $\dot{\mathbb{Q}}_i$ does not add new cofinal branches of T. Then $\mathbb{P}_\beta$ does not add new cofinal branches of T.*

Proof. We may assume by induction that for all $i \in \beta$, $\mathbb{P}_i$ forces that any cofinal branch of T in $V^{\mathbb{P}_i}$ lies in V. If β is a successor ordinal, then we are done by the assumptions of the corollary. Suppose that β is a limit ordinal. Then by Lemma 4.4, any cofinal branch of T in $V^{\mathbb{P}_\beta}$ contains a subtree lying in $V^{\mathbb{P}_i}$ for some $i \in \beta$. But any subtree of a cofinal branch is equal to that branch. By the inductive hypothesis, that cofinal branch is in V. $\square$

Abraham and Shelah [2] and Miyamoto [12] independently proved that for a given Suslin tree S, the property of a forcing poset that it is proper and preserves the Suslin property of S is preserved by countable support forcing iterations. This result has a number of significant applications, such as the consistency of the forcing axiom $\mathsf{PFA}(S)$ for a coherent Suslin tree S ([23]). We reprove this result as a special case of Lemma 4.4 by making use of the following well-known characterization of an ω_1-tree being Suslin.

Lemma 5.5. *Let S be an ω_1-tree. Then S is Suslin iff S contains no cofinal branch and for every subtree $U \subseteq S$, U contains a cone (that is, there exists some $x \in S$ such that the set $\{y \in S : x \leq_S y\}$ is a subset of U).*

Corollary 5.6. *Suppose that S is a Suslin tree. Assume that $(\mathbb{P}_i, \dot{\mathbb{Q}}_i : i \in \beta)$ is a countable support forcing iteration of proper forcings such that for all $i \in \beta$, $\mathbb{P}_i$ forces that S is Suslin. Then $\mathbb{P}_\beta$ forces that S is Suslin.*

Proof. If β is a successor ordinal, then the result is immediate from the assumptions of the corollary. Assume that β is a limit ordinal. By Corollary 5.4, $\mathbb{P}_\beta$ forces that S does not have a cofinal branch. So it suffices to prove that any subtree of S in $V^{\mathbb{P}_\beta}$ contains a cone. By Lemma 4.4, $\mathbb{P}_\beta$ forces that any subtree U of S in $V^{\mathbb{P}_\beta}$ contains a subtree W in $V^{\mathbb{P}_i}$ for some $i \in \beta$. Since S is Suslin in $V^{\mathbb{P}_i}$, W contains a cone in $V^{\mathbb{P}_i}$, which by upwards absoluteness implies that U contains a cone in $V^{\mathbb{P}_\beta}$. $\square$

For our final application of Lemma 4.4 in this section, we show that MRP is consistent with the statement that every ω_1-tree has a small subtree base. A standard way to obtain a model of MRP is to iterate forcings with countable support up to a supercompact cardinal κ, where each forcing is proper, has size less than κ, and adds a reflecting sequence for some open stationary set mapping, guided by a Laver function. By Theorem 3.5, each factor of the iteration preserves subtrees, and hence so does the entire iteration by Corollary 5.1. Let $(\mathbb{P}_i : i \in \kappa)$ denote such a forcing iteration. By standard arguments, any ω_1-tree T in $V^{\mathbb{P}_\kappa}$ lies in $V^{\mathbb{P}_i}$ for some $i \in \kappa$, and since the tail of the iteration in $V^{\mathbb{P}_i}$ also preserves subtrees by Corollary 5.1, any subtree of T in $V^{\mathbb{P}_\kappa}$ contains a subtree lying in $V^{\mathbb{P}_i}$. As the collection of subtrees of T lying in $V^{\mathbb{P}_i}$ has cardinality ω_1 in $V^{\mathbb{P}_\kappa}$, T has a small subtree base in $V^{\mathbb{P}_\kappa}$.

Corollary 5.7. *If there is a supercompact cardinal, then there is a proper forcing extension which satisfies the conjunction of MRP and "all ω_1-trees have a small subtree base."*

We note that in contrast, MA_{ω_1} implies that for every Aronszajn tree T, the minimum size of a subtree base for T is at least ω_2 ([7, Corollary 1.1]).

6. The Consistency Strength of φ

In this section, we resolve the issue of the consistency strength of φ. In [9], it was asked whether φ implies that ω_2 is a Mahlo cardinal in L. We answer this question negatively as follows.

Theorem 6.1. *The following are equiconsistent:*

(1) There exists an inaccessible cardinal.
(2) φ.

Lemma 6.2. *Suppose that κ is an inaccessible cardinal, $S \subseteq \kappa$ is stationary, and $(s_\alpha : \alpha \in S)$ is a $\diamondsuit(S)$-sequence. Let $(\mathbb{P}_i, \dot{\mathbb{Q}}_i : i \in \kappa)$ be a countable support forcing iteration of proper forcings, where each $\mathbb{P}_i$ has size less than κ and $\mathbb{P}_\kappa$ forces that $\check{\kappa}$ equals ω_2. Assume that:*

(1) $\mathbb{P}_\kappa$ forces that for every ω_1-tree T and for every family $\mathcal{F}$ of subtrees of T of size at most ω_1, $\varphi(T, \mathcal{F})$ holds;
(2) for all $\alpha \in S$, if s_α codes (in some canonical way) nice $\mathbb{P}_\alpha$-names for an ω_1-tree T with underlying set ω_1 and a predense family $\mathcal{F}$ of subtrees of T, then $\mathbb{P}_{\alpha+1}$ forces $\psi(T, \mathcal{F})$.

Then $\mathbb{P}_\kappa$ forces that for any ω_1-tree T and for any family of subtrees $\mathcal{F}$ of T, $\varphi(T, \mathcal{F})$ holds.

Proof. By Lemma 2.6 and (1), it suffices to show that $\mathbb{P}_\kappa$ forces that every ω_1-tree is saturated. If not, then there are nice $\mathbb{P}_\kappa$-names $\dot{T}$ and $\dot{U}_i$ for $i \in \kappa$ and some condition $p \in \mathbb{P}_\kappa$ such that $\mathbb{P}_\kappa$ forces that $\dot{T}$ is an ω_1-tree with underlying set ω_1, $\{\dot{U}_i : i \in \kappa\}$ is a predense family of subtrees of $\dot{T}$, and if p is in the generic filter, then $\{\dot{U}_i : i \in \kappa\}$ is a maximal antichain. By a straightforward argument using the fact that $\mathbb{P}_\kappa$ is κ-c.c., there exists a club $C \subseteq \kappa$ such that for all $\alpha \in C$:

- $\dot{T}$ is a nice $\mathbb{P}_\alpha$-name;
- for all $\gamma < \alpha$, $\dot{U}_\gamma$ is a nice $\mathbb{P}_\alpha$-name;
- for all $\beta < \alpha$ and for any nice $\mathbb{P}_\beta$-name $\dot{W}$ for a subtree of $\dot{T}$, $\mathbb{P}_\alpha$ forces that $\dot{W}$ has uncountable intersection with some member of $\{\dot{U}_\gamma : \gamma < \alpha\}$.

Now find $\alpha \in S$ such that s_α codes $\dot{T}$ and $\{\dot{U}_\gamma : \gamma < \alpha\}$.

We claim that $\mathbb{P}_\alpha$ forces that $\{\dot{U}_i : i < \alpha\}$ is predense. If not, then there is a condition $q \in \mathbb{P}_\alpha$ and a $\mathbb{P}_\alpha$-name $\dot{Y}$ for a subtree of $\dot{T}$ such that for all $\gamma < \alpha$, q forces that $\dot{Y} \cap \dot{U}_\gamma$ is countable. Applying Lemma 4.4, find $r \leq q$, $\beta < \alpha$, and a nice $\mathbb{P}_\beta$-name $\dot{Z}$ for a subtree of $\dot{T}$ such that r forces that $\dot{Z} \subseteq \dot{Y}$. But then r forces that for all $\gamma < \alpha$, $\dot{Z} \cap \dot{U}_\gamma$ is countable, which contradicts that $\alpha \in C$. By (2), $\mathbb{P}_{\alpha+1}$ forces $\psi(\{\dot{U}_\gamma : \gamma < \alpha\})$. By upwards absoluteness, $\mathbb{P}_\kappa$ forces $\psi(\{\dot{U}_\gamma : \gamma < \alpha\})$, and hence forces that $\{\dot{U}_\gamma : \gamma < \alpha\}$ is predense. This is impossible since p forces that $\{\dot{U}_i : i \in \kappa\}$ is an antichain. □

Proof of Theorem 6.1. The statement φ implies Aronszajn tree saturation, which in turn implies the non-existence of a Kurepa tree and hence that ω_2 is inaccessible in L. For the converse, assume that κ is inaccessible. Then in L, κ is inaccessible and $\diamondsuit_\kappa$ holds as witnessed by some diamond sequence $(s_\alpha : \alpha \in \kappa)$. Define a countable support forcing iteration $(\mathbb{P}_i, \dot{\mathbb{Q}}_i : i \in \kappa)$ of proper forcings by recursion as follows. Arrange that for cofinally many stages of the iteration we force with $\mathrm{Col}(\omega_1, \omega_2)$, thereby ensuring $\mathbb{P}_\kappa$ forces $\check{\kappa} = \dot{\omega}_2$. By standard bookkeeping and Proposition 3.7, we can ensure that for any ω_1-tree T and for any family $\mathcal{F}$ of subtrees of T of size at most ω_1 which appear in $V^{\mathbb{P}_\kappa}$, there is some $\alpha \in \kappa$ such that $\mathbb{P}_{\alpha+1}$ forces $\varphi(\mathcal{F})$. Finally, consider any limit ordinal $\alpha \in \kappa$ such that s_α codes nice $\mathbb{P}_\alpha$-names $\dot{T}$ and $\{\dot{U}_i : i < \alpha\}$ for an ω_1-tree with underlying ω_1 and a predense family of subtrees of $\dot{T}$. Apply Proposition 3.7 to find $\dot{\mathbb{Q}}_\alpha$ such that $\mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha$ forces $\psi(\{\dot{U}_i : i < \alpha\})$. Now we are done by Lemma 6.2. □

In the above construction, we can also bookkeep to force along the way with any proper forcing of size less than κ which appears at some stage less than κ in the iteration. In this way, Lemma 6.1(2) can be strengthened to include other statements, such as the proper forcing axiom for proper forcings of size at most ω_1 or the statement that any two ω_1-dense sets of reals are isomorphic. In fact, if we strengthen the inaccessibility assumption to a reflecting cardinal, this method will produce a model of φ together with BPFA ([6]).

Theorem 6.3. *The following are equiconsistent:*

(1) There exists a reflecting cardinal.
(2) BPFA and φ hold.

7. The Families $\mathscr{R}_n$ and $\mathscr{R}_n^\perp$

We now describe families of subtrees of products of a given coherent Aronszajn tree which play in a role in the analysis of partial orders involved in forcing an instance of CAT. We make use of the following basic result about coherent Aronszajn trees.

Lemma 7.1. *Let T be a special coherent Aronszajn tree. Assume that $\{X_\alpha : \alpha \in \omega_1\}$ is a pairwise disjoint family of members of $T^{\otimes n}$, where $n \in \omega$ is positive. Then there is an uncountable set $B \subseteq \omega_1$ such that for all $\alpha < \beta$ in B, every element of X_α is incomparable in T with every element of X_β, and for all $i < j < n$, $\Delta(X_\alpha(i), X_\beta(i)) = \Delta(X_\alpha(j), X_\beta(j))$.*

See Lemmas 4.2.4 and 4.2.5 of [26] for the proof.

For the remainder of this section, fix a special coherent Aronszajn tree T and a set $K \subseteq T$.

Definition 7.2 ([14])**.** Let $n \in \omega$ be positive. For any $X \in T^{\otimes n}$, define $K(X)$ as the set of γ less than the height of X such that for all $i < n$, $X(i) \restriction \gamma \in K$.

Definition 7.3 ([9])**.** Let $n \in \omega$ be positive. For any uncountable set $Z \subseteq T$, let R^n_Z be the set of all $X \in T^{\otimes n}$ for which there exists some $t \in T$ in the downward closure of Z with the same height as X such that $\Delta(Z, t) \cap K(X) = \varnothing$.

For any uncountable set $Z \subseteq T$, if $X \in R^n_Z$ as witnessed by t, then for all γ less than the height of X, $X \restriction \gamma \in R^n_Z$ as witnessed by $t \restriction \gamma$. So R^n_Z is downwards closed, and hence is a subtree provided that it is uncountable.

Definition 7.4 ([9])**.** Define $\mathscr{R}_n$ as the set of all subtrees of $T^{\otimes n}$ of the form R^n_Z for some uncountable set $Z \subseteq T$.

Lemma 7.5. *Let $n \in \omega$ be positive and let $S \in \mathscr{R}_n^\perp$. Suppose that $\{X_\alpha : \alpha \in \omega_1\}$ is a family of distinct elements of S. Then there exist $\alpha < \beta$ such that for all $i < n$, $X_\alpha(i)$ and $X_\beta(i)$ are incomparable in T and $X_\alpha(i) \wedge X_\beta(i) \in K$.*

Proof. Suppose for a contradiction that the conclusion of the lemma fails. Applying Lemma 7.1, find an uncountable set $B \subseteq \omega_1$ such that for all $\alpha < \beta$ in B and for all $i < j < n$, $\Delta(X_\alpha(i), X_\beta(i)) = \Delta(X_\alpha(j), X_\beta(j))$. Define $Z = \{X_\alpha(0) : \alpha \in B\}$. We prove that $\{X_\xi : \xi \in B\} \subseteq R^n_Z$, which is a contradiction since it then follows that $S \cap R^n_Z$ is uncountable. So let $\xi \in B$. Then $X_\xi(0)$ is in Z and has the same height as X, so it suffices to show that $\Delta(Z, X_\xi(0)) \cap K(X_\xi) = \varnothing$. Consider $\gamma \in \Delta(Z, X_\xi(0))$. Then there is $\alpha \in B$ such that $\gamma = \Delta(X_\alpha(0), X_\xi(0))$.

By our assumption for a contradiction, there is $i < n$ such that $X_\alpha(i) \wedge X_\xi(i) \notin K$. By the choice of B, $\gamma = \Delta(X_\alpha(i), X_\xi(i))$. So $X_\xi(i) \restriction \gamma = X_\alpha(i) \wedge X_\xi(i) \notin K$, and hence $\gamma \notin K(X_\xi)$. □

Lemma 7.6. *Let $n \in \omega$ be positive and suppose that U is subtree of $T^{\otimes n}$ which is contained in some member of $\mathscr{R}_n$. Then there exists a subtree $S \subseteq U$ such that, letting $Z = \{Y(0) : Y \in S\}$, for every $X \in S$, $\Delta(Z, X(0)) \cap K(X) = \varnothing$, and in particular, $S \subseteq R^n_Z$.*

Proof. Fix an uncountable set $Z_1 \subseteq T$ such that $U \subseteq R^n_{Z_1}$. Choose a sequence $(X_\alpha : \alpha \in \omega_1)$ which lists in increasing heights uncountably many elements of U. For each $\alpha \in \omega_1$, fix some t_α in the downward closure of Z_1 with the same height as X_α such that $\Delta(Z_1, t_\alpha) \cap K(X_\alpha) = \varnothing$. Applying Lemma 7.1 to the set of pairs $\{(t_\alpha, X_\alpha(0)) : \alpha \in \omega_1\}$, find an uncountable set $B \subseteq \omega_1$ such that for all $\alpha < \beta$ in B, $\Delta(t_\alpha, t_\beta) = \Delta(X_\alpha(0), X_\beta(0))$. Let S be the downward closure in $T^{\otimes n}$ of the set $\{X_\alpha : \alpha \in B\}$, and define $Z = \{Y(0) : Y \in S\}$. Note that $S \subseteq U$.

Consider $X \in S$ and we prove that $\Delta(Z, X(0)) \cap K(X) = \varnothing$. Fix $\alpha \in B$ such that $X \leq_{T^{\otimes n}} X_\alpha$. Since $\Delta(Z, X(0)) \subseteq \Delta(Z, X_\alpha(0))$ and $K(X) \subseteq K(X_\alpha)$, it suffices to show that $\Delta(Z, X_\alpha(0)) \cap K(X_\alpha) = \varnothing$. Let $\beta \in B \setminus \{\alpha\}$ and we claim $\Delta(X_\beta(0), X_\alpha(0)) \notin K(X_\alpha)$. By the choice of B, $\Delta(X_\beta(0), X_\alpha(0)) = \Delta(t_\beta, t_\alpha) \in \Delta(Z_1, t_\alpha)$. Since $\Delta(Z_1, t_\alpha) \cap K(X_\alpha) = \varnothing$, it follows that $\Delta(X_\beta(0), X_\alpha(0)) \notin K(X_\alpha)$. □

A combination of Lemmas 4.4 and 7.6 implies the next result.

Lemma 7.7. *Suppose that $(\mathbb{P}_i, \dot{\mathbb{Q}}_i : i \in \beta)$ is a countable support forcing iteration of proper forcings, where β is a limit ordinal. Let $n \in \omega$ be positive. Let $\dot{\mathcal{F}}$ be a $\mathbb{P}_\beta$-name for the set of elements of $\mathscr{R}_n \cup \mathscr{R}_n^\perp$ which lie in $V^{\mathbb{P}_i}$ for some $i \in \beta$. Then $\mathbb{P}_\beta$ forces that $\dot{\mathcal{F}}$ is predense.*

Proof. Consider a generic filter G on $\mathbb{P}_\beta$ and let $\mathcal{F} = \dot{\mathcal{F}}^G$. In $V[G]$, let U be a subtree of $T^{\otimes n}$. For each $i \in \beta$, define $G_i = G \cap \mathbb{P}_i$. Since $\mathscr{R}_n \cup \mathscr{R}_n^\perp$ is predense in $V[G]$, we can find some W in it such that $U \cap W$ is uncountable. First, assume that $W \in \mathscr{R}_n^\perp$. Applying Lemma 4.4, fix $i \in \beta$ and a subtree $S \subseteq U \cap W$ which lies in $V[G_i]$. Since $S \subseteq W$, clearly $S \in \mathscr{R}_n^\perp$ in $V[G]$. So $S \in \mathcal{F}$ and $S \cap U = S$ is uncountable.

Secondly, assume that $W \in \mathscr{R}_n$. Applying Lemma 7.6, find in $V[G]$ a subtree $S \subseteq U \cap W$ such that, letting $Z = \{Y(0) : Y \in S\}$, for every $X \in S$ it is the case that $\Delta(Z, X(0)) \cap K(X) = \varnothing$. Applying Lemma 4.4, fix $i \in \beta$ and a subtree $S_1 \subseteq S$ which lies in $V[G_i]$. Define $Z^* = \{X(0) : X \in S_1\}$, which is an uncountable subset of Z in $V[G_i]$. Let $R = R^n_{Z^*}$, which is in $V[G_i]$. As $S_1 \subseteq S$ and $Z^* \subseteq Z$, it follows that $S_1 \subseteq R$. Now it is easy to check that since $R \in \mathscr{R}_n$ in $V[G_i]$, it is also in $\mathscr{R}_n$ in $V[G]$ (see Lemma 8.1 below). So $R \in \mathcal{F}$, and $R \cap U$ is uncountable since it contains S_1. □

Lemma 7.8. *Suppose that there exists some $R \in \mathscr{R}_1$ which is uncountable. Then there is an uncountable antichain $X \subseteq T$ such that $\wedge(X) \subseteq T \setminus K$.*

Proof. Let $Z \subseteq T$ be uncountable such that $R = R^1_Z$. Choose a sequence $((x_\alpha) : \alpha \in \omega_1)$ with increasing heights consisting of members of R^1_Z. For each $\alpha \in \omega_1$, fix some t_α in the downwards closure of Z with the same height as x_α such that $\Delta(Z, t_\alpha) \cap K((x_\alpha)) = \varnothing$. Applying Lemma 7.1 to the sequence $((t_\alpha, x_\alpha) : \alpha \in \omega_1)$, find an uncountable set $B \subseteq \omega_1$ such that for all $\alpha < \beta$ in B, $\Delta(t_\alpha, t_\beta) = \Delta(x_\alpha, x_\beta)$. Let $X = \{x_\alpha : \alpha \in B\}$. We claim that $\wedge(X) \subseteq T \setminus K$. So consider $\alpha < \beta$ in B. Let $\gamma = \Delta(t_\alpha, t_\beta)$. Since $\gamma \in \Delta(Z, t_\alpha)$ and $\Delta(Z, t_\alpha) \cap K((x_\alpha)) = \varnothing$, $x_\alpha \restriction \gamma \notin K$. By the choice of B, $\gamma = \Delta(x_\alpha, x_\beta)$, so $x_\alpha \wedge x_\beta = x_\alpha \restriction \gamma \notin K$. □

8. Stability for T and K

In previous sections we considered a family $\mathcal{F}$ of subtrees of an ω_1-tree, together with its orthogonal $\mathcal{F}^\perp$, and described forcing φ and ψ for such a family. The situation with families of the form $\mathscr{R}_n$ and $\mathscr{R}_n^\perp$ is more complicated, since the meaning of these objects can change in forcing extensions.

For the remainder of this section, let T be a special coherent Aronszajn tree and let $K \subseteq T$. We first note that being a member of $\mathscr{R}_n$ is upwards absolute.

Lemma 8.1. *Suppose that $\mathbb{Q}$ is an ω_1-preserving forcing. Then for any positive $n \in \omega$:*

(1) For any subtree $R \subseteq T^{\otimes n}$, $R \in \mathscr{R}_n$ implies that $\Vdash_{\mathbb{Q}} \check{R} \in \mathscr{R}_n$.

(2) For any subtree $U \subseteq T^{\otimes n}$, if there exists some $q \in \mathbb{Q}$ which forces that $\check{U} \in \mathscr{R}_n^\perp$, then $U \in \mathscr{R}_n^\perp$.

Proof. (1) It suffices to note that for any uncountable set $Z \subseteq T$, the set R^n_Z is computed the same in both V and $V^{\mathbb{Q}}$. (2) follows easily from (1). $\square$

Definition 8.2. Let $\mathbb{Q}$ be an ω_1-preserving forcing. We say that $\mathbb{Q}$ is *stable for T and K* if for all positive $n \in \omega$, if $U \in \mathscr{R}_n^\perp$ then $\Vdash_{\mathbb{Q}} U \in \mathscr{R}_n^\perp$. If $q \in \mathbb{Q}$ and there exists some $U \in \mathscr{R}_n^\perp$ such that $q \Vdash_{\mathbb{Q}} \check{U} \notin \mathscr{R}_n^\perp$, then we say that q *destabilizes T and K*.

Lemma 8.3. *Suppose that $\mathbb{Q}$ is an ω_1-preserving forcing which preserves subtrees. Then $\mathbb{Q}$ is stable for T and K.*

Proof. Suppose for a contradiction that $n \in \omega$ is positive, $U \in \mathscr{R}_n^\perp$, G is a generic filter on $\mathbb{Q}$, and $V[G]$ satisfies that $U \notin \mathscr{R}_n^\perp$. Then in $V[G]$, there is some $R \in \mathscr{R}_n$ such that the subtree $U \cap R$ is uncountable. Applying Lemma 7.6, find in $V[G]$ a subtree $S \subseteq U \cap R$ such that, letting $Z = \{Y(0) : Y \in S\}$, for every $X \in S$ it is the case that $\Delta(Z, X(0)) \cap K(X) = \emptyset$. By the fact that $\mathbb{Q}$ preserves subtrees, fix a subtree $W \subseteq S$ which lies in V. Define $Z^* = \{X(0) : X \in W\}$, which is an uncountable subset of Z in V. The fact that $W \subseteq S$ and $Z^* \subseteq Z$ easily imply that $W \subseteq R^n_{Z^*}$. So in V, $U \cap R^n_{Z^*}$ contains the uncountable set W, and hence is uncountable. So $U \notin \mathscr{R}_n^\perp$ in V, which is a contradiction. $\square$

Definition 8.4. Let $\varphi(T, K)$ be the statement that for all positive $n \in \omega$, $\varphi(\mathscr{R}_n)$ holds.

Lemma 8.5. *There exists a proper forcing poset which preserves subtrees and forces $\varphi(T, K)$.*

Proof. Define a countable support forcing iteration $(\mathbb{P}_n, \dot{\mathbb{Q}}_n : n \in \omega)$ so that for all $n \in \omega$, $\mathbb{P}_n$ forces that $\dot{\mathbb{Q}}_n$ is the subtree preserving proper forcing of Corollary 5.2 which forces $\varphi(\mathscr{R}_{n+1})$. By Corollary 5.1, $\mathbb{P}_\omega$ preserves subtrees. To see that $\mathbb{P}_\omega$ forces that $\varphi(T, K)$ holds, consider a generic filter G on $\mathbb{P}_\omega$. Let $n \in \omega$ be positive and let $G_n = G \cap \mathbb{P}_n$. In $V[G_n]$, fix a sequence $(U_\xi : \xi \in \omega_1)$ and a club $C \subseteq \omega_1$ which witness that $\varphi(\mathscr{R}_n)$ holds. By Lemma 8.1(1), the same sequence and club witness that $\varphi(\mathscr{R}_n)$ holds in $V[G]$ provided that the elements of this sequence which are in $\mathscr{R}_n^\perp$ in $V[G_n]$ are still in $\mathscr{R}_n^\perp$ in $V[G]$. In other words, $\varphi(\mathscr{R}_n)$ holds in $V[G]$ provided that the quotient forcing $\mathbb{P}_\omega / G_n$ is stable for T and K in $V[G_n]$. But this fact holds by Corollary 5.1 applied to the tail of the iteration together with Lemma 8.3. $\square$

In the next section, we prove that it is consistent that every proper forcing is stable for T and K. We also need to know that this property can be preserved by certain forcings.

Lemma 8.6. *Suppose that every proper forcing is stable for T and K. Let $\mathcal{F}_n = \mathscr{R}_n \cup \mathscr{R}_n^\perp$ for each positive $n \in \omega$. Assume that $\mathbb{P}$ is a proper forcing which forces that $\psi(\mathcal{F}_n)$ holds for all positive $n \in \omega$. Then $\mathbb{P}$ forces that every proper forcing is stable for T and K.*

Proof. Let $\dot{\mathbb{Q}}$ be a $\mathbb{P}$-name for a proper forcing. Let G be a generic filter on $\mathbb{P}$ and let H be a $V[G]$-generic filter on $\mathbb{Q} = \dot{\mathbb{Q}}^G$. Suppose for a contradiction that for some positive $n \in \omega$ and some $S \in \mathscr{R}_n^\perp$ in $V[G]$, S is not in $\mathscr{R}_n^\perp$ in $V[G][H]$. Fix $R \in \mathscr{R}_n$ in $V[G][H]$ such that $S \cap R$ is uncountable. By upwards absoluteness, $\psi(\mathcal{F}_n)$ holds in $V[G][H]$, and hence $\mathcal{F}_n$ is predense in $V[G][H]$. So we can fix $U \in \mathcal{F}_n$ such that $U \cap S \cap R$ is uncountable. We claim that $U \in \mathscr{R}_n^\perp$ in V. If not, then since $\mathcal{F}_n = \mathscr{R}_n \cup \mathscr{R}_n^\perp$, $U \in \mathscr{R}_n$ in V. By Lemma 8.1, $U \in \mathscr{R}_n$ in $V[G]$. But $U \cap S$ is uncountable, which contradicts that $S \in \mathscr{R}_n^\perp$ in $V[G]$. Now in $V[G][H]$, $U \cap S \cap R \subseteq U \cap R$, and hence $U \cap R$ is uncountable. Therefore, $U \notin \mathscr{R}_n^\perp$ in $V[G][H]$. So in V, $\mathbb{P} * \dot{\mathbb{Q}}$ is a proper forcing which is not stable for T and K, which contradicts our assumptions. □

Starting with a model in which all proper forcings are stable for T and K and letting $\mathcal{F}_n = \mathscr{R}_n \cup \mathscr{R}_n^\perp$ for all positive $n \in \omega$, we can use the forcing of Lemma 5.3 to obtain a generic extension in which $\psi(\mathcal{F}_n)$ holds for all n. Since ψ is upwards absolute, any further proper forcing extension also satisfies the same. Applying Lemma 8.6, we can in this way make the property "all proper forcings are stable for T and K" indestructible with respect to proper forcings.

9. All Proper Forcings Can Be Stable

In this section, we prove that it is consistent that all proper forcings are stable.

Theorem 9.1. *Suppose that T is a special coherent Aronszajn tree and $K \subseteq T$. There is a proper poset which forces "every proper forcing is stable for T and K."*

Proof. We begin by recursively constructing $\dot{\mathbb{Q}}_\alpha$ for each ordinal α maintaining that for each α, $(\dot{\mathbb{Q}}_\xi : \xi \in \alpha)$ defines a countable support forcing iteration $(\mathbb{P}_\xi, \dot{\mathbb{Q}}_\xi : \xi \in \alpha)$ so that $\dot{\mathbb{Q}}_\alpha$ is a $\mathbb{P}_\alpha$-name for a proper forcing. We simultaneously recursively define ordinals γ_α and $\mathbb{P}_\alpha$-names $\dot{\mathcal{F}}_\alpha^n$ for each $n \in \omega$. Since the construction describes a proper class, we will be careful to avoid using global choice to ensure that the definition works within the framework of ZFC.

If $(\dot{\mathbb{Q}}_\xi : \xi \in \alpha)$ has been defined and $\mathbb{P}_\alpha$ is the countable support iteration, define $\dot{\mathcal{F}}_\alpha^n$ for each n as follows. $\dot{\mathcal{F}}_\alpha^n$ consists of all $(\dot{S}, p)$ such that:

- $p \in \mathbb{P}_\alpha$,
- $\dot{S}$ is a nice $\mathbb{P}_\alpha$-name for a subset of $T^{\otimes n}$ and if α is a limit ordinal, then moreover $\dot{S}$ is a nice $\mathbb{P}_\beta$-name for some $\beta < \alpha$, and
- $p \Vdash_\alpha \dot{S} \in \dot{\mathscr{R}}_n \cup \dot{\mathscr{R}}_n^\perp$.

We note that if $\mathbb{P}_\alpha$ is a complete suborder of a larger forcing $\mathbb{P}'$, we will also regard $\dot{\mathcal{F}}_\alpha^n$ as a $\mathbb{P}'$-name; with this convention $\dot{\mathcal{F}}_\alpha^n$ names the same set in both the intermediate extension by $\mathbb{P}_\alpha$ and in the full generic extension. Define γ_α to be the least limit ordinal such that:

- there is a $\mathbb{P}_\alpha$-name $\dot{\mathbb{Q}}$ of rank less than γ_α for a proper poset which forces $\forall n\, \psi(\dot{\mathcal{F}}_\alpha^n)$ and
- if there is a $\mathbb{P}_\alpha$-name $\dot{\mathbb{Q}}$ for a proper forcing such that there is $p \in \mathbb{P}_\alpha$, $n \in \omega$, and a $\mathbb{P}_\alpha$-name $\dot{S}$ for a member of $\dot{\mathscr{R}}_n^\perp$ such that
$$p \Vdash_{\mathbb{P}_\alpha * \dot{\mathbb{Q}}} \dot{S} \notin \dot{\mathscr{R}}_n^\perp \wedge \forall m\, \psi(\dot{\mathcal{F}}_\alpha^m)$$
then there is such a $\dot{\mathbb{Q}}$ of rank less than γ_α.

Roughly speaking, we define $\dot{\mathbb{Q}}_\alpha$ to be the $\mathbb{P}_\alpha$-name for the lottery sum of all proper partial orders having rank less than γ_α and which force $\forall n\, \psi(\dot{\mathcal{F}}_\alpha^n)$. Notice that by Lemma 7.7, each $\dot{\mathcal{F}}_\alpha^n$ is forced by $\mathbb{P}_\alpha$ to be predense and hence by Lemma 5.3, this lottery sum is nontrivial.

More formally, define $\dot{\mathbb{Q}}_\alpha$ to consist of $\check{\mathbb{1}}$ together with all $p \in \mathbb{P}_\alpha$ and all $(\dot{x}, p)$ such that:

- $\dot{x}$ is a $\mathbb{P}_\alpha$-name of rank less than γ_α which is forced to be an ordered pair whose first coordinate is a poset and whose second element is an element of this poset;
- if $\dot{\mathbb{Q}}$ is a $\mathbb{P}_\alpha$-name for the first coordinate of $\dot{x}$ and $\dot{q}$ is a $\mathbb{P}_\alpha$-name for the second coordinate, then $p \Vdash_\alpha \dot{q} \in \dot{\mathbb{Q}}$;

Define $\leq_{\dot{\mathbb{Q}}_\alpha}$ to consist of all $(\dot{z}, p)$ such that:

- $p \in \mathbb{P}_\alpha$,
- $\dot{z}$ is a $\mathbb{P}_\alpha$-name which has rank less than γ_α;
- p forces in $\mathbb{P}_\alpha$ that $\dot{z}$ is an ordered pair of elements of $\dot{\mathbb{Q}}_\alpha$;
- letting $\dot{x}$ and $\dot{y}$ be $\mathbb{P}_\alpha$-names for the first and second coordinates of $\dot{z}$, respectively, either:
  - p forces the second coordinate of $\dot{x}$ is $\check{\mathbf{1}}$ or
  - for some $\dot{\mathbb{Q}}$, $\leq_{\dot{\mathbb{Q}}}$, $\dot{q}_0$, and $\dot{q}_1$,

$$((\dot{\mathbb{Q}}, \leq_{\dot{\mathbb{Q}}}, \dot{q}_0), p), ((\dot{\mathbb{Q}}, \leq_{\dot{\mathbb{Q}}}, \dot{q}_1), p) \in \dot{\mathbb{Q}}_\alpha,$$

$$p \Vdash_\alpha \dot{q}_1 \leq \dot{q}_0$$

This completes the recursive construction.

Notice that since properness is preserved by countable support iterations, each $\mathbb{P}_\alpha$ is proper. The class of ordinals δ such that if $\alpha < \delta$, then $\gamma_\alpha < \delta$ is a closed unbounded class. Let θ be the least element of this class of uncountable cofinality. We first show that $\mathbb{P}_\theta$ has the property that if $\dot{\mathbb{Q}}$ is a $\mathbb{P}_\theta$-name for a proper partial order which forces $\forall n\, \psi(\dot{\mathcal{F}}^n_\theta)$, then $\mathbb{P}_\theta$ forces that $\dot{\mathbb{Q}}$ is stable for $\check{T}$ and $\check{K}$. Then we will prove that $\mathbb{P}_{\theta+1}$ satisfies the conclusion of the lemma.

Toward our first goal, suppose for contradiction that there is a $p \in \mathbb{P}_\theta$ and a $\mathbb{P}_\theta$-name $\dot{\mathbb{Q}}$ for a proper poset which forces $\forall n\, \psi(\dot{\mathcal{F}}^n_\theta)$ such that p forces that $\dot{\mathbb{Q}}$ is not stable for $\check{T}$ and $\check{K}$. By extending p if necessary, we may assume that for some positive n and $\dot{q}$ with $p * \dot{q} \in \mathbb{P}_\theta * \dot{\mathbb{Q}}$, there is a $\mathbb{P}_\theta$-name $\dot{S}$ such that

$$p \Vdash_\theta \dot{S} \in \dot{\mathscr{R}}_n^\perp$$

$$p * \dot{q} \Vdash_{\mathbb{P}_\theta * \dot{\mathbb{Q}}} \dot{S} \notin \dot{\mathscr{R}}_n^\perp.$$

Let $\dot{U}$ be a $\mathbb{P}_\theta * \dot{\mathbb{Q}}$-name such that $p * \dot{q}$ forces that $\dot{U}$ is a subtree of $\dot{S}$ which is contained in an element of $\dot{\mathscr{R}}_n$. Since $\psi(\dot{\mathcal{F}}^n_\theta)$ implies that $\dot{\mathcal{F}}^n_\theta$ is predense, we can find $p' * \dot{q}' \leq p * \dot{q}$ in $\mathbb{P}_\theta * \dot{\mathbb{Q}}$, an $\alpha < \theta$, and a nice $\mathbb{P}_\alpha$-name $\dot{S}'$ such that

$$p' \Vdash_\theta \dot{S}' \in \dot{\mathscr{R}}_n \cup \dot{\mathscr{R}}_n^\perp$$

$$p' * \dot{q}' \Vdash_{\mathbb{P}_\theta * \dot{\mathbb{Q}}} \dot{S}' \cap \dot{U} \text{ is uncountable.}$$

Since $\mathrm{cof}(\theta) > \omega$, by increasing α if necessary we may assume that it contains the support of p'.

Notice that since $p' * \dot{q}' \Vdash_{\mathbb{P}_\theta * \dot{\mathbb{Q}}} \dot{S}' \cap \dot{S}$ is uncountable, it follows that

$$p' \Vdash_\theta \dot{S}' \in \dot{\mathscr{R}}_n^\perp.$$

Also we have that

$$p' * \dot{q}' \Vdash_{\mathbb{P}_\theta * \dot{\mathbb{Q}}} \dot{S}' \notin \dot{\mathscr{R}}_n^\perp.$$

Since membership to $\mathscr{R}_n^\perp$ is downward absolute by Lemma 8.1,

$$p' \Vdash_\alpha \dot{S}' \in \dot{\mathscr{R}}_n^\perp.$$

Let $\dot{G}_\alpha$ be the canonical $\mathbb{P}_\alpha$-name for the generic filter and let $\dot{\mathbb{Q}}'$ be the $\mathbb{P}_\alpha|_{p'}$-name for the poset $(\mathbb{P}_\theta|_{p'}/\dot{G}_\alpha) * (\dot{\mathbb{Q}}|_{\dot{q}'})$. Then $(\mathbb{P}_\alpha|_{p'}) * \dot{\mathbb{Q}}'$ is forcing equivalent to $(\mathbb{P}_\theta|_{p'}) * (\dot{\mathbb{Q}}|_{\dot{q}'})$ and hence

$$p' \Vdash_{\mathbb{P}_\alpha * \dot{\mathbb{Q}}'} \dot{S}' \notin \dot{\mathscr{R}}_n^\perp.$$

Also, since $\dot{\mathbb{Q}}_\alpha$ is chosen so that it will force $\forall m\, \psi(\dot{\mathcal{F}}_\alpha^m)$, it is forced that $(\mathbb{P}_\theta|_{p'}/\dot{G}_\alpha) * \dot{\mathbb{Q}}$ is a proper poset which forces $\forall m\, \psi(\dot{\mathcal{F}}_\alpha^m)$. By definition of γ_α, there is a $\mathbb{P}_\alpha$-name $\dot{\mathbb{Q}}''$ of rank less than γ_α such that:

$$p' \Vdash_\alpha \dot{\mathbb{Q}}'' \text{ is a proper poset which forces } \dot{S}' \notin \dot{\mathscr{R}}_n^\perp \text{ and } \forall m\, \psi(\dot{\mathcal{F}}_\alpha^m).$$

Now extend p' to the condition p'' in $\mathbb{P}_{\alpha+1}$ whose $(\alpha+1)^{\text{st}}$-coordinate is

$$((\dot{\mathbb{Q}}'', \leq_{\dot{\mathbb{Q}}''}), \mathbf{1}_{\dot{\mathbb{Q}}''}).$$

Clearly,

$$p'' \Vdash_{\alpha+1} \dot{S}' \notin \dot{\mathscr{R}}_n^\perp,$$

and therefore

$$p'' \Vdash_\theta \dot{S}' \notin \dot{\mathscr{R}}_n^\perp,$$

contrary to $p'' \leq p'$ and $p' \Vdash_\theta \dot{S}' \in \dot{\mathscr{R}}_n^\perp$. This contradiction completes the proof that $\mathbb{P}_\theta$ forces that any proper forcing which forces $\forall n\, \psi(\dot{\mathcal{F}}_\theta^n)$ is stable for $\check{T}$ and $\check{K}$.

Finally, we will verify that $\mathbb{P}_{\theta+1}$ forces that if $\dot{\mathbb{Q}}$ is a proper poset, then $\dot{\mathbb{Q}}$ is stable for $\check{T}$ and $\check{K}$. First observe that by upwards absoluteness, $\mathbb{P}_\theta$ forces that $\dot{\mathbb{Q}}_\theta * \dot{\mathbb{Q}}$ is a proper poset which forces $\forall n\, \psi(\dot{\mathcal{F}}_\theta^n)$. Thus, $\mathbb{P}_\theta$ forces that $\dot{\mathbb{Q}}_\theta * \dot{\mathbb{Q}}$ is stable for $\check{T}$ and $\check{K}$.

Now suppose for a contradiction that $n \in \omega$, $\dot{S}$ is a $\mathbb{P}_{\theta+1}$-name, and $p * \dot{q} \in \mathbb{P}_{\theta+1} * \dot{\mathbb{Q}}$ is such that

$$p \Vdash_{\theta+1} \dot{S} \in \dot{\mathscr{R}}_n^\perp$$

$$p * \dot{q} \Vdash_{\mathbb{P}_{\theta+1} * \dot{\mathbb{Q}}} \dot{S} \notin \dot{\mathscr{R}}_n^\perp$$

Let $\dot{U}$ be a $\mathbb{P}_{\theta+1} * \dot{\mathbb{Q}}$-name for a subtree of $\dot{S}$ which $p * \dot{q}$ forces is contained in an element of $\dot{\mathscr{R}}_n$. Since $\mathbb{P}_{\theta+1} * \dot{\mathbb{Q}}$ forces $\dot{\mathcal{F}}_\theta^n$ is predense, there is a $\mathbb{P}_\theta$-name $\dot{S}'$ for an element of $\dot{\mathcal{F}}_\theta^n$ and a $p' * \dot{q}' \leq p * \dot{q}$ such that

$$p' * \dot{q}' \Vdash_{\mathbb{P}_{\theta+1} * \dot{\mathbb{Q}}} \dot{S}' \cap \dot{U} \text{ is uncountable.}$$

In particular,

$$p' * \dot{q}' \Vdash_{\mathbb{P}_{\theta+1} * \dot{\mathbb{Q}}} \dot{S}' \cap \dot{S} \text{ is uncountable.}$$

Since $p' \Vdash_{\theta+1} \dot{S} \in \mathscr{R}_n^\perp$, it must be that $p' \restriction \theta \Vdash_\theta \dot{S}' \in \mathscr{R}_n^\perp$. But now $\dot{U}$ witnesses that

$$p' * \dot{q}' \Vdash_{\mathbb{P}_\theta * \dot{\mathbb{Q}}_\theta * \dot{\mathbb{Q}}} \dot{S}' \notin \dot{\mathscr{R}}_n^\perp.$$

This contradicts our earlier observation that $\mathbb{P}_\theta$ forces that $\dot{\mathbb{Q}}_\theta * \dot{\mathbb{Q}}$ is stable for $\check{T}$ and $\check{K}$. $\square$

10. Rejection and the Key Lemma

The remaining tools which we need for proving the main theorems of the article are the forcing posets of [14] related to forcing an instance of CAT. These forcings involve the notion of rejection and the associated Key Lemma. We now review these ideas and their connection with the statement φ. For the remainder of this section and the next, fix a special Aronszajn tree T which is coherent, uniform, and binary, and fix a set $K \subseteq T$.

Definition 10.1. Define E_0 as the club subset of $[H(\omega_2)]^\omega$ consisting of all countable elementary submodels of $H(\omega_2)$ which contain T and K as elements. Let $\mathcal{E}$ be the family of all club subsets of $[H(\omega_2)]^\omega$ which are subsets of E_0.

Definition 10.2 ([14])**.** Let $n \in \omega$ be positive and let $X \in T^{\otimes n}$. For any $P \in E_0$, we say that P *rejects* X if X has height at least $P \cap \omega_1$ and there exists an uncountable set $Z \subseteq T$ in P and some $t \in T_{P \cap \omega_1}$ in the downward closure of Z such that $\Delta(Z,t) \cap K(X) = \varnothing$.

In the above, since $\Delta(Z,t) \subseteq P \cap \omega_1$, easily P rejects X iff P rejects $X \restriction (P \cap \omega_1)$. Note that P does not reject the empty tuple.

The next lemma connects the idea of rejection with the families $\mathscr{R}_n$, for positive $n \in \omega$. The proof is straightforward.

Lemma 10.3 ([9, Lemma 2.6])**.** *Let $n \in \omega$ be positive. Let $P \in E_0$ and let $X \in T^{\otimes n}$ have height at least $P \cap \omega_1$. Then P rejects X iff there exists some $R \in P \cap \mathscr{R}_n$ such that $X \restriction (P \cap \omega_1) \in R$.*

Lemma 10.4. *Let $n \in \omega$ be positive. Suppose that S is a subtree of $T^{\otimes n}$ and there exist stationarily many $P \in E_0$ such that P does not reject any member of S. Then $S \in \mathscr{R}_n^\perp$.*

Proof. Suppose for a contradiction that $S \notin \mathscr{R}_n^\perp$. Then there exists $R \in \mathscr{R}_n$ such that $S \cap R$ is uncountable. By the stationarity assumption, find $P \in E_0$ such that $R \in P$ and P does not reject any member of S. As $S \cap R$ is uncountable and downwards closed, we can fix some $X \in S \cap R$ with height $P \cap \omega_1$. By Lemma 10.3, P rejects X, which contradicts the fact that $X \in S$. $\square$

For any finite set $X \subseteq T$ and any $\gamma \in \omega_1$, let $X \restriction \gamma$ denote the tuple which enumerates the set $\{x \restriction \gamma : x \in X,\ \gamma \leq \mathrm{ht}_T(x)\}$ in lexicographically increasing order.

Definition 10.5. Let $P \in E_0$ and let $X \subseteq T$ be finite. We say that P *rejects* X if $X \restriction (P \cap \omega_1)$ is a non-empty tuple which P rejects.

Definition 10.6 ([14])**.** The *Key Lemma* is the statement that for any countable elementary submodel M of $H(2^{2^{\omega_1}+})$ which contains T and K as elements and for any finite set $X \subseteq T$, there exists $E \in M \cap \mathcal{E}$ such that either every element of $M \cap E$ rejects X, or no element of $M \cap E$ rejects X.

In [14], it is shown that MRP implies the Key Lemma.

Lemma 10.7. *Assume that $\varphi(T, K)$ holds. Then the Key Lemma holds.*

For a proof, see Lemma 2.7 of [9].

11. Instances of CAT

Our final step before proving the main theorems of the article is to describe the forcings of [14] which are related to forcing an instance of CAT, and how the analysis of these forcings in [14] is connected with the idea of stability introduced in Section 8.

Let χ denote $(2^{2^{\omega_1}})^+$ for the rest of the section. The following definitions all appear in [14].

Definition 11.1. Define E_1 as the set of all $N \in [H(2^{\omega_1+})]^\omega$ such that for some countable $M \prec H(\chi)$ with T and K in M, $N = M \cap H(2^{\omega_1+})$.

Note that E_1 is a club.

Definition 11.2. Let $\mathcal{H}(K)$ denote the set of all finite antichains $X \subseteq T$ such that $\wedge(X) \subseteq K$.

Definition 11.3. Define $\partial(K)$ as the forcing poset consisting of all pairs $p = (X_p, \mathcal{N}_p)$ such that:

(1) $\mathcal{N}_p$ is a finite $\in$-chain of members of E_1;
(2) $X_p \subseteq T$ is a finite antichain;
(3) for all $N \in \mathcal{N}_p$, there exists some $E \in N \cap \mathcal{E}$ such that X_p is not rejected by any member of $N \cap E$.

Let $q \leq p$ if $\mathcal{N}_p \subseteq \mathcal{N}_q$ and $X_p \subseteq X_q$.

Definition 11.4. Let $\partial\mathcal{H}(K)$ be the suborder of $\partial(K)$ consisting of conditions p such that $X_p \in \mathcal{H}(K)$.

Note that $\partial(K)$ and $\partial\mathcal{H}(K)$ are both in $H(\chi)$.

Definition 11.5. The forcing $\partial\mathcal{H}(K)$ is *canonically proper* if whenever M is a countable elementary submodel of $H(\chi)$ with T, K, and $\partial\mathcal{H}(K)$ members of M, if $p \in \partial(K)$ and $M \cap H(2^{\omega_1+})$ is in $\mathcal{N}_p$, then p is $(M, \partial(K))$-generic. A similar definition is made for $\partial(K)$ being canonically proper.

Lemma 11.6 ([14, Lemma 5.20])**.** *Assume that $\partial\mathcal{H}(K)$ is canonically proper. Then either there exists an uncountable antichain $X \subseteq T$ such that $\wedge(X) \subseteq T \setminus K$, or else there is a condition in $\partial\mathcal{H}(K)$ which forces that there is an uncountable antichain $Y \subseteq T$ such that $\wedge(Y) \subseteq K$.*

Proof. Fix $M \prec H(\chi)$ which is countable such that T, K, and $\partial\mathcal{H}(K)$ are in M, and let $N = M \cap H(2^{\omega_1+})$. Note that $N \in E_1$. Fix any $x \in T_{M\cap\omega_1}$ and define $p = (\{N\}, \{x\})$. First, assume that p is not in $\partial\mathcal{H}(K)$. It easily follows that there exists $P \in N \cap E_0$ which rejects $\{x\}$. Then P rejects the 1-tuple $(x \restriction (P \cap \omega_1))$, so Lemma 10.3 implies that $x \restriction (P \cap \omega_1) \in R$ for some $R \in P \cap \mathscr{R}_1$. It follows that R is uncountable, so by Lemma 7.8 we are done. Secondly, suppose that p is in $\partial\mathcal{H}(K)$. Let G be a generic filter on $\partial\mathcal{H}(K)$ such that $p \in G$. Since $\partial\mathcal{H}(K)$ is canonically proper, $M[G] \cap V = M$, and therefore $x \notin M[G]$. Since $M[G]$ is an elementary submodel of $H(\chi)^{V[G]}$, by elementarity the set $Y = \bigcup\{X_q : q \in G\}$ is in $M[G]$, and as $x \in Y \setminus M[G]$, Y is uncountable. By the definition of $\partial\mathcal{H}(K)$, Y is an uncountable antichain and $\wedge(Y) \subseteq K$. □

If $\partial\mathcal{H}(K)$ is not canonically proper, then there are two alternatives: either (1) $\partial(K)$ is canonically proper but $\partial\mathcal{H}(K)$ is not, or (2) both $\partial(K)$ and $\partial\mathcal{H}(K)$ are not canonically proper. In [14], it is shown that both alternatives are false assuming PFA. We now connect the core ideas of the proofs of these facts with the idea of stability.

Concerning alternative (1), the following lemma was proven as a part of the proof of Lemma 5.22 of [14].

Lemma 11.7. *Suppose that $\partial(K)$ is canonically proper but $\partial\mathcal{H}(K)$ is not. Then there exist $r \in \partial(K)$, $E \in \mathcal{E}$, and $n \in \omega$ positive such that r forces in $\partial(K)$ that there exists a sequence $(X_\alpha : \alpha \in \omega_1)$ of elements of $T^{\otimes n}$ satisfying:*

(1) for all $P \in E$ and for all $\alpha \in \omega_1$, P does not reject X_α;
(2) for all $\alpha < \beta$ there exists $i < n$ such that $X_\alpha(i) \wedge X_\beta(i) \notin K$.

Lemma 11.8. *Suppose that $\partial(K)$ is canonically proper but $\partial\mathcal{H}(K)$ is not. Then $\partial(K)$ is not stable for T and K.*

Proof. Fix r, E, and n as in Lemma 11.7, and let $(\dot{X}_\alpha : \alpha \in \omega_1)$ be a sequence of $\partial(K)$-names for which r forces properties (1) and (2) of that lemma to hold. Define S as the set of $X \in T^{\otimes n}$ such that for all $P \in E$, P does not reject X. Then r forces that for all $\alpha \in \omega_1$, $\dot{X}_\alpha \in \check{S}$. In particular, S is uncountable. By Lemma 10.4 and (1), $S \in \mathscr{R}_n^\perp$, and by Lemma 7.5 and (2), r forces that $S \notin \mathscr{R}_n^\perp$. So r destabilizes T and K. $\square$

Concerning alternative (2), a proof of the following lemma appears as a part of the proof of Lemma 5.31 of [14].

Lemma 11.9. *Assume that the Key Lemma holds and $\partial(K)$ is not canonically proper. Then there exist $E \in \mathcal{E}$, $n \in \omega$ positive, and a c.c.c. forcing $\mathbb{Q}$ such that $\mathbb{Q}$ forces that there exists a sequence $(X_\alpha : \alpha \in \omega_1)$ of elements of $T^{\otimes n}$ satisfying:*

(1) for all $P \in E$ and for all $\alpha \in \omega_1$, P does not reject X_α;
(2) for all $\alpha < \beta$ there exists $i < n$ such that $X_\alpha(i) \wedge X_\beta(i) \notin K$.

Using Lemma 10.7, the next lemma follows by the same argument as we gave for Lemma 11.8.

Proposition 11.10. *Suppose that $\varphi(T, K)$ holds and $\partial(K)$ is not canonically proper. Then there exists a c.c.c. forcing which is not stable for T and K.*

12. The Main Results

We now have all of the tools at our disposal which are needed to prove the main theorems of the article.

Theorem 12.1. *Let T be a special Aronszajn tree which is coherent, uniform, and binary. Then for any $K \subseteq T$, there exists a proper forcing which forces that there is an uncountable antichain $A \subseteq T$ such that either $\wedge(A) \subseteq K$ or $\wedge(A) \subseteq T \setminus K$.*

Proof. Applying Theorem 9.1, let $\mathbb{P}$ be a proper forcing which forces that every proper forcing is stable for T and K. For each positive $n \in \omega$, let $\dot{\mathcal{F}}_n$ be a $\mathbb{P}$-name for the set $\mathscr{R}_n \cup \mathscr{R}_n^\perp$ as computed in $V^{\mathbb{P}}$. By Lemma 5.3, we can fix a $\mathbb{P}$-name $\dot{\mathbb{Q}}$ for a proper forcing which forces $\psi(\dot{\mathcal{F}}_n)$ for all positive $n \in \omega$. Applying Lemma 8.5, let $\dot{\mathbb{R}}$ be a $\mathbb{P} * \dot{\mathbb{Q}}$-name for a proper forcing which forces $\varphi(T, K)$. Since ψ is upwards absolute, for all positive $n \in \omega$, $\mathbb{P} * \dot{\mathbb{Q}} * \dot{\mathbb{R}}$ forces $\psi(\mathcal{F}_n)$. By Lemma 8.6, $\mathbb{P} * \dot{\mathbb{Q}} * \dot{\mathbb{R}}$ forces that every proper forcing is stable for T and K. By Propositions 11.8 and 11.10, the forcing $\mathbb{P} * \dot{\mathbb{Q}} * \dot{\mathbb{R}}$ forces that $\partial\mathcal{H}(K)$ is canonically proper. Considering the two cases described in Lemma 11.6, let $\dot{\mathbb{S}}$ be a $\mathbb{P} * \dot{\mathbb{Q}} * \dot{\mathbb{R}}$-name which is forced to be equal to: (a) in the first case, the trivial forcing, and (b) in the second case, $\partial\mathcal{H}(K)|_r$ for some condition r which forces the existence of an uncountable antichain all of whose meets are in K. The forcing $\mathbb{P} * \dot{\mathbb{Q}} * \dot{\mathbb{R}} * \dot{\mathbb{S}}$ is as required. $\square$

The nontrivial content of the following two lemmas is stated without proof in [1]; details can be found in [24, Proposition 8.7].

Lemma 12.2. *Assume MA_{ω_1} and any two normal Aronszajn trees are club isomorphic. Then the following statements are equivalent:*

(1) There exists a normal Aronszajn tree T such that $\mathsf{CAT}(T)$ holds.
(2) CAT.

We note that by [1], the proper forcing axiom for proper forcings of size at most ω_1 implies the assumptions of Lemma 12.2.

By *Baumgartner's axiom* we mean the statement that any two ω_1-dense sets of reals are isomorphic ([4]).

Lemma 12.3. *Assume MA_{ω_1}, any two normal Aronszajn trees are club isomorphic, and Baumgartner's axiom. Then the following are equivalent:*

(1) CAT;
(2) Shelah's conjecture;
(3) there exists a five element basis for the uncountable linear orders consisting of ω_1 and its converse ω_1^*, any set of reals of size ω_1, and any Countryman line C and its converse C^*.

Putting these together, we obtain:

Theorem 12.4. *BPFA implies CAT.*

Proof. Let T be a special Aronszajn tree which is coherent, uniform, and binary. For any $K \subseteq T$, the statement that there exists an uncountable antichain whose meets are either all in K or all in $T \setminus K$ is a Σ_1-statement with parameters in $H(\omega_2)$. By Theorem 12.1, there exists a proper forcing which forces this statement, so by BPFA, this statement is true. As BPFA implies the assumptions of Lemma 12.2, we are done. $\square$

Since the assumptions of Lemma 12.3 follow from BPFA, we immediately obtain the following corollary.

Corollary 12.5. *BPFA implies Shelah's conjecture and the existence of a five element basis for the uncountable linear orders.*

Theorem 12.6. *If κ is inaccessible, then in L there exists a proper forcing which forces the proper forcing axiom for proper forcings of size at most ω_1, Baumgartner's axiom, Shelah's conjecture, and φ.*

Proof. Let κ be an inaccessible cardinal. Then κ is inaccessible in L and $\diamondsuit_\kappa$ holds. We define a countable support forcing iteration $\mathbb{P}$ of length κ of forcings of size less than κ. In order to arrange that $\mathbb{P}$ forces φ, we repeat the argument of Theorem 6.1. Since κ remains inaccessible after forcing with any initial segment of $\mathbb{P}$, and hence V_κ is a model of ZFC in any generic extension by such an initial segment, we can apply Theorem 12.1 to force instances of CAT with forcings of size less than κ, bookkeeping to make sure all T and K which appear in $V^{\mathbb{P}}$ are handled. Similarly, any required instance of Baumgartner's axiom or the proper forcing axiom for proper forcings of size at most ω_1 can be handled by standard bookkeeping. Now we are done by Lemma 12.3. $\square$

Corollary 12.7. *The following are equiconsistent:*

(1) There exists an inaccessible cardinal.
(2) There exists a five element basis for the uncountable linear orders and φ holds.

13. Open Problems

We close the article by stating some questions related to Shelah's conjecture which remain open.

Question 13.1 ([14])**.** *Does Shelah's conjecture have any large cardinal consistency strength?*

It is not known, for example, if the existence of a Kurepa tree implies the failure of Shelah's conjecture. If it does, then Corollary 12.7 provides the optimal consistency strength result.

Question 13.2. *Does Shelah's conjecture imply the non-existence of a Kurepa tree, or Aronszajn tree saturation?*

Similar questions can be asked by replacing Shelah's conjecture with any of the statements listed in the conclusion of Lemma 12.3.

BPFA implies the non-existence of a Kurepa tree, but we do not know if it implies the related statement of Aronszajn tree saturation. Under BPFA, φ is equivalent to Aronszajn tree saturation.

Question 13.3 ([9])**.** *Does BPFA imply Aronszajn tree saturation?*

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