

COMBINATORIAL PROPERTIES RELATED TO THE HIGHER BAUMGARTNER'S AXIOM

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ABSTRACT. We isolate two combinatorial properties, each expressible by a Π_2 -sentence over the structure $(H(\omega_3), \in, \omega_1, \omega_2, \text{NS}_{\omega_2})$, such that each property is consistent with CH, and their conjunction together with $2^\omega \leq \omega_2$ and $2^{\omega_1} = 2^{\omega_2} = \omega_3$ implies the existence of a c.c.c. forcing which forces the higher Baumgartner's axiom.

1. INTRODUCTION

For any uncountable cardinal κ , a set of reals X is κ -dense if for all $x < y$ in X , there exist exactly κ -many members of X between x and y . *Baumgartner's axiom for κ* , or BA_κ , is the property that any two κ -dense sets of reals without endpoints are isomorphic. The statement BA_{ω_1} , which is the only case of the above property which is currently known to be consistent, is known as *Baumgartner's axiom*. Baumgartner [Bau73] proved that assuming CH, any two ω_1 -dense sets of reals without endpoints can be made isomorphic by a c.c.c. forcing of size ω_1 . Thus, starting with a model of GCH, there exists a finite support forcing iteration of c.c.c. forcings of length ω_2 which forces that $2^\omega = \omega_2$ and BA_{ω_1} holds.

Baumgartner [Bau82] asked whether it is consistent that any two ω_2 -dense sets of reals without endpoints are isomorphic, a statement which we refer to as the *higher Baumgartner's axiom*. Despite the interest in this problem by Baumgartner and others (e.g. [ARS85]), no progress was made on it for more than three decades. By a result of Sierpiński, ZFC implies that there are at least $(2^\omega)^+$ -many distinct order types of sets of reals of size 2^ω ([Sie50]; also see [Bau82, Theorem 2.4(a)]). On the other hand, Abraham, Rubin, and Shelah [ARS85] proved the consistency of BA_{ω_1} together with $2^\omega > \omega_2$. Hence, a natural generalization of Baumgartner's axiom to higher cardinals is the statement that $2^\omega > \omega_2$ and for any uncountable cardinal $\kappa < 2^\omega$, BA_κ holds.

Recently, Moore and Todorćević [MT17] made a potential breakthrough on Baumgartner's question. They introduced a combinatorial property of ω_2 denoted by $(**)$ which, together with $2^\omega \leq \omega_2$ and $2^{\omega_1} = 2^{\omega_2} = \omega_3$, implies the existence of a c.c.c. forcing which forces $2^\omega = \omega_3$, MA_{ω_2} , BA_{ω_1} , and BA_{ω_2} . The general idea of their proof is to bootstrap a known c.c.c. forcing for adding an increasing function between two sets of reals of size ω_1 to sets of reals of size ω_2 by way of a sequence of bijections between ω_1 and uncountable ordinals less than ω_2 ([Tod89, Theorem 4.2]). They showed that assuming $(**)$ and $2^\omega \leq \omega_2$, there is a forcing which adds an increasing function from a set of reals of size ω_2 into another which can be written as an increasing union with length ω_2 of c.c.c. suborders, and hence is itself c.c.c. Around the time that these results were announced, there were indications that a proof of the consistency of $(**)$ from large cardinals would be forthcoming (see [MT17, Section 5]), but more than a decade later the status of the consistency of $(**)$ is still unresolved.

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In this article, we isolate two statements (A) and (B) , each consistent with CH , whose conjunction is equivalent to $(**)$. Both statements can be expressed by Π_2 sentences in the language of the structure $(H(\omega_3), \in, \omega_1, \omega_2, \text{NS}_{\omega_2})$. Statement (A) is the assertion that any family of ω_2 -many club subsets of ω_1 can be diagonalized. Statement (B) is a formal weakening of $(**)$. The consistency of (A) is well-known and can be obtained by a countable support forcing iteration of the classical fast club forcing of Jensen. Our proof of the consistency of (B) uses non-traditional techniques, and involves a countable support forcing iteration with uncountable models as side conditions.¹

A key feature of our approach is to fix in advance a nice family of models to use as side conditions, where the relevant relationship between the models is a feature of the family. In contrast, many side condition forcings include arbitrary elementary substructures as side conditions with additional restrictions on how models appearing in a condition interact (e.g. [Mit09]). This “off-the-shelf” approach to side conditions provides some simplifications, since it trivializes the parts of amalgamation arguments which purely involve models. We also used this idea in [Kru18] in our construction of a model of CH in which any two countably closed ω_2 -Aronszajn trees are club isomorphic ([Kru18]). A major difference between [Kru18] and this article is that here we define the family of models using $\square_{\omega_2}$ rather than from a large cardinal assumption. As a result, we prove the consistency of (B) assuming only the consistency of ZFC .

2. COMBINATORIAL PROPERTIES

We begin by reviewing the property $(**)$ and proving that it is equivalent to the conjunction of the statements (A) and (B) introduced below.

Definition 2.1 (Moore and Todorćević ([MT17])). Define $(**)$ to be the statement that for any family $\mathcal{F} \subseteq {}^{\omega_2}\omega_2$ of injective functions with $|\mathcal{F}| \leq \omega_2$, there exists an injective function $h : \omega_2 \rightarrow \omega_2$ such that for any $f \in \mathcal{F}$, there exists a countable set $D \subseteq \omega_2$ such that:

- for all $\alpha \in \omega_2 \setminus D$, $f(\alpha) \neq h(\alpha)$;
- for all distinct $\alpha, \beta \in \omega_2 \setminus D$, $f(h(\alpha)) \neq h(\beta)$.

Theorem 2.2 (Moore and Todorćević ([MT17])). *Assume that $2^\omega \leq \omega_2$, $2^{\omega_2} = \omega_3$, and $(**)$ holds. Then there exists a c.c.c. forcing which forces MA_{ω_1} , BA_{ω_1} , and BA_{ω_2} .*

We isolate two statements (A) and (B) , which jointly imply that $(**)$ holds.

Definition 2.3. Let (A) denote the statement that for any family $\mathcal{C}$ of at most ω_2 -many club subsets of ω_1 , there exists a club E such that for all $C \in \mathcal{C}$, $E \setminus C$ is countable.

The consistency of (A) is due to Jensen. Namely, assuming CH there exists an ω_1 -closed, ω_2 -centered forcing which adds club subset E of ω_1 such that for all clubs C in the ground model, $E \setminus C$ is countable ([DJ74, Chapter IX]). Starting with a model of GCH , this fast club forcing can be iterated with countable support up to ω_3 to obtain a generic extension satisfying (A) .

The following lemma is well-known. We leave the straightforward proof to the interested reader.

Lemma 2.4. *(A) is equivalent to the statement that for any family $\mathcal{F} \subseteq {}^{\omega_1}\omega_1$ of size at most ω_2 , there exists an injective function $g : \omega_1 \rightarrow \omega_1$ such that for all $f \in \mathcal{F}$, there exists $i < \omega_1$ such that for all $\gamma \in \omega_1 \setminus i$, $f(\gamma) < g(\gamma)$.*

¹The idea of using models as side conditions in a forcing iteration is due to Neeman and was originally used to construct a model of PFA with a finite support iteration ([Nee14]).

Define $S_1^2 = \{\alpha < \omega_2 : \text{cf}(\alpha) = \omega_1\}$.

Definition 2.5. Define (B) to be the statement that there exists a sequence $\vec{\pi} = \langle \pi_\beta : \omega_1 \leq \beta < \omega_2 \rangle$, where each $\pi_\beta : \beta \rightarrow \omega_1$ is a bijection, such that for any function $g \in {}^{\omega_1}\omega_1$, there exists an injective function $h : \omega_2 \rightarrow \omega_2$ and there exists a stationary set $I \subseteq S_1^2$ consisting of ordinals closed under h such that for all $\beta \in I$, there exists a countable set $D_\beta \subseteq \beta$ such that:

- for all $\alpha \in \beta \setminus D_\beta$,

$$g(\min\{\pi_\beta(\alpha), \pi_\beta(h(\alpha))\}) < \max\{\pi_\beta(\alpha), \pi_\beta(h(\alpha))\};$$

- for all distinct $\alpha, \gamma \in \beta \setminus D_\beta$,

$$g(\min\{\pi_\beta(h(\alpha)), \pi_\beta(h(\gamma))\}) < \max\{\pi_\beta(h(\alpha)), \pi_\beta(h(\gamma))\}.$$

Lemma 2.6. *The conjunction of (A) and (B) is equivalent to (**).*

Proof. For the forward direction, see Propositions 2.2 and 2.3 of [MT17]. For the reverse direction, let $\mathcal{F} \subseteq {}^{\omega_2}\omega_2$ be a family of injective functions with $|\mathcal{F}| \leq \omega_2$. For each $f \in \mathcal{F}$ and for each $\omega_1 \leq \beta < \omega_2$ such that β is closed under f and f^{-1} , define $g_{f,\beta,0}$ and $g_{f,\beta,1}$ in ${}^{\omega_1}\omega_1$ as follows. For each $i < \omega_1$, define:

- $g_{f,\beta,0}(i) = \pi_\beta(f(\pi_\beta^{-1}(i)))$;
- $g_{f,\beta,1}(i) = \pi_\beta(f^{-1}(\pi_\beta^{-1}(i)))$ provided that $\pi_\beta^{-1}(i) \in \text{ran}(f)$, and otherwise $g_{f,\beta,1}(i) = 0$.

Now let $\mathcal{G}$ be the set of all functions of the form $g_{f,\beta,j}$, where $f \in \mathcal{F}$, $\omega_1 \leq \beta < \omega_2$, β is closed under f and f^{-1} , and $j < 2$. Then $\mathcal{G} \subseteq {}^{\omega_1}\omega_1$ and $|\mathcal{G}| \leq \omega_2$.

Applying (A) and Lemma 2.4, fix an injective function $g \in {}^{\omega_1}\omega_1$ such that for all $k \in \mathcal{G}$, there exists $i(k) < \omega_1$ such that for all $\gamma \in \omega_1 \setminus i(k)$, $k(\gamma) < g(\gamma)$. Now apply (B) to fix an injective function $h : \omega_2 \rightarrow \omega_2$ and a stationary set $I \subseteq \omega_2$ consisting of ordinals closed under h such that for all $\beta \in I$, there exists a countable set $D_\beta \subseteq \beta$ such that:

- for all $\alpha \in \beta \setminus D_\beta$,

$$g(\min\{\pi_\beta(\alpha), \pi_\beta(h(\alpha))\}) < \max\{\pi_\beta(\alpha), \pi_\beta(h(\alpha))\};$$

- for all distinct $\alpha, \gamma \in \beta \setminus D_\beta$,

$$g(\min\{\pi_\beta(h(\alpha)), \pi_\beta(h(\gamma))\}) < \max\{\pi_\beta(h(\alpha)), \pi_\beta(h(\gamma))\}.$$

Consider $f \in \mathcal{F}$, and we verify the two required properties of Definition 2.1. For the first property, suppose for a contradiction that the set $X = \{\alpha < \omega_2 : f(\alpha) = h(\alpha)\}$ is uncountable. By the stationarity of I , we can find an ordinal $\beta \in I$ which is closed under f , f^{-1} , and h such that $X \cap \beta$ is uncountable. Then $g_{f,\beta,0}$ and $g_{f,\beta,1}$ are in $\mathcal{G}$. Since D_β is countable, we can fix $\alpha \in (X \cap \beta)$ which is not in D_β and satisfies that $\pi_\beta(\alpha) \geq i(g_{f,\beta,0})$ and $\pi_\beta(h(\alpha)) \geq i(g_{f,\beta,1})$. As $\alpha \in X$, $f(\alpha) = h(\alpha)$.

Case 1: $\pi_\beta(\alpha) < \pi_\beta(h(\alpha))$. Since $\alpha \in \beta \setminus D_\beta$, we have that $g(\pi_\beta(\alpha)) < \pi_\beta(h(\alpha))$. But also $\pi_\beta(\alpha) \geq i(g_{f,\beta,0})$, so $g_{f,\beta,0}(\pi_\beta(\alpha)) < \pi_\beta(h(\alpha))$. By the definition of $g_{f,\beta,0}$, this inequality is equivalent to $\pi_\beta(f(\alpha)) < \pi_\beta(h(\alpha))$, which contradicts that $f(\alpha) = h(\alpha)$.

Case 2: $\pi_\beta(h(\alpha)) < \pi_\beta(\alpha)$. Since $\alpha \in \beta \setminus D_\beta$, we have that $g(\pi_\beta(h(\alpha))) < \pi_\beta(\alpha)$. As $\pi_\beta(h(\alpha)) \geq i(g_{f,\beta,1})$, $g_{f,\beta,1}(\pi_\beta(h(\alpha))) < \pi_\beta(\alpha)$. Letting $i = \pi_\beta(h(\alpha))$, note that $\pi_\beta^{-1}(i) = h(\alpha) = f(\alpha)$ which is in the range of f . So by the definition of $g_{f,\beta,1}$, $g_{f,\beta,1}(\pi_\beta(h(\alpha))) = \pi_\beta(f^{-1}(\pi_\beta^{-1}(i))) = \pi_\beta(f^{-1}(h(\alpha))) = \pi_\beta(f^{-1}(f(\alpha))) = \pi_\beta(\alpha)$. Hence, the above inequality is the same as $\pi_\beta(\alpha) < \pi_\beta(\alpha)$, which is a contradiction.

For the second property, suppose for a contradiction that there does not exist a countable set $D_2 \subseteq \omega_2$ such that for all distinct $\alpha, \gamma \in \omega_2 \setminus D_2$, $f(h(\alpha)) \neq h(\gamma)$. Then it is easy to build a set $Y \subseteq \omega_2$ of size ω_1 such that for all countable $Y_0 \subseteq Y$, there are distinct $\alpha, \gamma \in Y \setminus Y_0$ such that $f(h(\alpha)) = h(\gamma)$. By the stationarity of I , we can fix $\beta \in I$ such that β is closed under f , f^{-1} , and h . Fix distinct $\alpha, \gamma \in \beta \setminus D_\beta$ such that $f(h(\alpha)) = h(\gamma)$, $\pi_\beta(h(\alpha)) > i(g_{f,\beta,0})$, and $\pi_\beta(h(\gamma)) > i(g_{f,\beta,1})$. Now the rest of the proof is similar to the above. $\square$

3. THE SINGLE FORCING

The rest of the article is devoted to proving the consistency of (B) . In this section, we describe how to force a single instance of (B) . For the remainder of the section, fix:

- a sequence $\vec{\pi} = \langle \pi_\beta : \omega_1 \leq \beta < \omega_2 \rangle$, where each $\pi_\beta : \omega_1 \rightarrow \beta$ is a bijection;
- an injective function $f : \omega_1 \rightarrow \omega_1$.

Definition 3.1. Define $\mathbb{B}(\vec{\pi}, f)$ to be the forcing whose conditions are ordered pairs $p = (h_p, I_p)$ satisfying:

- (1) h_p is an injective function whose domain is a countable subset of ω_2 which maps into ω_2 and satisfies that $h(\alpha) \neq \alpha$ for all $\alpha \in \text{dom}(h)$;
- (2) I_p is a countable subset of S_1^2 ;
- (3) for all $\beta \in I_p$, β is closed under h_p and h_p^{-1} .

Let $q \leq p$ if:

- (a) $h_p \subseteq h_q$;
- (b) $I_p \subseteq I_q$;
- (c) for all $\alpha \in \text{dom}(h_q) \setminus \text{dom}(h_p)$ and for all $\beta \in I_p$, if $\alpha < \beta$ then:
 - (i) $f(\min\{\pi_\beta(\alpha), \pi_\beta(h_q(\alpha))\}) < \max\{\pi_\beta(\alpha), \pi_\beta(h_q(\alpha))\}$;
 - (ii) for all $\gamma \in \text{dom}(h_q) \cap \beta$ different from α ,
$$f(\min\{\pi_\beta(h_q(\gamma)), \pi_\beta(h_q(\alpha))\}) < \max\{\pi_\beta(h_q(\gamma)), \pi_\beta(h_q(\alpha))\}.$$
²

Lemma 3.2. *The relation on $\mathbb{B}(\vec{\pi}, f)$ is transitive.*

Proof. Assume that $r \leq q \leq p$, and we show that $r \leq p$. (a, b) Clearly, $h_p \subseteq h_r$ and $I_p \subseteq I_r$. (c) Suppose that $\alpha \in \text{dom}(h_r) \setminus \text{dom}(h_p)$, $\beta \in I_p$, and $\alpha < \beta$. (i) If $\alpha \in \text{dom}(h_q)$, then $q \leq p$ implies (i). Suppose that $\alpha \notin \text{dom}(h_q)$. Then $r \leq q$ implies (i). (ii) Let $\gamma \in \text{dom}(h_r) \cap \beta$ be different from α . If $\alpha \notin \text{dom}(h_q)$, then the conclusion of (ii) follows from $r \leq q$. Suppose that $\alpha \in \text{dom}(h_q)$. If $\gamma \in \text{dom}(h_q)$, then $q \leq p$ implies the conclusion of (ii). Suppose that $\gamma \notin \text{dom}(h_q)$. Then the conclusion of (ii) follows from $r \leq q$ with the roles of α and γ reversed. $\square$

Lemma 3.3. *The forcing $\mathbb{B}(\vec{\pi}, f)$ is ω_1 -closed. In fact, if $\langle p_n : n < \omega \rangle$ is a descending sequence of conditions in $\mathbb{B}(\vec{\pi}, f)$, where each $p_n = (h_n, I_n)$, then $(\bigcup_n h_n, \bigcup_n I_n)$ is a condition in $\mathbb{B}(\vec{\pi}, f)$ which is the greatest lower bound of $\langle p_n : n < \omega \rangle$.*

The proof is easy.

Corollary 3.4. *The forcing $\mathbb{B}(\vec{\pi}, f)$ preserves ω_1 .*

Whether and under what circumstances $\mathbb{B}(\vec{\pi}, f)$ preserves ω_2 is a complex issue which we address over the remainder of the article.

²Note that the parameter f is only relevant in the order on $\mathbb{B}(\vec{\pi}, f)$, and not for membership in $\mathbb{B}(\vec{\pi}, f)$.

Lemma 3.5. *For any countable set $Y \subseteq \omega_2$, the set of $q \in \mathbb{B}(\vec{\pi}, f)$ such that $Y \subseteq \text{dom}(h_q)$ is dense open in $\mathbb{B}(\vec{\pi}, f)$.*

Proof. By Lemma 3.3, it suffices to prove the statement in the case that Y is a singleton. Let $p \in \mathbb{B}(\vec{\pi}, f)$ and let $\alpha \in \omega_2$. If $\alpha \in \text{dom}(h_p)$ then we are done, so assume not. We find $q \leq p$ such that $\alpha \in \text{dom}(h_q)$. For each $\xi \in I_p$ greater than α , define X_ξ as the set of $\nu < \xi$ such that at least one of the following is true:

- (1) $\pi_\xi(\nu) \leq \pi_\xi(\alpha)$;
- (2) $\pi_\xi(\nu) \leq f(\pi_\xi(\alpha))$;
- (3) there exists $\gamma \in \text{dom}(h_p) \cap \xi$ such that $\pi_\xi(\nu) \leq \pi_\xi(h_p(\gamma))$;
- (4) there exists $\gamma \in \text{dom}(h_p) \cap \xi$ such that $\pi_\xi(\nu) \leq f(\pi_\xi(h_p(\gamma)))$.

Since π_ξ is injective and $\text{dom}(h_p)$ is countable, X_ξ is countable. Let $X = \bigcup \{X_\xi : \xi \in I_p, \xi > \alpha\}$, which is countable. Let β be the least member of $I_p \cup \{\omega_2\}$ which is strictly greater than α . Then β has cofinality at least ω_1 , so we can find some $\zeta \in (\alpha, \beta)$ which is not in the range of h_p nor in X . Define q by letting $h_q = h_p \cup \{\langle \alpha, \zeta \rangle\}$ and $I_q = I_p$. It is straightforward to check that q is as required. $\square$

Lemma 3.6. *Suppose that $\mathbb{B}(\vec{\pi}, f)$ preserves ω_2 and forces that $\dot{I} = \bigcup \{I_p : p \in \dot{G}\}$ is stationary in ω_2 , where $\dot{G}$ is the canonical $\mathbb{B}(\vec{\pi}, f)$ -name for the generic filter. Then $\mathbb{B}(\vec{\pi}, f)$ forces that $(B)_{\vec{\pi}, f}$ holds as witnessed by $\dot{I}$.*

Proof. Let G be any generic filter on $\mathbb{B}(\vec{\pi}, f)$. Define

$$h = \bigcup \{h_p : p \in G\} \text{ and } I = \bigcup \{I_p : p \in G\}.$$

Using Lemma 3.5 together with Definition 3.1, h is a total injective function from ω_2 to ω_2 , $I \subseteq S_1^2$ is stationary in ω_2 , and for all $\beta \in I$ and for all $\alpha < \beta$, $h(\alpha) < \beta$.

Consider $\beta \in I$. Fix $p \in G$ such that $\beta \in I_p$. Let $D_\beta = \text{dom}(h_p) \cap \beta$. Now for any distinct $\alpha, \gamma \in \beta \setminus D_\beta$, there exists some $q \leq p$ in G such that $\alpha, \gamma \in \text{dom}(h_q) \setminus \text{dom}(h_p)$. By the definition of the order on $\mathbb{B}(\vec{\pi}, f)$ and the fact that $h(\alpha) = h_q(\alpha)$ and $h(\gamma) = h_q(\gamma)$, we have that

$$f(\min\{\pi_\beta(\alpha), \pi_\beta(h(\alpha))\}) < \max\{\pi_\beta(\alpha), \pi_\beta(h(\alpha))\}.$$

and

$$f(\min\{\pi_\beta(h(\alpha)), \pi_\beta(h(\gamma))\}) < \max\{\pi_\beta(h(\alpha)), \pi_\beta(h(\gamma))\}.$$

$\square$

4. ADDING THE BIJECTIONS

One component of our strategy for preserving ω_2 after forcing with $\mathbb{B}(\vec{\pi}, f)$ is to obtain the sequence of bijections $\vec{\pi}$ generically by forcing with countable conditions.

Definition 4.1. Define a forcing $\mathbb{C}$ to consist of conditions which are functions p with domain a countable subset of $\bigcup \{\{\beta\} \times \beta : \omega_1 \leq \beta < \omega_2\}$ such that for all $\omega_1 \leq \beta < \omega_2$, $p \restriction \{\beta\} \times \beta$ is an injective function mapping into ω_1 . Let $q \leq p$ in $\mathbb{C}$ if $p \subseteq q$.

The proofs of the next three lemmas are routine.

Lemma 4.2. *The forcing $\mathbb{C}$ is ω_1 -closed. In fact, if $\langle c_n : n < \omega \rangle$ is a descending sequence of conditions in $\mathbb{C}$, then $\bigcup_n c_n$ is in $\mathbb{C}$ and is the greatest lower bound of $\langle c_n : n < \omega \rangle$.*

Lemma 4.3. *CH implies that $\mathbb{C}$ is ω_2 -c.c.*

Lemma 4.4. *Suppose that G is a generic filter on $\mathbb{C}$. For each $\omega_1 \leq \beta < \omega_2$, define $\pi_\beta : \beta \rightarrow \omega_1$ by letting $\pi_\beta(\gamma) = c(\beta, \gamma)$ for some (any) $c \in G$ such that $(\beta, \gamma) \in \text{dom}(c)$. Then each π_β is a bijection of β onto ω_1 .*

5. A FAMILY OF MODELS FOR SIDE CONDITIONS

A second component for proving the preservation of ω_2 by forcings of the form $\mathbb{B}(\vec{\pi}, f)$, as well as for iterating forcings of this type, is to make use of a well-behaved family of models for which we can construct generic conditions. Such a family is described in the next theorem.

Theorem 5.1 (Essentially Mitchell ([Mit09])). *Assume CH , $2^{\omega_2} = \omega_3$, and $\square_{\omega_2}$. Then there exist stationary subsets $\mathcal{Y}$ and $\mathcal{Y}^+$ of $[H(\omega_3)]^{\omega_1}$ satisfying:*

- (1) *for all $M \in \mathcal{Y}$, $M \prec H(\omega_3)$, $|M| = \omega_1$, and $M^\omega \subseteq M$;*
- (2) *for all $M, N \in \mathcal{Y}$, $M \cap N \in \mathcal{Y}$;*
- (3) *$\mathcal{Y}^+ \subseteq \mathcal{Y}$;*
- (4) *for all $M \in \mathcal{Y}$ and for all $N \in \mathcal{Y}^+$, if $M \cap \omega_2 < N \cap \omega_2$ then $M \cap N \in N$.³*

While not stated in this exact manner in Mitchell's work, this result constitutes a small fragment of the information contained in his proof of the consistency that the approachability ideal $I[\omega_2]$ restricted to $\text{Cof}(\omega_1)$ can be trivial ([Mit09]). We include a proof for the convenience of the reader, which occupies the remainder of this section.

Fix a $\square_{\omega_2}$ -sequence $\vec{c} = \langle c_\alpha : \alpha < \omega_3, \alpha \text{ limit} \rangle$. So each c_α is a club subset of α whose order type is at most ω_2 , and if $\beta \in \lim(c_\alpha)$ then $c_\alpha \cap \beta = c_\beta$. A straightforward construction using the square sequence yields the following lemma. For a proof, see [Mit09, pp. 17–18].

Lemma 5.2. *There exists a sequence $\vec{A} = \langle A_{\eta, \xi} : \eta < \omega_3, \xi < \omega_2 \rangle$ satisfying:*

- (1) *for all $\eta < \omega_3$, $\langle A_{\eta, \xi} : \xi < \omega_2 \rangle$ is an increasing and continuous sequence of sets of size less than ω_2 with union equal to η ;*
- (2) *whenever $\beta \in \lim(c_\alpha)$, for all $\xi < \omega_2$, $A_{\beta, \xi} = A_{\alpha, \xi} \cap \beta$.*

Definition 5.3. For each $\beta < \omega_3$, for each $\xi < \omega_2$, and for each $\gamma < \text{ot}(A_{\beta, \xi})$, define $a_{\beta, \xi, \gamma}$ to be the γ -th element of $A_{\beta, \xi}$.

Note that whenever $\beta \in \lim(c_\alpha)$, the fact that $A_{\beta, \xi} = A_{\alpha, \xi} \cap \beta$ implies that for all $\gamma < \text{ot}(A_{\beta, \xi})$, $a_{\beta, \xi, \gamma} = a_{\alpha, \xi, \gamma}$.

Definition 5.4. For each $\beta < \omega_3$, define $g_\beta : \omega_2 \times \omega_2 \rightarrow \beta$ by

- $g_\beta(\xi, \gamma) = a_{\beta, \xi, \gamma}$ if $\gamma < \text{ot}(A_{\beta, \xi})$,
- $g_\beta(\xi, \gamma) = 0$ if $\text{ot}(A_{\beta, \xi}) \leq \gamma$.

Since $2^{\omega_2} = \omega_3$, we can fix a bijection $f : \omega_3 \rightarrow H(\omega_3)$. Define $\mathcal{A}$ to be the structure $(H(\omega_3), \in, f, \vec{c}, \vec{A})$. Due to f being a part of the structure, $\mathcal{A}$ has definable Skolem functions. Observe that for all $\beta < \omega_3$, g_β is definable in $\mathcal{A}$.

For simplicity in notation, for any $N \subseteq H(\omega_3)$, we abbreviate $\sup(N \cap \omega_3)$ as $\sup(N)$.

Definition 5.5. Let $\mathcal{Y}$ denote the set of all $N \prec \mathcal{A}$ satisfying that $|N| = \omega_1$ and $N^\omega \subseteq N$.

Note that for all $M, N \in \mathcal{Y}$, $M \cap N \in \mathcal{Y}$.

Lemma 5.6. *For all $M \in \mathcal{Y}$, $\lim(c_{\sup(M)}) \cap M$ is cofinal in $\sup(M)$.*

³The reason we need two families of models instead of one is because $\mathcal{Y}^+$ is not closed under intersections.

Proof. This follows easily from the fact that $M^\omega \subseteq M$. $\square$

Lemma 5.7. *Let $M \in \mathcal{Y}$. If $\beta \in \lim(M \cap \omega_3)$, then $\text{ot}(c_\beta) \leq M \cap \omega_2$.*

Proof. First, assume that $\beta = \sup(M)$. Then $\lim(c_\beta) \cap M$ is cofinal in β by Lemma 5.6. By the coherence of the square sequence and elementarity, this easily implies that for cofinally many $\gamma \in c_\beta$, $\text{ot}(c_\beta \cap \gamma) \in M \cap \omega_2$, which in turn implies that $\text{ot}(c_\beta) \leq M \cap \omega_2$. Secondly, assume that $\beta < \sup(M)$. Let $\alpha = \min((M \cap \omega_3) \setminus \beta)$. Then by elementarity, β is a limit point of c_α , so $c_\beta = c_\alpha \cap \beta$. Hence, $\text{ot}(c_\beta) \leq \text{ot}(c_\alpha) \in M \cap \omega_2$. $\square$

Definition 5.8. Define $\mathcal{Y}^+$ to be the set of all $N \in \mathcal{Y}$ such that $\text{ot}(c_{\sup(N)}) = N \cap \omega_2$.

Proposition 5.9. *The families $\mathcal{Y}$ and $\mathcal{Y}^+$ are stationary subsets of $[H(\omega_3)]^{\omega_1}$.*

Proof. Since $\mathcal{Y}^+ \subseteq \mathcal{Y}$, it suffices to show that $\mathcal{Y}^+$ is stationary. Let $H : H(\omega_3)^{<\omega} \rightarrow H(\omega_3)$ be a given function. Fix $L \prec \mathcal{A}$ which is closed under H and satisfies that $|L| = \omega_2$, $\omega_2 \subseteq L$, $L^\omega \subseteq L$, and $L \cap \omega_3$ has cofinality ω_2 (using CH, it is easy to show that such a set exists, using an argument similar to that in the next paragraph). Let $\delta = L \cap \omega_3$. Note that c_δ has order type ω_2 .

Define by recursion an increasing and continuous sequence $\langle M_\alpha : \alpha < \omega_1 \rangle$ of subsets of L of size ω_1 as follows. Let M_0 be an elementary substructure of L of size ω_1 such that $\omega_1 \subseteq M_0$. At limit stages take unions. Suppose that $\alpha < \omega_1$ and M_α is defined and is an elementary substructure of L of size ω_1 . Then $\sup(M_\alpha) < \delta$. Since $\text{ot}(c_\delta) = \omega_2$, we can pick $\gamma_\alpha > \sup(M_\alpha)$ which is a limit point of c_δ with cofinality ω_1 satisfying that $\text{ot}(c_{\gamma_\alpha}) > M_\alpha \cap \omega_2$. Now let $M_{\alpha+1}$ be the closure under the function H and the definable Skolem functions of $\mathcal{A}$ of the set $M_\alpha \cup M_\alpha^\omega \cup \{\gamma_\alpha, \text{ot}(c_{\gamma_\alpha})\}$. Note that since $M_\alpha \subseteq L$ and $L^\omega \subseteq L$, $M_{\alpha+1}$ is an elementary substructure of L of size ω_1 . This completes the construction.

Let $M = \bigcup \{M_\alpha : \alpha < \omega_1\}$. Then M has size ω_1 , and by construction, $M^\omega \subseteq M$, $M \prec \mathcal{A}$, and M is closed under H . So $M \in \mathcal{Y}$. Let $\theta = \sup(M)$. By construction, θ is a limit point of c_δ , so $c_\theta = c_\delta \cap \theta$. We claim that $\text{ot}(c_\theta) = M \cap \omega_2$, and hence $M \in \mathcal{Y}^+$. By Lemma 5.7, $\text{ot}(c_\theta) \leq M \cap \omega_2$. On the other hand, for all $\beta < M \cap \omega_2$ there exists $\alpha < \omega_1$ such that $\beta < M_\alpha \cap \omega_2 < \text{ot}(c_{\gamma_\alpha}) < M_{\alpha+1} \cap \omega_2 < M \cap \omega_2$. But c_{γ_α} is a limit point of c_δ which is less than θ , and hence is a limit point of $c_\delta \cap \theta = c_\theta$. Therefore, $\beta < \text{ot}(c_{\gamma_\alpha}) < \text{ot}(c_\theta)$. So $M \cap \omega_2 = \text{ot}(c_\theta)$. $\square$

Lemma 5.10. *Let $M \in \mathcal{Y}$ and let $\alpha = \sup(M)$. Then $M \cap \omega_3 = \{g_\alpha(\xi, \gamma) : \xi, \gamma \in M \cap \omega_2\}$.*

Proof. ($\subseteq$) Consider $\tau \in M \cap \omega_3$. By Lemma 5.6, fix $\beta \in \lim(c_\alpha) \cap M$ greater than τ . Then by elementarity, there is $\xi \in M \cap \omega_2$ such that $\tau \in A_{\beta, \xi}$. Let $\gamma = \text{ot}(A_{\beta, \xi} \cap \tau)$. Then $\gamma \in M$ and $\tau = a_{\beta, \xi, \gamma}$. But $A_{\beta, \xi} = A_{\alpha, \xi} \cap \beta$, so $\tau = a_{\beta, \xi, \gamma} = a_{\alpha, \xi, \gamma} = g_\alpha(\xi, \gamma)$.

($\supseteq$) Suppose that $\xi, \gamma \in M \cap \omega_2$ and we show that $g_\alpha(\xi, \gamma) \in M$. If $\gamma \geq \text{ot}(A_{\alpha, \xi})$, then $g_\alpha(\xi, \gamma) = 0 \in M$. Assume that $\gamma < \text{ot}(A_{\alpha, \xi})$. Then $g_\alpha(\xi, \gamma) = a_{\alpha, \xi, \gamma} < \alpha$. Since $\lim(c_\alpha) \cap M$ is cofinal in α by Lemma 5.6, we can fix $\beta \in \lim(c_\alpha) \cap M$ which is greater than $a_{\alpha, \xi, \gamma}$. Then $A_{\beta, \xi} = A_{\alpha, \xi} \cap \beta$. So $a_{\alpha, \xi, \gamma} = a_{\beta, \xi, \gamma}$, and by elementarity, $a_{\beta, \xi, \gamma} \in M$. $\square$

Lemma 5.11. *Suppose that $M \in \mathcal{Y}$, $\beta \in \lim(M \cap \omega_3)$, $\beta < \sup(M)$, and $\alpha = \min((M \cap \omega_3) \setminus \beta)$. Then for all $\xi, \gamma < M \cap \omega_2$, $g_\beta(\xi, \gamma) = g_\alpha(\xi, \gamma)$.*

Proof. The statement is immediate if $\beta = \alpha$, so assume that $\beta < \alpha$. Then an easy argument by elementarity shows that $\beta \in \lim(c_\alpha)$. Consequently, for all $\xi < \omega_2$, $A_{\beta, \xi} = A_{\alpha, \xi} \cap \beta$, and hence

$\text{ot}(A_{\beta,\xi}) \leq \text{ot}(A_{\alpha,\xi})$. Now consider $\xi, \gamma < M \cap \omega_2$. If $\gamma \geq \text{ot}(A_{\alpha,\xi})$, then also $\gamma \geq \text{ot}(A_{\beta,\xi})$, so both $g_\beta(\xi, \gamma)$ and $g_\alpha(\xi, \gamma)$ are equal to 0. Suppose that $\gamma < \text{ot}(A_{\alpha,\xi})$. Since $\xi, \gamma \in M$, $g_\alpha(\xi, \gamma) = a_{\alpha,\xi,\gamma} \in M \cap \alpha = M \cap \beta$. So $a_{\alpha,\xi,\gamma} \in A_{\alpha,\xi} \cap \beta = A_{\beta,\xi}$. As $A_{\beta,\xi}$ is an initial segment of $A_{\alpha,\xi}$, $a_{\alpha,\xi,\gamma} = a_{\beta,\xi,\gamma}$, that is, $g_\alpha(\xi, \gamma) = g_\beta(\xi, \gamma)$. $\square$

Proposition 5.12. *Let $M \in \mathcal{Y}$, $N \in \mathcal{Y}^+$, and suppose that $M \cap \omega_2 < N \cap \omega_2$. Then $M \cap N \in N$.*

Proof. Since $M \cap N$ is an elementary substructure of $\mathcal{A}$, $M \cap N = f[M \cap N \cap \omega_3]$. So by the elementarity of N in $\mathcal{A}$, it suffices to show that $M \cap N \cap \omega_3 \in N$. Let $\beta = \sup(M \cap N)$. We claim that $N \setminus \beta$ is non-empty. Otherwise, $\sup(N) = \beta$ and so $\text{ot}(c_\beta) = N \cap \omega_2$. On the other hand, β is a limit point of $M \cap \omega_3$ and hence by Lemma 5.7, $\text{ot}(c_\beta) \leq M \cap \omega_2 < N \cap \omega_2$, which is a contradiction. Let $\alpha = \min((N \cap \omega_3) \setminus \beta)$. Since $M \cap N \in \mathcal{Y}$, by Lemma 5.10 we have that $M \cap N \cap \omega_3 = \{g_\beta(\xi, \gamma) : \xi, \gamma \in M \cap N \cap \omega_2\}$. By Lemma 5.11 and the fact that $M \cap N \cap \omega_2 = M \cap \omega_2$, it follows that $M \cap N \cap \omega_3 = \{g_\alpha(\xi, \gamma) : \xi, \gamma \in M \cap \omega_2\}$. As α and $M \cap \omega_2$ are in N , so is $M \cap N \cap \omega_3$. $\square$

This completes the proof of Theorem 5.1.

6. THE DEFINITION OF THE FORCING ITERATION

For the remainder of the article, assume that GCH holds and there exist families $\mathcal{Y}$ and $\mathcal{Y}^+$ as described in Theorem 5.1. For example, these statements hold if $V = L$.

We define by recursion a sequence of posets $\langle \mathbb{P}_\delta : \delta \leq \Delta \rangle$, where $\Delta \leq \omega_3$ is some ordinal whose value is to be determined. Our goal is to show that the recursion does not terminate, and in that case, $\Delta = \omega_3$ and $\mathbb{P}_{\omega_3}$ preserves all cardinals. Until we finish our work, we cannot conclude that the recursion does not fail at some successor ordinal $\delta + 1$, where $\delta < \omega_3$, and if so, then $\Delta = \delta$. Roughly speaking, the forcing iteration begins with the forcing $\mathbb{C}$ to produce a generic sequence $\vec{\pi}$ of bijections, and then iterates forcings of the form $\mathbb{B}(\vec{\pi}, f)$, bookkeeping to handle every injective function f from ω_1 to ω_1 .

Assuming that the recursion succeeds, we achieve our objective as follows. Each $\mathbb{P}_\delta$ has size at most ω_3 and is ω_3 -c.c., and $\mathbb{P}_{\omega_3} = \bigcup \{\mathbb{P}_\delta : \delta < \omega_3\}$. Consequently, any nice $\mathbb{P}_{\omega_3}$ -name for a subset of ω_1 is a nice $\mathbb{P}_\delta$ -name for some $\delta < \omega_3$. By straightforward bookkeeping, we can arrange that for every injective function f from ω_1 to ω_1 in the final generic extension, there is some stage of the iteration at which we forced with $\mathbb{B}(\vec{\pi}, f)$. We also need to show that each $\mathbb{B}(\vec{\pi}, f)$ satisfies the assumptions of Lemma 3.6, and it then follows that $\mathbb{P}_{\omega_3}$ forces that (B) holds.

We use the following natural abbreviations associated with these posets, for each $\delta \leq \Delta$:

- the order on $\mathbb{P}_\delta$ is written as $\leq_\delta$;
- we write $\dot{G}_\delta$ for the canonical $\mathbb{P}_\delta$ -name for a generic filter on $\mathbb{P}_\delta$;
- we write the forcing relation for $\mathbb{P}_\delta$ as $\Vdash_\delta$.

Alongside the construction of the sequence $\langle \mathbb{P}_\delta : \delta \leq \Delta \rangle$, we also specify sequences $\langle \dot{f}_\tau : \tau < \Delta \rangle$ and $\langle \dot{Q}_\tau : \tau < \Delta \rangle$.

Base case:

Let $\mathbb{P}_0$ be the forcing whose conditions are ordered triples $(c, \emptyset, \emptyset)$, where $c \in \mathbb{C}$. Define $(d, \emptyset, \emptyset) \leq (c, \emptyset, \emptyset)$ if $c \subseteq d$. Note that $\mathbb{P}_0$ is isomorphic to $\mathbb{C}$.

Having defined $\mathbb{P}_0$, we fix some notation. Suppose that G is a generic filter on $\mathbb{P}_0$. For each $\omega_1 \leq \beta < \omega_2$, define $\pi_\beta : \beta \rightarrow \omega_1$ by letting $\pi_\beta(\gamma) = c(\beta, \gamma)$ for some (any) $(c, \emptyset, \emptyset) \in G$

such that $(\beta, \gamma) \in \text{dom}(c)$. By Lemma 4.4, each π_β is a bijection of β onto ω_1 . We write $\dot{\pi}_\beta$ as a $\mathbb{P}_0$ -name for this function for each $\omega_1 \leq \beta < \omega_2$, and let $\dot{\pi}$ be a $\mathbb{P}_0$ -name for the sequence $\langle \dot{\pi}_\beta : \omega_1 \leq \beta < \omega_2 \rangle$.

Successor case:

Assume that $\delta < \omega_3$ and $\mathbb{P}_\xi$ is defined for all $\xi \leq \delta$. For the recursion to continue, we assume inductively that $\mathbb{P}_0$ is a regular suborder of $\mathbb{P}_\delta$ and $\mathbb{P}_\delta$ preserves all cardinals. If these assumptions fail, then the recursion ends and $\mathbb{P}_{\delta+1}$ is not defined. Fix a $\mathbb{P}_\delta$ -name $\dot{f}_\delta$ for a function from ω_1 to ω_1 and let $\dot{\mathbb{Q}}_\delta$ be a $\mathbb{P}_\delta$ -name for the forcing $\mathbb{B}(\dot{\pi}, \dot{f}_\delta)$.

A condition in $\mathbb{P}_{\delta+1}$ is any ordered triple $p = (c_p, s_p, A_p)$ satisfying:

- (1) s_p is a function whose domain is a countable subset of $\delta + 1$;
- (2) A_p is a countable set of pairs of the form (ξ, M) , where $\xi \leq \delta$, ξ is a limit ordinal, and $M \in \mathcal{Y}$;
- (3) the ordered triple

$$p \restriction \delta = (c_p, s_p \restriction \delta, A_p)$$

is a member of $\mathbb{P}_\delta$;⁴

- (4) if $\delta \in \text{dom}(s_p)$, then $s_p(\delta)$ is an ordered pair $(h_{p,\delta}, I_{p,\delta})$ such that:
 - (a) $h_{p,\delta}$ is an injective function whose domain is a countable subset of ω_2 which maps into ω_2 and satisfies that $h_{p,\delta}(\alpha) \neq \alpha$ for all $\alpha \in \text{dom}(h_{p,\delta})$;
 - (b) $I_{p,\delta}$ is a countable subset of S_1^2 ;
 - (c) for all $\beta \in I_{p,\delta}$, β is closed under $h_{p,\delta}$ and $h_{p,\delta}^{-1}$.

Let $q \leq_{\delta+1} p$ if:

- (a) $A_p \subseteq A_q$;
- (b) $q \restriction \delta \leq_\delta p \restriction \delta$;
- (c) if $\delta \in \text{dom}(s_p)$, then $\delta \in \text{dom}(s_q)$ and $q \restriction \delta \Vdash_\delta s_q(\delta) \leq_{\dot{\mathbb{Q}}_\delta} s_p(\delta)$.

Note that for any $p \in \mathbb{P}_{\delta+1}$ with $\delta \in \text{dom}(s_p)$, $\Vdash_\delta s_p(\delta) \in \mathbb{B}(\dot{\pi}, \dot{f}_\delta)$.

Limit case:

Assume that $\delta \leq \omega_3$ is a limit ordinal and $\mathbb{P}_\xi$ is defined for all $\xi < \delta$. Since the recursion has continued through all ordinals less than δ , it follows that for every $\xi < \delta$, $\mathbb{P}_\xi$ preserves all cardinals.

Define $\mathbb{P}_\delta$ as follows. A condition in $\mathbb{P}_\delta$ is any ordered triple $p = (c_p, s_p, A_p)$ satisfying:

- (1) A_p is a countable set of pairs of the form (σ, M) , where $\sigma \leq \delta$, $\sigma < \omega_3$, σ is a limit ordinal, and $M \in \mathcal{Y}$;
- (2) s_p is a function whose domain is a countable subset of δ ;
- (3) for all $\xi < \delta$, the ordered triple

$$p \restriction \xi = (c_p, s_p \restriction \xi, \{(\sigma, M) \in A_p : \sigma \leq \xi\})$$

is a member of $\mathbb{P}_\xi$;

- (4) for all $(\delta, M) \in A_p$ and for all $\tau \in M \cap \text{dom}(s_p) \cap \delta$, $M \cap \omega_2 \in I_{p,\tau}$.

Let $q \leq p$ if:

- (a) $A_p \subseteq A_q$;

⁴For clarity, we remark that we do not need to restrict A_p to δ since by (2), $(\xi, M) \in A_p$ implies that $\xi < \delta + 1$.

- (b) for all $\xi < \delta$, $q \restriction \xi \leq_\xi p \restriction \xi$.

This completes the definition of the forcing iteration.

Definition 6.1. Let Δ be the least ordinal $\delta \leq \omega_3$ such that either $\delta = \omega_3$, or else $\delta < \omega_3$, $\mathbb{P}_\delta$ is defined, and $\mathbb{P}_{\delta+1}$ is not defined.

Definition 6.2. Let $\xi < \delta \leq \Delta$. For any $p = (c_p, s_p, A_p)$ in $\mathbb{P}_\delta$, define

$$p \restriction \xi = (c_p, s_p \restriction \xi, \{(\sigma, M) \in A_p : \sigma \leq \xi\}).$$

In the next section, we prove that $\mathbb{P}_\xi$ is a regular suborder of $\mathbb{P}_\delta$ when $\xi < \delta \leq \Delta$. We point out that the map defined above is only a reduction mapping when restricted to a dense subset of $\mathbb{P}_\delta$.

We give a useful non-inductive characterization of the forcing iteration, which can be proved with a routine argument by induction. Going forward we always refer to this lemma to verify membership in $\mathbb{P}_\delta$ and being related by $\leq_\delta$, rather than using the recursive definition.

Lemma 6.3. Let $\delta \leq \Delta$. Then $p \in \mathbb{P}_\delta$ iff $p = (c_p, s_p, A_p)$ is an ordered triple satisfying:

- (1) $c_p \in \mathbb{C}$;
- (2) s_p is a function whose domain is a countable subset of δ ;
- (3) for all $\tau \in \text{dom}(s_p)$, $s_p(\tau)$ is an ordered pair $(h_{p,\tau}, I_{p,\tau})$ such that:
 - (a) $h_{p,\tau}$ is an injective function whose domain is a countable subset of ω_2 which maps into ω_2 and satisfies that $h_{p,\tau}(\alpha) \neq \alpha$ for all $\alpha \in \text{dom}(h_{p,\tau})$;
 - (b) $I_{p,\tau}$ is a countable subset of S_1^2 ;
 - (c) for all $\beta \in I_{p,\tau}$, β is closed under $h_{p,\tau}$ and $h_{p,\tau}^{-1}$;
- (4) A_p is a countable set of pairs of the form (ξ, M) , where $\xi \leq \delta$, $\xi < \omega_3$, ξ is a limit ordinal, and $M \in \mathcal{Y}$;
- (5) for all $(\xi, M) \in A_p$ and for all $\tau \in M \cap \text{dom}(s_p) \cap \xi$, $M \cap \omega_2 \in I_{p,\tau}$.

For $p, q \in \mathbb{P}_\delta$, $q \leq_\delta p$ iff:

- (a) $c_p \subseteq c_q$;
- (b) $A_p \subseteq A_q$;
- (c) $\text{dom}(s_p) \subseteq \text{dom}(s_q)$;
- (d) for all $\tau \in \text{dom}(s_p)$, $q \restriction \tau \Vdash_\tau s_q(\tau) \leq_{\dot{Q}_\tau} s_p(\tau)$.

Definition 6.4. Let $\delta \leq \Delta$. For any $p \in \mathbb{P}_\delta$ and for any $\tau \in \text{dom}(s_p)$, write $s_p(\tau) = (h_{p,\tau}, I_{p,\tau})$.

7. BASIC FACTS ABOUT THE ITERATION

In this section, we establish some basic information about the forcing iteration just introduced.

Lemma 7.1. Let $\delta \leq \Delta$. Then $\mathbb{P}_\delta$ is ω_1 -closed. In fact, suppose that $\langle p_n : n < \omega \rangle$ is a descending sequence of conditions in $\mathbb{P}_\delta$. Define $q = (c_q, s_q, A_q)$, where:

- $c_q = \bigcup_n c_{p_n}$;
- $A_q = \bigcup_n A_{p_n}$;
- $\text{dom}(s_q) = \bigcup_n \text{dom}(s_{p_n})$;
- for all $\tau \in \text{dom}(s_q)$, $s_q(\tau) = (h_{q,\tau}, I_{q,\tau})$, where

$$h_{q,\tau} = \bigcup \{h_{p_n,\tau} : n < \omega, \tau \in \text{dom}(s_{p_n})\}$$

and

$$I_{q,\tau} = \bigcup \{I_{p_n,\tau} : n < \omega, \tau \in \text{dom}(s_{p_n})\}.$$

Then $q \in \mathbb{P}_\delta$ and q is the greatest lower bound of $\langle p_n : n < \omega \rangle$.

Proof. The proof is straightforward using Lemmas 3.3 and 4.2. $\square$

Lemma 7.2. Let $\tau < \delta \leq \Delta$. The set of $q \in \mathbb{P}_\delta$ such that $\tau \in \text{dom}(s_q)$ is dense in $\mathbb{P}_\delta$.

Proof. Let $p \in \mathbb{P}_\delta$, and we find $q \leq_\delta p$ with $\tau \in \text{dom}(s_q)$. If $\tau \in \text{dom}(s_p)$, then we are done, so assume not. Define $q = (c_q, s_q, A_q)$, where $c_q = c_p$, $A_q = A_p$, $\text{dom}(s_q) = \text{dom}(s_p) \cup \{\tau\}$, $s_q \upharpoonright \text{dom}(s_p) = s_p$, and

$$s_q(\tau) = (\emptyset, \{M \cap \omega_2 : \exists \xi (\xi, M) \in A_p, \tau \in M \cap \xi\}).$$

It easily follows by Lemma 6.3 that $q \in \mathbb{P}_\delta$ and $q \leq_\delta p$. $\square$

Definition 7.3. Let $\omega \leq \zeta < \delta \leq \Delta$. Define $D_{\delta, \zeta}$ as the set of $p \in \mathbb{P}_\delta$ such that, letting ζ^- be the largest limit ordinal less than or equal to ζ ,

- for all $\zeta^- \leq \sigma < \zeta$, $\sigma \in \text{dom}(s_p)$;
- for any $(\xi, M) \in A_p$ with $\zeta < \xi \leq \delta$, $(\zeta^-, M) \in A_p$.

Lemma 7.4. Let $\omega \leq \zeta < \delta \leq \Delta$. Then $D_{\delta, \zeta}$ is dense in $\mathbb{P}_\delta$.

Proof. It is easy to show that whenever $p \in \mathbb{P}_\delta$, $\gamma < \xi \leq \delta \leq \Delta$, γ is a limit ordinal, and $(\xi, M) \in A_p$, then $(c_p, s_p, A_p \cup \{(\gamma, M)\})$ is in $\mathbb{P}_\delta$ and extends p . Now we are done by Lemmas 7.2 and 7.1. $\square$

Lemma 7.5. Let $\zeta < \delta \leq \Delta$. Let $p \in \mathbb{P}_\delta$, and in the case that $\zeta \geq \omega$, also assume that $p \in D_{\delta, \zeta}$. Suppose that $u \leq_\zeta p \upharpoonright \zeta$. Then p and u are compatible in $\mathbb{P}_\delta$. More specifically, define $q = (c_q, s_q, A_q)$ where $c_q = c_u$, $s_q = s_u \cup s_p \upharpoonright [\zeta, \delta)$, and $A_q = A_u \cup A_p$. Then q is in $\mathbb{P}_\delta$ and extends u and p .

Proof. The only non-trivial thing to check is that q satisfies property (5) of Lemma 6.3 in the case that $\omega \leq \zeta$. Define ζ^- to be the largest limit ordinal less than or equal to ζ . Let $(\xi, M) \in A_q$ and let $\tau \in M \cap \text{dom}(s_q) \cap \xi$, and we show that $M \cap \omega_2 \in I_{q, \tau}$. If $(\xi, M) \in A_u$, then $\xi \leq \zeta$ and hence $\tau \in \text{dom}(s_u)$, so we are done since u is a condition. Suppose that $(\xi, M) \in A_p \setminus A_u$, and in particular, $\xi > \zeta$. If $\tau \in \text{dom}(s_p)$, then we are done since p is a condition. Assume that $\tau \in \text{dom}(s_u) \setminus \text{dom}(s_p)$. Then $\tau < \zeta^-$, for otherwise $\tau \in \text{dom}(s_p)$ since $p \in D_{\delta, \zeta}$. Since $p \in D_{\delta, \zeta}$, $(\zeta^-, M) \in A_p$. So $(\zeta^-, M) \in A_p \upharpoonright \zeta \subseteq A_u$. As $\tau \in M \cap \text{dom}(s_u) \cap \zeta^-$, $M \cap \omega_2 \in I_{u, \tau} = I_{q, \tau}$. $\square$

Lemma 7.6. Let $\zeta < \delta \leq \Delta$. Then $\mathbb{P}_\zeta$ is a regular suborder of $\mathbb{P}_\delta$.

Proof. Using Lemma 6.3, it is straightforward to show the following:

- $\mathbb{P}_\zeta \subseteq \mathbb{P}_\delta$;
- $\leq_\zeta$ equals $\leq_\delta \cap \mathbb{P}_\zeta^2$;
- for all $p, q \in \mathbb{P}_\zeta$ and for all $r \in \mathbb{P}_\delta$, $r \leq_\delta p, q$ implies $r \upharpoonright \zeta \leq_\zeta p, q$.

Suppose that A is a maximal antichain of $\mathbb{P}_\zeta$. Then the third statement above implies that A is an antichain of $\mathbb{P}_\delta$, and a routine argument using Lemma 7.5 shows that A is a maximal antichain of $\mathbb{P}_\delta$. $\square$

In particular, $\mathbb{P}_0$ is a regular suborder of $\mathbb{P}_\delta$ for all $\delta \leq \Delta$.

Lemma 7.7. Let $\delta < \Delta$. Define $\mathcal{D}$ as the set of $q \in \mathbb{P}_{\delta+1}$ such that $\delta \in \text{dom}(s_q)$ and define $\mathcal{E}$ as the set of $p * \dot{a}$ in $\mathbb{P}_\delta * \mathbb{B}(\dot{\pi}, \dot{f}_\delta)$ such that for some h and I , $p \Vdash_\delta \dot{a} = (\check{h}, \check{I})$. Then $\mathcal{D}$

and $\mathcal{E}$ are dense in $\mathbb{P}_{\delta+1}$ and $\mathbb{P}_\delta * \mathbb{B}(\dot{\pi}, \dot{f}_\delta)$ respectively. Moreover, letting $F : \mathcal{D} \rightarrow \mathcal{E}$ be the function defined by $F(q) = (q \restriction \delta) * \dot{a}$, where $\dot{a}$ is a $\mathbb{P}_\delta$ -name for $(h_{q,\delta}, I_{q,\delta})$, then F is a dense embedding.

The proof is straightforward.

Proposition 7.8. *Let $\tau < \delta \leq \Delta$. Assume that $\mathbb{P}_\delta$ preserves ω_2 and forces that the set $\bigcup\{I_{p,\tau} : p \in \dot{G}_\delta, \tau \in \text{dom}(s_p)\}$ is stationary in ω_2 , where $\dot{G}_\delta$ is the canonical $\mathbb{P}_\delta$ -name for the generic filter. Then $\mathbb{P}_\delta$ forces $(B)_{\dot{\pi}, \dot{f}_\tau}$.*

Proof. Suppose that G is a generic filter on $\mathbb{P}_\delta$. Let $\tilde{\pi} = (\dot{\pi})^G$ and $f_\tau = \dot{f}_\tau^G$. By Lemma 7.6, $G_{\tau+1} = G \cap \mathbb{P}_{\tau+1}$ is a generic filter on $\mathbb{P}_{\tau+1}$. Note that $\mathcal{I}_\tau = \bigcup\{I_{p,\tau} : p \in G, \tau \in \text{dom}(s_p)\}$ is equal to $\bigcup\{I_{u,\tau} : u \in G_{\tau+1}, \tau \in \text{dom}(s_u)\}$. By Lemmas 3.6 and 7.7, it is routine to check that $(B)_{\tilde{\pi}, f_\tau}$ holds in $V[G]$ as witnessed by $\mathcal{I}_\tau$. $\square$

A similar argument combined with Lemma 7.5 shows the following.

Lemma 7.9. *Let $\zeta < \delta \leq \Delta$. Suppose that $\dot{D}$ is a $\mathbb{P}_\zeta$ -name for a dense subset of $\dot{\mathbb{Q}}_\zeta$. Then for all $p \in \mathbb{P}_\delta$, there exists $q \leq_\delta p$ such that $\zeta \in \text{dom}(s_q)$ and $q \restriction \zeta \Vdash_\zeta s_q(\zeta) \in \dot{D}$.*

The next two lemmas follow by Lemmas 3.5 and 7.9.

Lemma 7.10. *Let $\delta \leq \Delta$. Let Y be a countable subset of ω_2 . Then for any $p \in \mathbb{P}_\delta$, there exists $q \leq_\delta p$ such that for any τ in $\text{dom}(s_p)$, $Y \subseteq \text{dom}(h_{q,\tau})$.*

Lemma 7.11. *Let $\delta \leq \Delta$. Let X be a countable set of pairs of the form (β, γ) , where $\omega_1 \leq \beta < \omega_2$ and $\gamma < \beta$. Then for any $p \in \mathbb{P}_\delta$, there exists $q \leq p$ such that $X \subseteq \text{dom}(c_q)$.*

8. PRESERVING CARDINALS, 1

In order to prove that $\Delta = \omega_3$ and the recursion succeeds, we need to show that each $\mathbb{P}_\delta$ preserves all cardinals. By Lemma 7.1, we immediately have the preservation of ω_1 . Preserving cardinals greater than or equal to ω_3 is handled next. The preservation of ω_2 is the most difficult challenge of the article and is proven in Section 10.

Lemma 8.1. *Assume that $\delta \leq \Delta$ and $\delta < \omega_3$. Then $\mathbb{P}_\delta$ is ω_2 -centered.*

Proof. We associate to any condition $p \in \mathbb{P}_\delta$ the following parameters:

- (1) c_p ;
- (2) $\text{dom}(s_p)$;
- (3) $\langle (h_{p,\tau}, I_{p,\tau}) : \tau \in \text{dom}(s_p) \rangle$.

By CH, there are ω_2 -many possibilities for these three objects.

Assume that $p_0, \dots, p_{n-1}$ have the same such parameters. For each $i < n$, write $p_i = (c_i, s_i, A_i)$ and for each $\tau \in \text{dom}(s_i)$, write $s_i(\tau) = (h_{i,\tau}, I_{i,\tau})$. Define q as follows:

- $c_q = c_0$, $\text{dom}(s_q) = \text{dom}(s_0)$, and $A_q = \bigcup_{i < n} A_i$;
- for all $\tau \in \text{dom}(s_q)$, $s_q(\tau) = (h_{0,\tau}, I_{0,\tau})$.

Using Lemma 6.3, it is simple to check that q is in $\mathbb{P}_\delta$ and extends each of $p_0, \dots, p_{n-1}$. $\square$

Proposition 8.2. *For all $\delta \leq \Delta$, $\mathbb{P}_\delta$ is ω_3 -Knaster.*

Proof. If $\delta < \omega_3$, then the statement is immediate by Lemma 8.1. Assume that $\delta = \Delta = \omega_3$. Consider a sequence $\langle p_i : i < \omega_3 \rangle$ of conditions in $\mathbb{P}_{\omega_3}$. For each $i < \omega_3$, let:

- $p_i = (c_i, s_i, A_i)$ and for all $\tau \in \text{dom}(s_i)$, $s_i(\tau) = (h_{i,\tau}, I_{i,\tau})$;
- $\mathcal{M}_i = \bigcup \{M : \exists \xi (\xi, M) \in A_i\}$.

Using CH and $2^{\omega_1} = \omega_2$, we can in four successive steps:

- fix a stationary set $A_0 \subseteq \omega_3$ consisting of ordinals of cofinality ω_2 and fix $\mathcal{M}$ such that for all $i < j$ in A_0 :
 - $\mathcal{M}_i \subseteq j$;
 - $\mathcal{M}_i \cap i = \mathcal{M}$ and $\mathcal{M}_j \cap j = \mathcal{M}$.
- fix a stationary set $A_1 \subseteq A_0$ and fix d such that for all $i < j$ in A_1 :
 - $\text{dom}(s_i) \subseteq j$;
 - $\text{dom}(s_i) \cap i = \text{dom}(s_j) \cap j = d$;
- fix a stationary set $A_2 \subseteq A_1$ and fix $c \in \mathbb{C}$ such that for all $i \in A_2$, $c_i = c$;
- fix a stationary set $A_3 \subseteq A_2$ and fix a sequence $\langle (h_\tau, I_\tau) : \tau \in d \rangle$ such that for all $i \in A_3$ and for all $\tau \in d$, $h_{i,\tau} = h_\tau$ and $I_{i,\tau} = I_\tau$.

We claim that for all $i < j$ in A_3 , p_i and p_j are compatible. So consider $i < j$ in A_3 . Define $q = (c_q, s_q, A_q)$ as follows. Let $c_q = c$, $A_q = A_i \cup A_j$, and $\text{dom}(s_q) = \text{dom}(s_i) \cup \text{dom}(s_j)$. For all $\tau \in \text{dom}(s_q)$:

- if $\tau \in \text{dom}(s_i) \setminus \text{dom}(s_j)$, let $s_q(\tau) = s_i(\tau)$;
- if $\tau \in \text{dom}(s_j) \setminus \text{dom}(s_i)$, let $s_q(\tau) = s_j(\tau)$;
- if $\tau \in \text{dom}(s_i) \cap \text{dom}(s_j)$, let $s_q(\tau) = (h_\tau, I_\tau)$.

We show that q is in $\mathbb{P}_\delta$ and extends p_i and p_j . Referring to Lemma 6.3, the only non-trivial thing to check is that for all $(\xi, M) \in A_q$ and for all $\tau \in M \cap \text{dom}(s_q) \cap \xi$, $M \cap \omega_2 \in I_{q,\tau}$ (where $s_q(\tau) = (h_{q,\tau}, I_{q,\tau})$). If $(\xi, M) \in A_i$ and $\tau \in \text{dom}(s_i)$, or if $(\xi, M) \in A_j$ and $\tau \in \text{dom}(s_j)$, then we are done since p_i and p_j are conditions. Assume that $(\xi, M) \in A_j$ and $\tau \in \text{dom}(s_i) \setminus \text{dom}(s_j)$. Then $\tau < j$ so $\tau \in M \cap j \subseteq \mathcal{M} \subseteq i$. So $\tau \in \text{dom}(s_i) \cap i = d \subseteq \text{dom}(s_j)$, which is a contradiction. Now assume that $(\xi, M) \in A_i$ and $\tau \in \text{dom}(s_j) \setminus \text{dom}(s_i)$. Then $\tau \in M \subseteq \mathcal{M}_i \subseteq j$, so $\tau \in \text{dom}(s_j) \cap j = d \subseteq \text{dom}(s_i)$, which again is a contradiction. $\square$

Lemma 8.3. *Suppose that for all $\delta \leq \Delta$ with $\delta < \omega_3$, $\mathbb{P}_\delta$ preserves ω_2 . Then $\Delta = \omega_3$ and $\mathbb{P}_{\omega_3}$ preserves ω_2 .*

Proof. The assumption clearly implies that $\Delta = \omega_3$, so it suffices to show that $\mathbb{P}_{\omega_3}$ preserves ω_2 . Let $\dot{H}$ be a nice $\mathbb{P}_{\omega_3}$ -name for a function from ω_1 into ω_2 . Now every antichain of $\mathbb{P}_{\omega_3}$ has size at most ω_2 by Proposition 8.2, and by definition every member of $\mathbb{P}_{\omega_3}$ is a member of $\mathbb{P}_\delta$ for some $\delta < \omega_3$. Consequently, every antichain of $\mathbb{P}_{\omega_3}$ is a subset of $\mathbb{P}_\delta$ for some $\delta < \omega_3$. Therefore, $\dot{H}$ is actually a nice $\mathbb{P}_\beta$ -name for some $\beta < \omega_3$. Since $\mathbb{P}_\beta$ preserves ω_2 , it forces that $\dot{H}$ is bounded below ω_2 . As $\mathbb{P}_\beta$ is a regular suborder of $\mathbb{P}_{\omega_3}$, $\mathbb{P}_{\omega_3}$ forces the same. $\square$

9. PREPARATORY LEMMAS

We now move towards the final goal of the article which is the preservation of ω_2 by the forcing iteration. Before we jump into the proof, we first do some preparation by isolating several technical lemmas which we use in the next section.

For the remainder of the section, fix some $\delta \leq \Delta$ with $\delta < \omega_3$.

Lemma 9.1. *For any $p \in \mathbb{P}_\delta$, there exists $q \leq_\delta p$ such that for all $(\beta, \gamma) \in \text{dom}(c_p)$ and for all $\tau \in \text{dom}(s_p)$, if there exists some $w \leq_\delta q$ such that $\gamma \in \text{ran}(h_{w,\tau})$, then $\gamma \in \text{ran}(h_{q,\tau})$.*

Proof. Let X be the set of all pairs (τ, γ) such that τ is in $\text{dom}(s_p)$ and there exists some β such that $(\beta, \gamma) \in \text{dom}(c_p)$. Injectively enumerate X as $\langle (\tau_n, \gamma_n) : n < k \rangle$ where $k \leq \omega$. Define by

induction a sequence of conditions $\langle p_n : n \leq k \rangle$ as follows. Let $p_0 = p$. Suppose that $n < k$ and p_n is defined. If there exists some $w \leq_\delta p_n$ such that $\gamma_n \in \text{ran}(h_{w,\tau_n})$, then let p_{n+1} be any such condition, and otherwise let $p_{n+1} = p_n$. This completes the construction. If $k < \omega$, let $q = p_k$, and if $k = \omega$, then let q be the greatest lower bound of the sequence $\langle p_n : n < \omega \rangle$ using Lemma 7.1. It is straightforward to check that q is as required. $\square$

Definition 9.2. For any $p \in \mathbb{P}_\delta$, define $I_p = \bigcup \{I_{p,\tau} : \tau \in \text{dom}(s_p)\}$.

Definition 9.3. For any $p \in \mathbb{P}_\delta$ and for any $\beta \in I_p$, define $c_{p,\beta}$ to be the function whose domain is equal to the set $\{\gamma : (\beta, \gamma) \in \text{dom}(c_p)\}$ such that for any γ in the domain of $c_{p,\beta}$, $c_{p,\beta}(\gamma) = c_p(\beta, \gamma)$.

Definition 9.4. Define E_δ to be the set of conditions $r \in \mathbb{P}_\delta$ satisfying that for all $\beta \in I_r$ and for all $\tau \in \text{dom}(s_r)$:

- (1) $\text{dom}(c_{r,\beta}) \subseteq \text{dom}(h_{r,\tau})$;
- (2) $\text{dom}(h_{r,\tau}) \cap \beta$ and $\text{ran}(h_{r,\tau}) \cap \beta$ are subsets of $\text{dom}(c_{r,\beta})$;
- (3) for all $\gamma \in \text{dom}(c_{r,\beta})$, if there exists some $w \leq_\delta r$ such that $\gamma \in \text{ran}(h_{w,\tau})$, then $\gamma \in \text{ran}(h_{r,\tau})$.

Lemma 9.5. The set E_δ is dense in $\mathbb{P}_\delta$.

Proof. Define $\langle p_n : n < \omega \rangle$ by recursion as follows. Let $p_0 = p$. Let $n < \omega$ and assume that p_n is defined. Define Y_n to be the set of all γ such that for some $\beta \in I_{p_n}$, $(\beta, \gamma) \in \text{dom}(c_{p_n})$. By Lemma 7.10, fix $u_n \leq_\delta p_n$ such that for any $\tau \in \text{dom}(s_{p_n})$, $Y_n \subseteq \text{dom}(h_{u_n,\tau})$. Define X_n to be the set of all pairs (β, γ) such that $\beta \in I_{u_n}$ and for some $\tau \in \text{dom}(s_{u_n})$, $\gamma \in (\text{dom}(h_{u_n,\tau}) \cup \text{ran}(h_{u_n,\tau})) \cap \beta$. By Lemma 7.11, fix $v_n \leq_\delta u_n$ such that $X_n \subseteq \text{dom}(c_{v_n})$. Now apply Lemma 9.1 to fix $p_{n+1} \leq_\delta v_n$ satisfying that for all $\beta \in I_{v_n}$, for all $\gamma \in \text{dom}(c_{v_n,\beta})$, and for all $\tau \in \text{dom}(s_{v_n})$, if there exists some $w \leq_\delta p_{n+1}$ such that $\gamma \in \text{ran}(h_{w,\tau})$, then $\gamma \in \text{ran}(h_{p_{n+1},\tau})$. Let r be the greatest lower bound of $\langle p_n : n < \omega \rangle$. It is routine to check that $r \in E_\delta$. $\square$

Lemma 9.6. Suppose that $\omega \leq \delta$ and let δ^- be the largest limit ordinal less than or equal to δ . Let $p \in E_\delta$ and suppose that $(\delta^-, N) \in A_p$. Then there exists $q \leq_\delta p$ such that $q \in E_\delta$ and for all $(\xi, M) \in A_q$, if $M \cap \omega_2 < N \cap \omega_2$ then $(\sup(M \cap N \cap \xi), M \cap N) \in A_q$.

Proof. Define $q = (c_q, s_q, A_q)$ by letting $c_q = c_p$, $s_q = s_p$, and

$$A_q = A_p \cup \{(\sup(K \cap N \cap \sigma), K \cap N) : (\sigma, K) \in A_p, K \cap \omega_2 < N \cap \omega_2\}.$$

We claim that q is in $\mathbb{P}_\delta$ and extends p . The only non-trivial thing to check is (5) of Lemma 6.3. Consider $(\sigma, K) \in A_p$ with $K \cap \omega_2 < N \cap \omega_2$ and $\tau \in (K \cap N) \cap \text{dom}(s_q) \cap \sup(K \cap N \cap \sigma)$. Then $\tau \in K \cap \text{dom}(s_p) \cap \sigma$, so $K \cap \omega_2 = K \cap N \cap \omega_2$ is in $I_{p,\tau} = I_{q,\tau}$. As $c_q = c_p$, $s_q = s_p$, and $p \in E_\delta$, by the definition of E_δ it easily follows that $q \in E_\delta$.

Suppose that $(\xi, M) \in A_q$ and $M \cap \omega_2 < N \cap \omega_2$. We show that $(\sup(M \cap N \cap \xi), M \cap N) \in A_q$. If $(\xi, M) \in A_p$, then we are done by the definition of q . Otherwise, for some $(\sigma, K) \in A_p$ with $K \cap \omega_2 < N \cap \omega_2$, $(\xi, M) = (\sup(K \cap N \cap \sigma), K \cap N)$. Hence, $\xi = \sup(K \cap N \cap \sigma)$ and $M = K \cap N$. So $\sup(M \cap N \cap \xi) = \sup(K \cap N \cap \sup(K \cap N \cap \sigma))$, which is easily shown to be equal to $\sup(K \cap N \cap \sigma) = \xi$. And also $M \cap N = M$. So $(\sup(M \cap N \cap \xi), M \cap N) = (\xi, M) \in A_q$. $\square$

Lemma 9.7. Let $p \in \mathbb{P}_\delta$ and let x be a subset of $\text{dom}(s_p)$. Let $\zeta < \omega_1$. Then there exist $q \leq_\delta p$, countable limit ordinals ζ_0 and ζ_1 , and a sequence $\langle F_\tau : \tau \in x \rangle$ satisfying:

- (1) $\zeta \leq \zeta_0 < \zeta_0 + \omega \leq \zeta_1$;

- (2) for all $\tau \in x$:
- (a) $F_\tau : \zeta_1 \rightarrow \zeta_1$;
 - (b) $q \restriction \tau \Vdash_\tau \dot{f}_\tau \restriction \zeta_1 = \check{F}_\tau$;
 - (c) $F_\tau[\zeta] \subseteq \zeta_0$;
- (3) for all $j < \zeta_1$, the set $\{F_\tau(j) : \tau \in x\}$ is bounded below ζ_1 .

Proof. Note that by Lemmas 7.4 and 7.5, for any $i < \omega_1$ and for any $\tau < \delta$, the set of $q \in \mathbb{P}_\delta$ such that $q \restriction \tau$ decides (in $\mathbb{P}_\tau$) $\dot{f}_\tau \restriction i$ is dense in $\mathbb{P}_\delta$. Define by recursion sequences $\langle p_n : n < \omega \rangle$, $\langle \iota_n : n < \omega \rangle$, and $\langle F_{\tau,n} : \tau \in x, n < \omega \rangle$ as follows. Using the fact stated above together with Lemma 7.1, we can find $p_0 \leq_\delta p$ such that for all $\tau \in x$, there exists $F_{\tau,0}$ such that $p_0 \restriction \tau \Vdash_\tau \dot{f}_\tau \restriction \zeta = F_{\tau,0}$. Let $\iota_0 = \zeta$.

Let $n < \omega$ and assume that we have defined p_n , ι_n , and $F_{\tau,n}$ for all $\tau \in x$. Also, assume as an inductive hypothesis that $\iota_n < \omega_1$ and for all $\tau \in x$, $p_n \restriction \tau \Vdash_\tau \dot{f}_\tau \restriction \iota_n = F_{\tau,n}$. Choose a countable ordinal $\iota_{n+1} > \iota_n$ large enough so that for all $\tau \in x$, $F_{\tau,n}[\iota_n] \subseteq \iota_{n+1}$. Now apply the fact described in the previous paragraph to fix some $p_{n+1} \leq_\delta p_n$ and a sequence $\langle F_{\tau,n+1} : \tau \in x \rangle$ such that for all $\tau \in x$, $p_{n+1} \restriction \tau \Vdash_\tau \dot{f}_\tau \restriction \iota_{n+1} = F_{\tau,n+1}$.

This complete the construction. Let q be the greatest lower bound of $\langle p_n : n < \omega \rangle$ and for each $\tau \in x$, let $F_\tau = \bigcup_n F_{\tau,n}$. Define $\zeta_0 = \iota_1$ and $\zeta_1 = \sup_n \iota_n$. It is straightforward to check that these objects satisfy the required properties. $\square$

10. PRESERVING CARDINALS, 2

We now prove the preservation of ω_2 by the forcing iteration, which completes our work.

Definition 10.1. For any $\delta \leq \Delta$, define δ^- as follows:

- if $\delta \geq \omega$, then δ^- is the largest limit ordinal less than or equal to δ ;
- if $\delta < \omega$, then $\delta^- = 0$.

Definition 10.2. Let $1 \leq \delta \leq \Delta$. Suppose that $N \in \mathcal{Y}^+$, $p \in N \cap \mathbb{P}_\delta$, and $[\delta^-, \delta) \subseteq \text{dom}(s_p)$. Define $p + (\delta, N)$ to be the ordered triple (c, A, s) satisfying:

- (1) $c = c_p$;
- (2) if $\delta < \omega$ then $A = A_p$, and if $\delta \geq \omega$ then $A = A_p \cup \{(\delta^-, N)\}$;
- (3) $\text{dom}(s) = \text{dom}(s_p)$;
- (4) for all $\tau \in \text{dom}(s)$:
 - (a) if $\tau \notin N$, then $s(\tau) = s_p(\tau)$;
 - (b) if $\tau \in N$, then $s(\tau) = (h_{p,\tau}, I_{p,\tau} \cup \{N \cap \omega_2\})$.

Lemma 10.3. Let $1 \leq \delta \leq \Delta$. For any $N \in \mathcal{Y}^+$ and for any $p \in N \cap \mathbb{P}_\delta$, $p + (\delta, N)$ is a condition in $\mathbb{P}_\delta$ which extends p .

The proof is routine using Lemma 6.3.

Lemma 10.4. Let $1 \leq \delta \leq \Delta$. Suppose that $N \in \mathcal{Y}^+$, $p \in N \cap \mathbb{P}_\delta$, and $[\delta^-, \delta) \subseteq \text{dom}(s_p)$. If $r \leq_\delta p + (\delta, N)$, then for all $\tau \in N \cap \text{dom}(s_r)$, $N \cap \omega_2 \in I_{r,\tau}$.

Proof. Let $\tau \in N \cap \text{dom}(s_r)$. If $\tau \in \text{dom}(s_p)$, then by the definition of $p + (\delta, N)$, $N \cap \omega_2 \in I_{p+(\delta,N),\tau} \subseteq I_{r,\tau}$. Suppose that $\tau \in \text{dom}(s_r)$. If $\delta \geq \omega$ and $\tau < \delta^-$, then since $(\delta^-, N) \in A_r$, $N \cap \omega_2 \in I_{r,\tau}$ since r is a condition. Otherwise, either $\delta^- = 0 \leq \tau < \delta < \omega$, or else $\delta \geq \omega$ and $\delta^- \leq \tau < \delta$. In either case, $\tau \in \text{dom}(s_p)$. $\square$

The next proposition is essentially the heart of the article.

Theorem 10.5. *Let $1 \leq \delta \leq \Delta$ and let $\chi > \omega_3$ be a regular cardinal. Suppose that $N^* \prec H(\chi)$ satisfies that $N = N^* \cap H(\omega_3) \in \mathcal{Y}^+$ and $\langle \mathbb{P}_\beta : \beta \leq \delta \rangle \in N^*$. Assume that $p \in N \cap \mathbb{P}_\delta$ and $[\delta^-, \delta] \subseteq \text{dom}(s_p)$. Then $p + (\delta, N)$ is $(N^*, \mathbb{P}_\delta)$ -generic.*

The proof of this result actually shows that $p + (\delta, N)$ is strongly $(N^*, \mathbb{P}_\delta)$ -generic, but we do not need this fact for our purposes.

Before embarking on the difficult proof of Theorem 10.5, we first observe that by standard proper forcing arguments this theorem allows us to complete the proof of the main result of the article.

Proposition 10.6. *For any $\delta \leq \Delta$, $\mathbb{P}_\delta$ preserves ω_2 .*

Proof. By Lemma 8.3, we may assume that $\delta < \omega_3$. If $\delta = 0$, then $\mathbb{P}_\delta$ is isomorphic to $\mathbb{C}$ which is ω_2 -c.c. So assume that $1 \leq \delta < \omega_3$. Let $\dot{F}$ be a $\mathbb{P}_\delta$ -name for a function from ω_1 into ω_2 . We show that for all $p \in \mathbb{P}_\delta$, there exists $q \leq_\delta p$ which forces that $\dot{F}$ is bounded below ω_2 . Let $p \in \mathbb{P}_\delta$. By extending p further if necessary using Lemma 7.2, we may assume that $[\delta^-, \delta] \subseteq \text{dom}(s_p)$. Fix a regular cardinal $\chi > \omega_3$ such that $\dot{F} \in H(\chi)$. As $\mathcal{Y}^+$ is a stationary subset of $[H(\omega_3)]^{\omega_1}$, we can fix $N^* \prec H(\chi)$ such that $\langle \mathbb{P}_\beta : \beta \leq \delta \rangle$, $\dot{F}$, and p are members of N^* and $N = N^* \cap H(\omega_3) \in \mathcal{Y}^+$.

Since $\delta < \omega_3$, $p \in N^* \cap H(\omega_3) = N$. So $p + (\delta, N)$ is defined, is a member of $\mathbb{P}_\delta$, and extends p . By Theorem 10.5, $p + (\delta, N)$ is $(N^*, \mathbb{P}_\delta)$ -generic. We claim that $p + (\delta, N)$ forces that $\dot{F}$ is bounded below ω_2 . Consider a generic filter G on $\mathbb{P}_\delta$ which contains $p + (\delta, N)$. Let $F = \dot{F}^G$, which is a member of $N^*[G]$. By a standard fact, $N^*[G]$ is an elementary substructure of $H(\chi)^{V[G]}$ and hence is closed under F . Since $\omega_1 \subseteq N^*[G]$, $F[\omega_1] \subseteq N^*[G] \cap \omega_2$. As $p + (\delta, N)$ is in G and is (N^*, G) -generic, it follows that $N^*[G] \cap \omega_2 = N \cap \omega_2 < \omega_2$, so F is bounded by $N \cap \omega_2$. $\square$

Corollary 10.7. $\Delta = \omega_3$ and $\mathbb{P}_{\omega_3}$ preserves all cardinals.

Proof. By Lemmas 7.1 and 8.3 together with Propositions 8.2 and 10.6. $\square$

Proposition 10.8. *For all $\tau < \omega_3$, $\mathbb{P}_{\omega_3}$ forces that the set $\dot{I}_\tau = \bigcup \{I_{p,\tau} : p \in \dot{G}_{\omega_3}, \tau \in \text{dom}(s_p)\}$ is stationary in ω_2 , where $\dot{G}_{\omega_3}$ is the $\mathbb{P}_{\omega_3}$ -name for the generic filter.*

Proof. Let $\dot{C}$ be a nice $\mathbb{P}_{\omega_3}$ -name for a club subset of ω_2 . Fix $\delta < \omega_3$ large enough so that $\dot{C}$ is a $\mathbb{P}_\delta$ -name. We show that for all $p \in \mathbb{P}_\delta$, there exists $q \leq_\delta p$ which forces that $\dot{I}_\tau \cap \dot{C}$ is non-empty. Let $p \in \mathbb{P}_\delta$. By extending p further if necessary using Lemma 7.2, we may assume that $[\delta^-, \delta] \subseteq \text{dom}(s_p)$ and also $\tau \in \text{dom}(s_p)$. Fix a regular cardinal $\chi > \omega_3$ such that $\dot{C} \in H(\chi)$. Since $\mathcal{Y}^+$ is a stationary subset of $[H(\omega_3)]^{\omega_1}$, we can fix $N^* \prec H(\chi)$ such that $\langle \mathbb{P}_\beta : \beta \leq \delta \rangle$, $\dot{C}$, τ , and p are members of N^* and $N = N^* \cap H(\omega_3) \in \mathcal{Y}^+$. Let $q = p + (\delta, N)$. Then $q \in \mathbb{P}_\delta$, $q \leq_\delta p$, and by Theorem 10.5, q is $(N^*, \mathbb{P}_\delta)$ -generic. We have that $N \cap \omega_2 \in I_{q,\tau}$, so q forces that $N \cap \omega_2 \in \dot{I}_\tau$. Let G be a generic filter on $\mathbb{P}_\delta$ containing q and let $C = \dot{C}^G$. Then $C \in N^*[G]$. Since $N^*[G]$ is an elementary substructure of $H(\chi)^{V[G]}$ and q is $(N^*, \mathbb{P}_\delta)$ -generic, $N^* \cap \omega_2 = N \cap \omega_2$ is a limit point of C and hence is a member of C . $\square$

Corollary 10.9. *For all $\delta < \omega_3$, $\mathbb{P}_{\omega_3}$ forces that $(B)_{\dot{\pi}, \dot{f}_\delta}$ holds.*

Proof. By Propositions 7.8 and 10.8. $\square$

Proof of Theorem 10.5. Let $\theta = N \cap \omega_2$. Then $\theta \in S_1^2$. Fix a dense open subset D of $\mathbb{P}_\delta$ which is a member of N^* , and we prove that $N^* \cap D$ is predense below $p + (\delta, N)$. Let $r \leq p + (\delta, N)$

be given, and we find a member of $N^* \cap D$ which is compatible with r . By extending r further if necessary using Lemmas 9.5 and 9.6, we may assume without loss of generality that $r \in D \cap E_\delta$ and for all $(\xi, M) \in A_r$, if $M \cap \omega_2 < N \cap \omega_2$ then $(\sup(M \cap N \cap \xi), M \cap N) \in A_r$. Define $Z = \{\gamma : \exists \beta \in I_r (\beta, \gamma) \in \text{dom}(c_r)\}$.

Fix $\theta_0 < \theta$ and $\zeta < \omega_1$ both large enough so that:

- (1) $I_r \cap \theta \subseteq \theta_0$;
- (2) $Z \cap \theta \subseteq \theta_0$;
- (3) for all $\tau \in \text{dom}(s_r)$, $\text{dom}(h_{r,\tau}) \cap \theta$ and $\text{ran}(h_{r,\tau}) \cap \theta$ are subsets of θ_0 ;
- (4) $\text{ran}(c_r) \subseteq \zeta$.

Since $r \leq p + (\delta, N)$, by Lemma 10.4 it follows that for all $\tau \in N \cap \text{dom}(s_r)$, $\theta \in I_{r,\tau}$, and hence θ is closed under both $h_{r,\tau}$ and $h_{r,\tau}^{-1}$.

Define:

- (5) $c_N = c_r \upharpoonright N$;
- (6) $s_N = \text{dom}(s_r) \cap N$;
- (7) $A_N = A_r \cap N$;
- (8) $I_N = I_r \cap N$;
- (9) $Z_N = Z \cap N$;
- (10) for all $\tau \in s_N$, $h_{N,\tau} = h_{r,\tau} \upharpoonright N$ and $I_{N,\tau} = I_{r,\tau} \cap N$.

Applying the elementarity of N^* , fix $\bar{r}$ and $\bar{\theta}$ which are members of N^* and satisfy:

- (11) $\bar{r} \in D \cap E_\delta$;
- (12) $s_N \subseteq \text{dom}(s_{\bar{r}})$ and $A_N \subseteq A_{\bar{r}}$;
- (13) for all $\gamma \in Z_N$, there exists $\beta \in I_{\bar{r}}$ such that $(\beta, \gamma) \in \text{dom}(c_{\bar{r}})$;
- (14) $\bar{\theta} \in S_1^2$, $\theta_0 < \bar{\theta}$, and for all $\tau \in s_N$, $\bar{\theta} \in I_{\bar{r},\tau}$;
- (15) $c_{\bar{r}} \upharpoonright \bar{\theta}^2 = c_N$;
- (16) for all $\tau \in s_N$:
 - (a) $\bar{\theta}$ is closed under $h_{\bar{r},\tau}$ and $h_{\bar{r},\tau}^{-1}$;
 - (b) $h_{\bar{r},\tau} \upharpoonright \bar{\theta} = h_{N,\tau}$;
 - (c) $I_{\bar{r},\tau} \cap \bar{\theta} = I_{N,\tau}$.

Namely, the properties described in (11)–(16) can be expressed by a first-order formula with parameters in N^* (using the fact that $N^\omega \subseteq N$) which is satisfied by r and θ in $H(\chi)$. So by elementarity, these properties are satisfied by some $\bar{r}$ and $\bar{\theta}$ in N^* . Note that $\bar{r}$ and $\bar{\theta}$ are in $H(\omega_3)$ and hence are in N .

Applying Lemma 9.7, fix in N^* a condition $v \leq_\delta \bar{r}$, countable limit ordinals ζ_0 and ζ_1 , and a sequence $\langle F_\tau : \tau \in s_N \rangle$ satisfying:

- (17) $\zeta \leq \zeta_0 < \zeta_0 + \omega \leq \zeta_1$;
- (18) for all $\tau \in s_N$:
 - (a) $F_\tau : \zeta_1 \rightarrow \zeta_1$;
 - (b) $v \upharpoonright \tau \Vdash_\tau \check{f}_\tau \upharpoonright \zeta_1 = \check{F}_\tau$;
 - (c) $F_\tau[\zeta] \subseteq \zeta_0$;
- (19) for all $j < \zeta_1$, the set $\{F_\tau(j) : \tau \in s_N\}$ is bounded below ζ_1 .

Since D is open, $v \in D \cap N^*$. So we are done provided we can show that v and r are compatible.

Claim A: Suppose that $\beta \in I_r \setminus \theta$, $\tau \in s_N$, and $\alpha \in \text{dom}(h_{v,\tau})$. If $h_{v,\tau}(\alpha) \in \text{dom}(c_{r,\beta})$, then $\alpha \in \text{dom}(c_{r,\beta})$.

Proof: Assume that $h_{v,\tau}(\alpha) \in \text{dom}(c_{r,\beta})$. Then $h_{v,\tau}(\alpha) \in Z_N$. So by (13), there exists $\bar{\beta} \in I_{\bar{r}}$ such that $h_{v,\tau}(\alpha) \in \text{dom}(c_{\bar{r},\bar{\beta}})$. Since $v \leq_\delta \bar{r}$ and $h_{v,\tau}(\alpha) \in \text{ran}(h_{v,\tau})$, the fact that $\bar{r} \in E_\delta$ implies by Definition 9.4(3) that $h_{v,\tau}(\alpha) \in \text{ran}(h_{\bar{r},\tau})$. Since $h_{\bar{r},\tau} \subseteq h_{v,\tau}$ and $h_{v,\tau}$ is injective, it follows that $\alpha \in \text{dom}(h_{\bar{r},\tau})$ and $h_{\bar{r},\tau}(\alpha) = h_{v,\tau}(\alpha)$. Now $h_{v,\tau}(\alpha) \in Z_N \subseteq \theta_0 < \bar{\theta}$. Since $\bar{\theta} \in I_{\bar{r},\tau} \subseteq I_{v,\tau}$, $\alpha < \bar{\theta}$ since v being a condition implies that $\bar{\theta}$ is closed under $h_{v,\tau}^{-1}$. But $h_{\bar{r},\tau} \upharpoonright \bar{\theta} = h_{N,\tau} = h_{r,\tau} \upharpoonright \bar{\theta}$. So $\alpha \in \text{dom}(h_{r,\tau})$ and $\alpha < \beta$. Since $r \in E_\delta$, by Definition 9.4(2) we have that $\alpha \in \text{dom}(c_{r,\beta})$. This completes the proof of Claim A.

Definitions of C and functions $C_\beta : \{\gamma_{\beta,n} : n < k_\beta\} \rightarrow \zeta_1$ for each $\beta \in I_r \setminus \theta$. For each $\beta \in I_r \setminus \theta$, we define a function C_β with domain equal to the set

$$\left(\bigcup \{ \text{dom}(h_{v,\tau}) \cup \text{ran}(h_{v,\tau}) : \tau \in s_N \} \right) \setminus \text{dom}(c_{r,\beta})$$

and mapping into the interval $[\zeta_0, \zeta_1)$ as follows. Injectively enumerate the domain of C_β as $\langle \gamma_{\beta,n} : n < k_\beta \rangle$, where $k_\beta \leq \omega$. We define $C_\beta(\gamma_{\beta,n})$ by induction on $n < k_\beta$. Suppose that $n < k_\beta$ and for all $0 \leq m < n$, $C_\beta(\gamma_{\beta,m})$ is defined. Define $C_\beta(\gamma_{\beta,n})$ to be any ordinal in the interval $[\zeta_0, \zeta_1)$ which is greater than any member of the set

$$\{C_\beta(\gamma_{\beta,m}) : m < n\} \cup \{F_\tau(C_\beta(\gamma_{\beta,m})) : m < n, \tau \in s_N\}.$$

This is possible by statement (19) above. This completes the definition of C_β . Now define C with domain equal to the set of all (β, γ) with $\beta \in I_r \setminus \theta$ and $\gamma \in \text{dom}(C_\beta)$ so that for all such (β, γ) , $C(\beta, \gamma) = C_\beta(\gamma)$.

Claim B: For all $\beta \in I_r \setminus \theta$, for all $\gamma \in \text{dom}(C_\beta)$, and for all $\tau \in s_N$, if $\gamma \in \text{dom}(h_{v,\tau})$ then $h_{v,\tau}(\gamma) \in \text{dom}(C_\beta)$.

Proof: Immediate by Claim A.

Claim C: For all $\beta \in I_r \setminus \theta$ and for all $\tau \in s_N$:

(1) if $\gamma \in \text{dom}(h_{v,\tau}) \cap \text{dom}(C_\beta)$, then

$$F_\tau(\min\{C_\beta(\gamma), C_\beta(h_{v,\tau}(\gamma))\}) < \max\{C_\beta(\gamma), C_\beta(h_{v,\tau}(\gamma))\};$$

(2) if α, γ are distinct elements of $\text{dom}(C_\beta) \cap \text{dom}(h_{v,\tau})$, then

$$F_\tau(\min\{C_\beta(h_{v,\tau}(\alpha)), C_\beta(h_{v,\tau}(\gamma))\}) < \max\{C_\beta(h_{v,\tau}(\alpha)), C_\beta(h_{v,\tau}(\gamma))\}.$$

Proof: (1) Fix $n < k_\beta$ such that $\gamma = \gamma_{\beta,n}$. By Claim B, $h_{v,\tau}(\gamma_{\beta,n}) \in \text{dom}(C_\beta)$, so we can fix $m < k_\beta$ different from n such that $h_{v,\tau}(\gamma_{\beta,n}) = \gamma_{\beta,m}$. Case 1: $n < m$. By the definition of C_β , $C_\beta(\gamma_{\beta,n})$ and $F_\tau(C_\beta(\gamma_{\beta,n}))$ are both less than $C_\beta(\gamma_{\beta,m}) = C_\beta(h_{v,\tau}(\gamma_{\beta,n}))$, so (1) holds. Case 2: $m < n$. By the definition of C_β , $C_\beta(\gamma_{\beta,m})$ and $F_\tau(C_\beta(\gamma_{\beta,m}))$ are both less than $C_\beta(\gamma_{\beta,n})$. But this is the same as $C_\beta(h_{v,\tau}(\gamma_{\beta,n}))$ and $F_\tau(C_\beta(h_{v,\tau}(\gamma_{\beta,n})))$ both being less than $C_\beta(\gamma_{\beta,n})$, so (1) holds.

(2) Fix distinct $m, n < k_\beta$ such that $\alpha = \gamma_{\beta,m}$ and $\gamma = \gamma_{\beta,n}$. Since $h_{v,\tau}$ is injective, by Claim B we can fix distinct $g, l < k_\beta$ such that $h_{v,\tau}(\gamma_{\beta,m}) = \gamma_{\beta,g}$ and $h_{v,\tau}(\gamma_{\beta,n}) = \gamma_{\beta,l}$. By symmetry, we may assume that $g < l$. Then by the definition of C_β , both $C_\beta(\gamma_{\beta,g})$ and $F_\tau(C_\beta(\gamma_{\beta,g}))$ are less than $C_\beta(\gamma_{\beta,l})$. So $C_\beta(h_{v,\tau}(\gamma_{\beta,m}))$ and $F_\tau(C_\beta(h_{v,\tau}(\gamma_{\beta,m})))$ are less than $C_\beta(h_{v,\tau}(\gamma_{\beta,n}))$ and (2) holds.

We are now ready to define a condition w which extends v and r .

Definition of w : Define $w = (c_w, s_w, A_w)$ as follows. Let $c_w = c_v \cup c_r \cup C$ and $A_w = A_v \cup A_r$. Let the domain of s_w be equal to $\text{dom}(s_v) \cup \text{dom}(s_r)$. For each $\tau \in \text{dom}(s_w)$, define $s_w(\tau)$ as follows.

(I) If $\tau \in \text{dom}(s_v) \setminus \text{dom}(s_r)$, then let

$$s_w(\tau) = (h_{v,\tau}, I_{v,\tau} \cup \{K \cap \omega_2 : \exists \beta (\beta, K) \in A_r, \theta \leq K \cap \omega_2, \tau \in K \cap \beta\}).$$

(II) If $\tau \in \text{dom}(s_r) \setminus \text{dom}(s_v)$, let $s_w(\tau) = s_r(\tau)$.

(III) If $\tau \in \text{dom}(s_v) \cap \text{dom}(s_r)$, let $s_w(\tau) = (h_{v,\tau} \cup h_{r,\tau}, I_{v,\tau} \cup I_{r,\tau})$.

We split up the proof that w is a condition which extends v and r into a series of claims.

Claim D: $c_w \in \mathbb{C}$ and c_v and c_r are subsets of c_w .

Proof: We start by showing that c_w is a function, that is, whenever (β, γ) is in the domain of at least two of c_v , c_r , or C , then these functions have the same value at (β, γ) . The domain of C is disjoint from the domains of c_v and c_r . Namely, if $(\beta, \gamma) \in \text{dom}(C)$, then $\beta \notin N$ so (β, γ) is not in the domain of c_v , and also by the definition of the domain of C_β , $\gamma \notin \text{dom}(c_{r,\beta})$ so $(\beta, \gamma) \notin \text{dom}(c_r)$. We are left with the case that $(\beta, \gamma) \in \text{dom}(c_v) \cap \text{dom}(c_r)$. But then $(\beta, \gamma) \in \text{dom}(c_v) \cap N \subseteq \text{dom}(c_N)$, so $c_r(\beta, \gamma) = c_N(\beta, \gamma) = c_{\bar{r}}(\beta, \gamma) = c_v(\beta, \gamma)$.

Now we show that for any distinct (β, α) and (β, γ) in the domain of c_w , $c_w(\beta, \alpha) \neq c_w(\beta, \gamma)$. This is clear if (β, α) and (β, γ) are both in the domain of any of c_v , c_r , or C . Case 1: $\beta \notin N$. Then neither (β, α) nor (β, γ) are in $\text{dom}(c_v)$. So without loss of generality we may assume that $(\beta, \alpha) \in \text{dom}(c_r)$ and $(\beta, \gamma) \in \text{dom}(C)$. Then $c_w(\beta, \alpha) = c_r(\beta, \alpha) < \xi \leq \xi_0 \leq C(\beta, \gamma) = c_w(\beta, \gamma)$. Case 2: $\beta \in N$. Then neither (β, α) nor (β, γ) are in $\text{dom}(C)$. Without loss of generality we may assume that $(\beta, \alpha) \in \text{dom}(c_r)$ and $(\beta, \gamma) \in \text{dom}(c_v)$. So $\beta \in I_r \cap N \subseteq \theta$. Hence, $\alpha < \theta$ so $(\beta, \alpha) \in N$. Therefore, $(\beta, \alpha) \in \text{dom}(c_N) \subseteq \text{dom}(c_v)$ and we are done. Finally, c_v and c_r are subsets of c_w by definition. This completes the proof of Claim D.

The next two claims are simple to verify.

Claim E: s_w is a function whose domain is a countable subset of δ , and $\text{dom}(s_v)$ and $\text{dom}(s_r)$ are subsets of $\text{dom}(s_w)$.

Claim F: A_w is a countable set of pairs of the form (ξ, M) , where $\xi \leq \delta$, $\xi < \omega_3$, ξ is a limit ordinal, and $M \in \mathcal{Y}$. Moreover, A_v and A_r are subsets of A_w .

Claim G: For all $\tau \in \text{dom}(s_w)$, $s_w(\tau)$ is an ordered pair $(h_{w,\tau}, I_{w,\tau})$ satisfying:

- (i) $h_{w,\tau}$ is an injective function whose domain is a countable subset of ω_2 , which maps into ω_2 , and satisfies that for all $\alpha \in \text{dom}(h_{w,\tau})$, $h_{w,\tau}(\alpha) \neq \alpha$;
- (ii) $I_{w,\tau}$ is a countable subset of S_1^2 ;
- (iii) for all $\alpha \in \text{dom}(h_{w,\tau})$ and for all $\beta \in I_{w,\tau}$, if $\alpha < \beta$ then $h_{w,\tau}(\alpha) < \beta$;
- (iv) for all $\gamma \in \text{ran}(h_{w,\tau})$ and for all $\beta \in I_{w,\tau}$, if $\gamma < \beta$ then $h_{w,\tau}^{-1}(\gamma) < \beta$.

Proof: (i) The fact that $h_{w,\tau}$ satisfies (i) is immediate in cases (I) and (II) of the definition of $s_w(\tau)$. For case (III), suppose that $\tau \in \text{dom}(s_v) \cap \text{dom}(s_r)$ and so $h_w(\tau) = h_{v,\tau} \cup h_{r,\tau}$. Recall that $h_{r,\tau} \upharpoonright \theta = h_{N,\tau} \subseteq h_{v,\tau}$, so $h_{w,\tau} = h_{v,\tau} \cup (h_{r,\tau} \upharpoonright [\theta, \omega_2))$. Therefore, $h_{w,\tau}$ is indeed a function. Now the range of $h_{v,\tau}$ is a subset of θ , whereas for all $\alpha \in \text{dom}(h_{r,\tau}) \setminus \theta$, $h_{r,\tau}(\alpha) \geq \theta$. Since $h_{v,\tau}$ and $h_{r,\tau}$ are injective, so is $h_{w,\tau}$. Finally, $h_{w,\tau}(\alpha) \neq \alpha$ for all $\alpha \in \text{dom}(h_{w,\tau})$ is obvious.

(ii) is immediate.

(iii) Suppose that $\alpha \in \text{dom}(h_{w,\tau})$, $\beta \in I_{w,\tau}$, and $\alpha < \beta$. We show that $h_{w,\tau}(\alpha) < \beta$. This is immediate in cases (I) and (II) of the definition of s_w . So assume that $\tau \in \text{dom}(s_v) \cap \text{dom}(s_r)$. We are done if either $\alpha \in \text{dom}(h_{v,\tau})$ and $\beta \in I_{v,\tau}$, or if $\alpha \in \text{dom}(h_{r,\tau})$ and $\beta \in I_{r,\tau}$. Case 1: $\alpha \in \text{dom}(h_{v,\tau})$ and $\beta \in I_{r,\tau}$. Then $h_{w,\tau}(\alpha) = h_{v,\tau}(\alpha) < \theta$. If $\beta \geq \theta$ we are done, so assume that $\beta < \theta$. Then $\beta \in I_{r,\tau} \cap \theta = I_{N,\tau} \subseteq I_{v,\tau}$. Case 2: $\alpha \in \text{dom}(h_{r,\tau})$ and $\beta \in I_{v,\tau}$. Then $\alpha < \beta < \theta$. So $\alpha \in \text{dom}(h_{r,\tau}) \cap N = \text{dom}(h_{N,\tau}) \subseteq \text{dom}(h_{v,\tau})$.

(iv) Suppose that $\gamma \in \text{ran}(h_{w,\tau})$, $\beta \in I_{w,\tau}$, and $\gamma < \beta$. We show that $h_{w,\tau}^{-1}(\gamma) < \beta$. This is immediate in cases (I) and (II) of the definition of s_w . So assume that $\tau \in \text{dom}(s_v) \cap \text{dom}(s_r)$. We are done if either $\gamma \in \text{ran}(h_{v,\tau})$ and $\beta \in I_{v,\tau}$, or if $\gamma \in \text{ran}(h_{r,\tau})$ and $\beta \in I_{r,\tau}$. Case 1: $\gamma \in \text{ran}(h_{v,\tau})$ and $\beta \in I_{r,\tau}$. If $\beta \geq \theta$ then we are done since $h_{v,\tau}^{-1}(\gamma) < \theta$. Suppose that $\beta < \theta$. Then $\beta \in I_{r,\tau} \cap N = I_{N,\tau} \subseteq I_{v,\tau}$. Case 2: $\gamma \in \text{ran}(h_{r,\tau})$ and $\beta \in I_{v,\tau}$. Then $\gamma < \beta < \theta$. Let $\alpha = h_{r,\tau}^{-1}(\gamma)$. Then $\alpha < \theta$ since $\theta \in I_{r,\tau}$. So $\alpha \in \text{dom}(h_{r,\tau} \upharpoonright \theta) = \text{dom}(h_{N,\tau}) \subseteq \text{dom}(h_{v,\tau})$ and $h_{v,\tau}(\alpha) = h_{r,\tau}(\alpha) = \gamma$. So $\gamma \in \text{ran}(h_{v,\tau})$. This completes the proof of Claim G.

Claim H: For all $(\xi, M) \in A_w$ and for all $\tau \in M \cap \text{dom}(s_w) \cap \xi$, $M \cap \omega_2 \in I_{w,\tau}$.

Proof: If $(\xi, M) \in A_v$ and $\tau \in \text{dom}(s_v)$, or if $(\xi, M) \in A_r$ and $\tau \in \text{dom}(s_r)$, then we are done since v and r are conditions. Case 1: $(\xi, M) \in A_v$ and $\tau \in \text{dom}(s_r) \setminus \text{dom}(s_v)$. Then $M \in N$, so $M \subseteq N$. Therefore, $\tau \in \text{dom}(s_r) \cap N = s_N \subseteq \text{dom}(s_v)$, which is a contradiction. Case 2: $(\xi, M) \in A_r$ and $\tau \in \text{dom}(s_v) \setminus \text{dom}(s_r)$. So we are in case (I) in the definition of $s_w(\tau)$. If $M \cap \omega_2 \geq \theta$, then $M \cap \omega_2 \in I_{w,\tau}$ by the definition of $s_w(\tau)$. Suppose that $M \cap \omega_2 < \theta$. Then by the choice of r , $(\sup(M \cap N \cap \xi), M \cap N) \in A_r$. Now $M \cap N \in N$ and $M \cap N \cap \xi$ is an initial segment of $M \cap N \cap \delta$. Since $M \cap N \cap \delta$ is in N and has ω_1 -many initial segments, $M \cap N \cap \xi$ is in N and so $\sup(M \cap N \cap \xi) \in N$. Hence, $(\sup(M \cap N \cap \xi), M \cap N) \in A_r \cap N = A_N \subseteq A_v$. As $\tau \in (M \cap N) \cap \text{dom}(s_v) \cap \sup(M \cap N \cap \xi)$, it follows that $M \cap N \cap \omega_2 = M \cap \omega_2$ is in $I_{v,\tau} \subseteq I_{w,\tau}$. This completes the proof of Claim H.

Putting Claims (D)–(H) together, by Lemma 6.3 we have that $w \in \mathbb{P}_\delta$.

Claim I: For all $\tau \in \text{dom}(s_v)$, $w \upharpoonright \tau \Vdash_\tau s_w(\tau) \leq_{\dot{\mathbb{Q}}_\tau} s_v(\tau)$.

Proof: Case 1: $\tau \in \text{dom}(s_v) \setminus \text{dom}(s_r)$. Then by definition, $h_{w,\tau} = h_{v,\tau}$ and $I_{v,\tau} \subseteq I_{w,\tau}$. So Definition 3.1(a, b) are immediate, and (c) is vacuous because $\text{dom}(h_{w,\tau}) \setminus \text{dom}(h_{v,\tau})$ is empty. Case 2: $\tau \in \text{dom}(s_v) \cap \text{dom}(s_r)$. Then by definition, $h_{v,\tau} \subseteq h_{w,\tau}$ and $I_{v,\tau} \subseteq I_{w,\tau}$. So Definition 3.1(a, b) are immediate, and (c) is trivial because if $\alpha \in \text{dom}(h_{w,\tau}) \setminus \text{dom}(h_{v,\tau})$ and $\beta \in I_{v,\tau}$, then $\beta < \theta \leq \alpha$ since $\text{dom}(h_{r,\tau}) \cap \theta = \text{dom}(h_{N,\tau}) \subseteq \text{dom}(h_{v,\tau})$. This completes the proof of Claim J.

Claims (D), (E), (F), and (I) establish that $w \leq_\delta v$.

Claim J: For all $\tau \in \text{dom}(s_r)$, $w \upharpoonright \tau \Vdash_\tau s_w(\tau) \leq_{\dot{\mathbb{Q}}_\tau} s_r(\tau)$.

Proof: If $\tau \in \text{dom}(s_r) \setminus \text{dom}(s_v)$, then $s_w(\tau) = s_r(\tau)$ so the claim is immediate. So we may assume that $\tau \in \text{dom}(s_r) \cap \text{dom}(s_v)$, which implies that $\tau \in \text{dom}(s_r) \cap N = s_N$ and we are in case (III) of the definition of $s_w(\tau)$. By definition, $h_{r,\tau} \subseteq h_{w,\tau}$ and $I_{r,\tau} \subseteq I_{w,\tau}$, so (a) and (b) of Definition 3.1 are immediate.

For (c), suppose that $\alpha \in \text{dom}(h_{w,\tau}) \setminus \text{dom}(h_{r,\tau})$, $\beta \in I_{r,\tau}$, and $\alpha < \beta$. We need to show that $w \upharpoonright \tau$ forces (i) and (ii) of Definition 3.1(c). By case (III) in the definition of $s_w(\tau)$, we have that $s_w(\tau) = (h_{v,\tau} \cup h_{r,\tau}, I_{v,\tau} \cup I_{r,\tau})$ and so $\alpha \in \text{dom}(h_{v,\tau})$.

Case 1: $\beta < \theta$. Then $\beta \in I_{r,\tau} \cap \theta = I_{N,\tau} \subseteq I_{\bar{r},\tau}$ and $\alpha < \beta < \theta_0$. If $\alpha \in \text{dom}(h_{\bar{r},\tau})$, then since $h_{\bar{r},\tau} \upharpoonright \theta_0 = h_{r,\tau} \upharpoonright \theta_0$, $\alpha \in \text{dom}(h_{r,\tau})$, which is false. So $\alpha \in \text{dom}(h_{v,\tau}) \setminus \text{dom}(h_{\bar{r},\tau})$. As $v \leq_\delta \bar{r}$, we have that:

$$v \upharpoonright \tau \Vdash_\tau \dot{f}_\tau(\min\{\dot{\pi}_\beta(\alpha), \dot{\pi}_\beta(h_{v,\tau}(\alpha))\}) < \max\{\dot{\pi}_\beta(\alpha), \dot{\pi}_\beta(h_{v,\tau}(\alpha))\}.$$

But $w \leq_\delta v$, so $w \upharpoonright \tau$ forces the same. Also, $h_{v,\tau}(\alpha) = h_{w,\tau}(\alpha)$. So

$$w \upharpoonright \tau \Vdash_\tau \dot{f}_\tau(\min\{\dot{\pi}_\beta(\alpha), \dot{\pi}_\beta(h_{w,\tau}(\alpha))\}) < \max\{\dot{\pi}_\beta(\alpha), \dot{\pi}_\beta(h_{w,\tau}(\alpha))\},$$

which proves (i). The same argument shows that for all $\gamma \in \text{dom}(h_{v,\tau}) \cap \beta$ different from α ,

$$w \upharpoonright \tau \Vdash_\tau (\min(\dot{\pi}_\beta(h_{w,\tau}(\gamma)), \dot{\pi}_\beta(h_{w,\tau}(\alpha))) < \max(\dot{\pi}_\beta(h_{w,\tau}(\gamma)), \dot{\pi}_\beta(h_{w,\tau}(\alpha))).$$

Assume that $\gamma \in \text{dom}(h_{w,\tau}) \cap \beta$. Then either $\gamma \in \text{dom}(h_{v,\tau})$, or else $\gamma \in \text{dom}(h_{r,\tau}) \cap \beta \subseteq \text{dom}(h_{v,\tau})$. So in either case, $\gamma \in \text{dom}(h_{v,\tau}) \cap \beta$ and the proof of (ii) is complete.

Case 2: $\beta \geq \theta$. If $\alpha \in \text{dom}(c_{r,\beta})$, then since $r \in E_\delta$ it follows by Definition 9.4(1) that $\alpha \in \text{dom}(h_{r,\tau})$, which is false. So $\alpha \in \text{dom}(h_{v,\tau}) \setminus \text{dom}(c_{r,\beta})$. Hence, $\alpha \in \text{dom}(C_\beta)$. So $c_w(\beta, \alpha) = C_\beta(\alpha)$ and $c_w(\beta, h_{v,\tau}(\alpha)) = C_\beta(h_{v,\tau}(\alpha))$. By Claim (C), we have that

$$F_\tau(\min\{c_w(\beta, \alpha), c_w(\beta, h_{v,\tau}(\alpha))\}) < \max\{c_w(\beta, \alpha), c_w(\beta, h_{v,\tau}(\alpha))\}.$$

Now $v \upharpoonright \tau$ forces (in $\mathbb{P}_\tau$) that $\dot{f}_\tau \upharpoonright \zeta_1 = F_\tau$, and hence so does $w \upharpoonright \tau$. Also, $w \upharpoonright 0$ forces (in $\mathbb{P}_0$, and hence in $\mathbb{P}_\tau$) that $\dot{\pi}_\beta \upharpoonright \text{dom}(c_{w,\beta}) = c_{w,\beta}$. So

$$w \upharpoonright \tau \Vdash_\tau \dot{f}_\tau(\min\{\dot{\pi}_\beta(\alpha), \dot{\pi}_\beta(h_{w,\tau}(\alpha))\}) < \max\{\dot{\pi}_\beta(\alpha), \dot{\pi}_\beta(h_{w,\tau}(\alpha))\}.$$

which proves (i).

Now consider $\gamma \in \text{dom}(h_{w,\tau}) \cap \beta$ which is different from α . First, assume that $\gamma \in \text{dom}(h_{r,\tau})$. Since $\beta \in I_{r,\tau}$, $h_{r,\tau}(\gamma) < \beta$. Since $r \in E_\delta$, by Definition 9.4(2) both γ and $h_{r,\tau}(\gamma)$ are in $\text{dom}(c_{r,\beta})$. By statement (4) above, both $c_{r,\beta}(\gamma)$ and $c_{r,\beta}(h_{r,\tau}(\gamma))$ are less than ζ . By statement (18c) above, $F_\tau(c_{r,\beta}(\gamma))$ and $F_\tau(c_{r,\beta}(h_{r,\tau}(\gamma)))$ are less than ζ_0 . On the other hand, α and $h_{v,\tau}(\alpha)$ are in the domain of C_β and consequently $F_\tau(\alpha)$ and $F(h_{v,\tau}(\alpha))$ are both greater than or equal to ζ_0 . So $w \upharpoonright \tau$ forces (in $\mathbb{P}_\tau$) that

$$\dot{f}_\tau(\min\{\dot{\pi}_\beta(h_{w,\tau}(\gamma)), \dot{\pi}_\beta(h_{w,\tau}(\alpha))\}) = \dot{f}_\tau(\dot{\pi}_\beta(h_{w,\tau}(\gamma)))$$

is strictly less than

$$\dot{\pi}_\beta(h_{w,\tau}(\alpha)) = \max\{\dot{\pi}_\beta(h_{w,\tau}(\gamma)), \dot{\pi}_\beta(h_{w,\tau}(\alpha))\}.$$

Secondly, assume that $\gamma \in \text{dom}(h_{w,\tau}) \setminus \text{dom}(h_{r,\tau})$. Then $\gamma \in \text{dom}(h_{v,\tau})$. If $\gamma \in \text{dom}(c_{r,\beta})$, then since $r \in E_\delta$ by Definition 9.4(1) it follows that $\gamma \in \text{dom}(h_{r,\tau})$, which is false. So $\gamma \in \text{dom}(C_\beta)$. By Claim (C),

$$F_\tau(\min\{C_\beta(h_{v,\tau}(\gamma)), C_\beta(h_{v,\tau}(\alpha))\}) < \max\{C_\beta(h_{v,\tau}(\gamma)), C_\beta(h_{v,\tau}(\alpha))\}.$$

So

$$w \upharpoonright \tau \Vdash_\tau \dot{f}_\tau(\min\{\dot{\pi}_\beta(h_{w,\tau}(\gamma)), \dot{\pi}_\beta(h_{w,\tau}(\alpha))\}) < \max\{\dot{\pi}_\beta(h_{w,\tau}(\gamma)), \dot{\pi}_\beta(h_{w,\tau}(\alpha))\}.$$

This completes the proof of Claim J.

Claims (D), (E), (F), and (J) confirms that $w \leq_\delta r$ and completes the proof of the proposition. $\square$

In light of the results of this article, a natural idea for proving the consistency of $(**)$ would be to interleave the fast club forcing into the iteration defined above. At first glance, this approach seems feasible due to the fast club forcing being ω_1 -closed and 2^ω -centered. Unfortunately, we are unable to prove that such an iteration preserves ω_2 .

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