COUNTRYMAN LINES AND THE CONTINUUM HYPOTHESIS

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ABSTRACT. We explore the prospect of basis-like results for the class of Aronszajn lines which are consistent with the Continuum Hypothesis (CH). In particular, we prove that each of the following statements is consistent with CH: Any two Countryman lines contain isomorphic or anti-isomorphic uncountable suborders; for any coherent Aronszajan tree $T \subseteq {}^{\omega_1}\omega$, the filter $\mathcal{U}(T)$ is an ultrafilter. The first result confirms a conjecture of Shelah in the context of CH which was previously shown to follow from the Proper Forcing Axiom. On the other hand, the weak diamond principle $2^\omega < 2^{\omega_1}$ implies that there does not exist a two element basis for the Countryman lines.

1. INTRODUCTION

An *Aronszajn line* is an uncountable linear order which contains no uncountable well-ordered or conversely well-ordered suborder nor an uncountable separable suborder. Aronszajn lines are closely connected to the idea of an *Aronszajn tree*, which is a tree of height ω_1 with countable levels and no uncountable chain. An uncountable linear order is an Aronszajn line iff it is the lexicographical ordering of some Aronszajn tree ([5, 15]). A distinguished subclass of the Aronszajn lines are the *Countryman lines*, which are uncountable linear orders whose square is a countable union of chains in the componentwise order. The existence of Countryman lines was proved by Shelah [12], solving a problem of Countryman [6]. Countryman lines are particularly relevant to basis problems for uncountable linear orders.[1] Since a Countryman line cannot contain uncountable anti-isomorphic suborders, any basis for the Aronszajn lines must have size at least two. Shelah asked whether it is consistent that every Aronszajn line contains a Countryman line, a statement known as *Shelah's conjecture*. Moore [10] confirmed this conjecture by proving that the Proper Forcing Axiom implies that there exists a two element basis for the Aronszajn lines consisting of a Countryman line and its converse.[2]

A notable feature of the two element basis theorem for Aronszajn lines, including both the work of Moore [10] as well as the reduction of the problem to Shelah's conjecture ([1, 16]), is that every forcing involved in the proof adds reals. As a result, the methods used to prove this theorem are antithetical to the construction of models of CH. This fact raises the question whether the violation of CH is essential, or if it is just a consequence of the specific techniques used. Relevant to this question is work of Abraham and Shelah [1], who used the weak diamond principle $2^\omega < 2^{\omega_1}$, which follows from CH, to derive anti-basis results for Aronszajn trees. For instance, they showed that the weak diamond principle implies that for every Aronszajn tree T, there is an Aronzajn tree

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[1]A *basis* for a class $\mathcal{L}$ of linear orders is a subclass $\mathcal{B} \subseteq \mathcal{L}$ such that every member of $\mathcal{L}$ contains a suborder which is isomorphic to a member of $\mathcal{B}$.

[2]More broadly, when combined with work of Baumgartner [4], the Proper Forcing Axiom implies the existence of a five element basis for the uncountable linear orders.

U such that T does not club embed into U. Other authors have used their method to derive related consequences of the weak diamond principle, including:

- There does not exist a two element basis for the Aronszajn lines ([5, Cor. 4.11(a)]).
- There exists an Aronszajn line which is the lexicographical ordering of a coherent Aronszajn tree which is not *minimal*, in the sense that it contains an uncountable suborder into which it does not embed (P. Larson, unpublished (see [11, Sec. 8])).
- There does not exist a two element basis for the Countryman lines ([9]).

This situation suggests the question of whether, and to what extent, basis-like statements for the Aronszajn and Countryman lines are consistent with CH. To simplify language, we introduce some terminology. Uncountable linear orders L and M are said to be *near* if there exist uncountable suborders $L_0 \subseteq L$ and $M_0 \subseteq M$ such that L_0 and M_0 are isomorphic, and they are *compatible* if there exist uncountable suborders $L_0 \subseteq L$ and $M_0 \subseteq M$ such that L_0 and M_0 are either isomorphic or anti-isomorphic. Since the existence of a two element basis for the Countryman lines is inconsistent with CH, we focus on formally weaker statements whose consistency was asked in the early literature on the subject:

(A) (Shelah's conjecture) Every Aronszajn line contains a Countryman line ([12, Conjecture (A)]).
(B) Any two Countryman lines are compatible ([12, Conjecture (B)]).
(C) Any two Aronszajn lines are compatible ([15, 5]).

The consistency of (B) was proven by Todorčević using the Proper Forcing Axiom for posets of size at most $\aleph_1$. Assuming that MA_{ω_1} holds and any two normal Aronszajn trees are club isomorphic, if we let C denote any of the Countryman lines $C(\rho_0)$, $C(\rho_1)$, $C(\rho_3)$, or $C(\rho)$ defined from characteristics of walks on ordinals, then C and its converse C^* constitute a two element basis for the Countryman lines.[3] This two element basis theorem for the Countryman lines is derived from the the conjunction of two statements: (i) C is minimal, that is, it embeds into all of its uncountable suborders, and (ii) every Countryman line is compatible with C (see e.g. [17, Cor. 2.1.13]).[4]

The conjunction of (i) and (ii) is inconsistent with CH, but we can ask whether versions of (i) or (ii) are consistent with CH individually. For (i), Baumgartner [5, Th. 4.15] proved that $\diamondsuit^+$ implies that there exists a minimal Aronszajn line. Baumgartner's example is a Suslin line (that is, a non-separable linear order which has no uncountable family of pairwise disjoint non-empty open intervals), and hence is not Countryman. More recently, Cummings, Eisworth, and Moore [7] proved that $\diamondsuit$ implies the existence of a minimal Countryman line. Concerning (ii), Shelah [12] observed that $\diamondsuit$ implies the existence of 2^{ω_1} many pairwise non-compatible Countryman lines. In contrast, in this article we prove:

Theorem 1. It is consistent that CH holds and any two Countryman lines are compatible.

Note that (C) follows from (A) and (B), so it is natural to approach the consistency of (C) by way of Shelah's conjecture. The *coloring axiom for trees* (CAT) is the statement that for any Aronszajn tree T and for any $K \subseteq T$, there exists an uncountable antichain of T such that all meets of pairs of elements in the antichain lie in K or all lie in $T \setminus K$. When this property holds for a specific Aronszajn tree T, we say that CAT holds for T. The coloring axiom for trees was introduced by

[3] MA_{ω_1} is the version of Martin's axiom which asserts the existence of a filter on any c.c.c. forcing which meets a given family of at most ω_1-many dense sets.

[4] As pointed out to the first author by Justin Moore, there is an error in [17] whereby it is asserted that these statements follow only from MA_{ω_1}. Statement (ii) needs the additional assumption about club isomorphisms.

Abraham and Shelah [1], who stated (without proof) that, under additional hypotheses, CAT is equivalent to Shelah's conjecture. Later, a proof of this equivalence appeared in [16, Prop. 8.7], where it is shown in particular that CAT implies Shelah's conjecture under the assumptions:

(1) MA_{ω_1};
(2) Any two normal Aronszajn trees are club isomorphic.

Each of these hypotheses implies the failure of CH. However, an analysis of the proof of [16, Prop. 8.7] reveals that, with a mild adjustment of the argument, CAT implies Shelah's conjecture under the weaker assumptions:

(1′) Any normal infinitely splitting Aronszajn tree contains a normal downwards closed binary subtree.
(2′) Any two normal Aronszajn trees contain normal downwards closed subtrees which are club isomorphic.

The consistency of (2′) with CH was pointed out by Abraham and Shelah [1] (without proof), and the consistency of (1′) with CH was proven in [8]. These observations suggest that if CH is consistent with CAT, then it is likely that CH is also consistent with Shelah's conjecture.

Moore's proof of the consistency of Shelah's conjecture involved showing that CAT holds for a particular coherent Aronszajn tree and was inspired in part by an earlier and related result of Todorčević ([17]). Recall that a tree T is *coherent* if it is a downwards closed subtree of ${}^{\omega_1}\omega$ such that any two elements of the tree agree as functions on all but finitely many members of the intersection of their domains.[5] For a given coherent Aronszajn tree T, define $\mathcal{U}(T)$ to be the collection of sets $X \subseteq \omega_1$ for which there exists an uncountable antichain $A \subseteq T$ such that for all distinct $x, y \in A$, the height of the meet of x and y is in X. Assuming that T is *non-Suslin* (that is, any uncountable subset of T contains an uncountable antichain), $\mathcal{U}(T)$ is a filter. Todorčević [17, Th. 4.1.13] proved that MA_{ω_1} implies that $\mathcal{U}(T)$ is an ultrafilter.[6] The statement that $\mathcal{U}(T)$ is an ultrafilter is a weak form of CAT in which we restrict the required dichotomy to sets $K \subseteq T$ of the form $T \restriction X$ for some $X \subseteq \omega_1$. This connection of $\mathcal{U}(T)$ with Shelah's conjecture motivates the second main theorem of the article.

Theorem 2. It is consistent with CH that for any coherent Aronszajn tree $T \subseteq {}^{\omega_1}\omega$, $\mathcal{U}(T)$ is an ultrafilter.

A number of open problems are suggested by the above discussion, which we list at the end of the article. For now we highlight one of the strongest possibilities, which can be thought of as an analogue of the two element basis theorem for Aronszajn lines in the context of CH.

Problem 1. Is it consistent with CH that for any Aronszajn lines L and J, there exists a Countryman line C such that each of L and J contain either a copy of C or its converse C^*?

One scenario in which a positive solution to Problem 1 could be realized would be if a model of Theorem 1 is combined with the family of all uncountable suborders of $C(\rho_3)$ and $C(\rho_3)^*$ constituting a basis for the Aronszajn lines.

We give a short overview of the article. In Section 2, we provide some background information about trees. In Sections 3–9 we prove the first main theorem, and in Sections 10–16 we prove the second. These two parts can be read independently of each other. In Section 17, we state further results and some open problems.

[5] For any cardinal κ and set X, ${}^{\underline{\kappa}}X$ is the collection of all functions of the form $f : \alpha \to X$, where $\alpha < \kappa$.

[6] The weaker hypothesis that the c.c.c. property of posets is productive also implies that $\mathcal{U}(T)$ is an ultrafilter. This hypothesis is inconsistent with CH ([14]).

2. Background on Trees and Linear Orders

We assume that the reader is familiar with trees and lexicographical orderings of trees, but for clarity we review some prominent ideas, facts, and notation which we use. For more background see [15], and for notation see [8, Sec. 1]. If L is a linear order, L^* denotes the *converse* of L which is obtained by reversing the order of elements of L (that is, $x <_L y$ iff $y <_{L^*} x$). Linear orders L and M are *anti-isomorphic* if L^* and M are isomorphic. An uncountable linear order L is a *Countryman line* if L^2 is a countable union of chains in the componentwise order. If L is a Countryman line, then: (1) L is an Aronszajn line, (2) L does not contain anti-isomorphic uncountable suborders, and (3) for all $2 \leq n < \omega$, L^n is a countable union of chains ([15, Th. 5.4]). Note that any uncountable suborder of a Countryman line is also a Countryman line, and if L is a Countryman line then so is L^*.

An ω_1-tree (that is, a tree of height ω_1 with countable levels) is *normal* if (1) it has a root, (2) every element has incomparable elements above it, (3) is Hausdorff (that is, distinct elements of the same limit height have different sets of predecessors), and (4) every element has uncountably many elements above it. In this article, we are concerned almost exclusively with ω_1-trees which are uncountable downwards closed subtrees of $({}^{\omega_1}\omega, \subsetneq)$. For any such tree, (1) and (3) are automatic, and (2) follows from (4) if it is Aronszajn. Any ω_1-tree contains an uncountable downwards closed subtree which satisfies (4). Hence, any downwards closed Aronszajn subtree of ${}^{\omega_1}\omega$ contains a downwards closed normal subtree.

Suppose that T is an ω_1-tree. For any $n < \omega$ and set X, X^n denotes the n-dimensional Cartesian product of X. Let $T^{\otimes n}$ denote the subset of T^n consisting of tuples whose elements all have the same height in T. The set $T^{\otimes n}$ ordered componentwise is itself a tree. Similarly, if T and S are ω_1-trees, then $T \otimes S$ is the tree of pairs (x, y), where $x \in T_\alpha$ and $y \in S_\alpha$ for some $\alpha < \omega_1$. For any $x \in T$, we let $T_x = \{y \in T : x \leq_T y\}$. An element $x \in T$ is said to *split* if it has at least two immediate successors. An ω_1-tree is *non-Suslin* if every uncountable subset of it contains an uncountable antichain (for Aronszajn trees, this property is equivalent to the tree not containing a downwards closed Suslin subtree). For each $x \in T$ and $\xi \leq \mathrm{ht}_T(x)$, let $x \upharpoonright \xi$ denote the unique member of T with height ξ such that $x \upharpoonright \xi \leq_T x$. If $X \subseteq T_\alpha$ for some α and $\xi \leq \alpha$, we say that X has *unique drop-downs to* ξ if the function which maps any $x \in X$ to $x \upharpoonright \xi$ is injective on X.

Assume that T is a downwards closed uncountable subtree of ${}^{\omega_1}\omega$. The height of any element of T is equal to its domain. So if $\xi \leq \mathrm{ht}_T(x)$, then $x \upharpoonright \xi$ as defined above is the same as the restriction of the function x to ξ. We say that T is *coherent* if any two elements of T agree as functions on all but finitely many elements of the intersection of their domain. For incomparable $x, y \in T$, let $\Delta(x, y)$ denote the largest ordinal $\gamma < \min(\mathrm{ht}_T(x), \mathrm{ht}_T(y))$ such $x \upharpoonright \gamma = y \upharpoonright \gamma$. Equivalently, γ is the least ordinal such that $x(\gamma) \neq y(\gamma)$, and γ is the height of the meet $x \wedge y$ of x and y. For an antichain $A \subseteq T$, let $\Delta(A) = \{\Delta(x, y) : x \neq y,\ x, y \in A\}$. Define $\mathcal{U}(T)$ to be the collection of all sets $X \subseteq \omega_1$ such that for some uncountable antichain $A \subseteq T$, $\Delta(T) \subseteq X$. If T is coherent, Aronszajn, and non-Suslin, then $\mathcal{U}(T)$ is a uniform filter on ω_1. The *lexicographical ordering of* T is the linear order L defined by letting $x <_L y$ if either $x <_T y$, or else x and y are incomparable and $x(\Delta(x, y)) < y(\Delta(x, y))$. If T is Aronszajn, then the lexicographical ordering of T is an Aronszajn line (for a converse, see the beginning of Section 3 below). Note that if x and y are incomparable, $x <_T x^+$, and $y <_T y^+$, then $\Delta(x, y) = \Delta(x^+, y^+)$. This fact easily implies the following useful lemma which we use frequently.

Lemma 2.1. *Let T be a downwards closed subtree of ${}^{\omega_1}\omega$ with lexicographical ordering L. If $a_0 <_T a_1$, $b_0 <_T b_1$, and a_0 and b_0 are incomparable in T, then $a_0 <_L b_0$ iff $a_1 <_L b_1$. In* *particular, for all incomparable $a, b \in T$, for all $\Delta(a,b) < \xi \le ht_T(a)$, and for all $\Delta(a,b) < \gamma \le ht_T(b)$, $a <_L b$ iff $a \restriction \xi <_L b \restriction \gamma$.*

The following lemma has a straightforward proof.

Lemma 2.2. *Suppose that T is a normal Aronszajn tree which is a downwards closed subtree of ${}^{\omega_1}\omega$. Let L be the lexicographical ordering of T. Then there exists a club $E \subseteq \omega_1$ such that for all $\eta \in E$ and for all $x \in T \restriction \eta$, the set of successors of x in T_η, considered as a suborder of L, is isomorphic to the rationals.*

3. Compatibility of Countryman Lines

In this section, we give an overview of our approach for proving the consistency of the statement that CH holds and any two Countryman lines are compatible.

We begin by reviewing one way of representing an Aronszajn line as a lexicographically ordered Aronszajn tree. Suppose that L is an Aronszajn line. Then there exists an Aronszajn tree T satisfying the following properties.

(1) The elements of T of successor height are intervals of L which have a right endpoint.
(2) The elements of T of limit height are non-empty intervals of L.
(3) T has a root which is L.
(4) $I <_T J$ iff $J \subseteq I$, for all $I, J \in T$.
(5) Incomparable elements of T are disjoint.
(6) T is Hausdorff.
(7) Every element of T with limit height has an immediate successor.
(8) Linearly order the set of immediate successors of any element of T by letting I be below J iff $\max(I) <_L \max(J)$, and let M denote the corresponding lexicographical ordering of T. Then:
   (a) The set of immediate successors of any element of T is either empty, a singleton, or is isomorphic to $(\omega, \in)$.
   (b) The suborder of M consisting of elements of T with successor height is isomorphic to L.

The proof of the above is a slight modification of the augmented version of the tree construction of Theorem 4.2 mentioned in the comments before Theorem 4.7 of [5] (we use right endpoints instead of left endpoints to obtain (8a)). Note that any lexicographically ordered tree satisfying the above properties is isomorphic to a downwards closed lexicographically ordered subtree of ${}^{\omega_1}\omega$, so going forward we only consider trees of this kind.

Lemma 3.1. *Suppose that T is a downwards closed subtree of ${}^{\omega_1}\omega$ which is special, every element of limit height has an immediate successor, and the lexicographical ordering of T restricted to its successor levels is a Countryman line. Then the lexicographical ordering of T is a Countryman line.*

Proof. Let L denote the lexicographical ordering of T and let C denote the suborder of L consisting of elements of T with successor height. Fix a specializing function $f : T \to \omega$ and fix a function $g : C^2 \to \omega$ such that for all $n < \omega$, $g^{-1}(n)$ is a chain in C^2. For each $c \in T$ not of successor height, pick an arbitrary immediate successor c^+ of c, and for $c \in T$ of successor height, let $c^+ = c$. Define $h : L^2 \to \omega^3$ by $h(c,d) = (f(c), f(d), g(c^+, d^+))$. Using Lemma 2.1, it is easy to check that for all $x \in \omega^3$, $h^{-1}(x)$ is a chain in L^2. $\square$

Lemma 3.2. *Let T be an Aronszajn tree which is a downwards closed subtree of ${}^{\omega_1}\omega$ and let L be the lexicographical ordering of T. Suppose that X is a dense subset of T. Then any uncountable suborder of L which is an antichain of T is isomorphic to an uncountable suborder of X.*

Proof. Let $A \subseteq T$ be an uncountable antichain. For each $a \in A$, pick some $x_a \in X$ such that $a \leq_T x_a$. Since A is an antichain, Lemma 2.1 implies that for all distinct $a, b \in A$, $a <_L b$ iff $x_a <_L x_b$. □

The next lemma describes our strategy for proving the first main theorem of the article.

Lemma 3.3. *Assume that all Aronszajn trees are special. Suppose that whenever L and J are Countryman lexicographical orderings of normal downwards closed subtrees T and S of ${}^{\omega_1}\omega$, respectively, then L and J are compatible. Then any two Countryman lines are compatible.*

Proof. Let X and Y be Countryman lines. Fix downwards closed subtrees T_0 and S_0 of ${}^{\omega_1}\omega$ satisfying properties (1)–(8) listed at the beginning of the section such that the lexicographical ordering of the successor levels of T_0 and S_0 are isomorphic to X and Y, respectively. Let L_0 and J_0 be the lexicographical orderings of T_0 and S_0, respectively. By Lemma 3.1, L_0 and J_0 are Countryman lines. Fix normal subtrees T and S of T_0 and S_0, respectively, and let L and J denote the lexicographical orderings of T and S, respectively. Since $L \subseteq L_0$ and $J \subseteq J_0$, L and J are also Countryman lines.

By the assumptions of the lemma, L and J are compatible. So fix uncountable sets $A_0 \subseteq L$ and $B_0 \subseteq J$ such that A_0 and B_0 are either isomorphic or anti-isomorphic as witnessed by an order preserving or order reversing bijection $G : A_0 \to B_0$. Applying the fact that $A_0 \subseteq T$ and T is special, fix an uncountable set $A_1 \subseteq A_0$ which is an antichain of T. Similarly, as S is special fix an uncountable set $B \subseteq G[A_1]$ which is an antichain of S. Let $A = G^{-1}[B]$. Then A and B are either isomorphic or anti-isomorphic. Now Lemma 3.2 applied to each of T_0 and S_0 implies that A is isomorphic to a subset of X and B is isomorphic to a subset of Y. Therefore, X and Y contain uncountable subsets which are either isomorphic or anti-isomorphic. □

We work with normal special Aronszajn trees T and S which are downwards closed subtrees of ${}^{\omega_1}\omega$ with corresponding lexicographical orderings L and J which are each Countryman lines. We consider two possibilities. The first is that L^* and J are near. In that case, T and S are already handled. Assume that L^* and J are not near. Then our goal becomes to design a forcing $\mathbb{P}(T, S)$ which adds a club set $C \subseteq \omega_1$ and a function $F : T \upharpoonright C \to S \upharpoonright C$ satisfying:

- F is level preserving (that is, for all $x \in T \upharpoonright C$, $\mathrm{ht}_S(F(x)) = \mathrm{ht}_T(x)$);
- for all x and y in $T \upharpoonright C$:
  - $x <_T y$ implies $F(x) <_S F(y)$;
  - $x <_L y$ implies $F(x) <_J F(y)$.

The forcing $\mathbb{P}(T, S)$ adds the function as above using countable approximations. Specifically, conditions include a *working part* which is a function $f : c_f \to S$, where c_f is a closed countable subset of ω_1 and f satisfies the description of F above for members of its domain. There are two main types of properties which we prove of this forcing. The first type is *extension*, which means extending the working part of a condition arbitrarily high in the tree while satisfying various requirements. The second type is the ability to build generic conditions over countable elementary substructures to prove properness and related properties. For this second purpose, the conditions in our forcing involve a second component, which is a countable set of subtrees of product trees called *promises* which put additional restrictions on the working part. The method of promises was introduced by Shelah in his forcing for specializing an Aronszajn tree while not adding reals ([13, Ch. V, Sec. 6]).

In order for $\mathbb{P}(T, S)$ to preserve CH, we prove that it is *totally proper*, which means that it is proper and does not add reals. Equivalently, for any appropriate countable elementary substructure N, every condition in N has an extension q which is a *total master condition*, meaning that for every dense set $D \subseteq \mathbb{P}(T, S)$ in N, there is some condition in $N \cap D$ above q. A general approach for proving total properness is to enumerate all dense sets in N as $\langle D_n : n < \omega \rangle$, build a descending sequence $\langle p_n : n < \omega \rangle$ of conditions such that for each n, $p_{n+1} \in D_n \cap N$, and then define a lower bound of this sequence. A minor adjustment in this argument shows that $\mathbb{P}(T, S)$ satisfies a stronger property called $({<}\,\omega_1)$*-proper*, which means that total master conditions exists not only for a single countable model, but for all models appearing in an $\in$-chain of countable models of any countable length. Another adjustment shows that $\mathbb{P}(T, S)$ has the ω_2-p.i.c. ([13, Ch. VIII, Def. 2.1]).

In order to obtain the desired consistency result, we need to iterate the above forcing so that the iteration itself does not add reals or collapse ω_2. Working over a model of $2^\omega = \omega_1$ and $2^{\omega_1} = \omega_2$, we use a countable support iteration of length ω_2 of forcings of the type above together with forcings which specialize Aronszajn trees without adding reals. The fact that the forcing iteration is totally proper is a consequence of the following iteration theorem of Shelah: Any countable support forcing iteration of forcings which are $({<}\,\omega_1)$-proper and Dee-complete is totally proper ([13, Ch. V, Th. 7.1]). Finally, for the preservation of cardinals greater than ω_1 as well as for the usual bookkeeping process, the forcings we use satisfy the ω_2-p.i.c. This property is a strong form of the ω_2-chain condition which is preserved, over a model of CH, by any countable support forcing iteration of length less than ω_2, and guarantees that such an iteration of length ω_2 is ω_2-c.c. ([13, Ch. VIII, Lem. 2.4]). Finally, in order to ensure a successful bookkeeping of all trees which appear in the final model, we use the fact that our forcings have size $2^{\omega_1} = \omega_2$ and the iteration is ω_2-c.c.

To summarize, in order to prove the first consistency result we need forcings of the form $\mathbb{P}(T, S)$ to be $({<}\,\omega_1)$-proper, Dee-complete, and ω_2-p.i.c. We also use the fact due to Shelah [13] that any Aronszajn tree can be made special using a forcing which has these properties as well. But the core piece of information about the forcing $\mathbb{P}(T, S)$ is that it is totally proper. The verification of $({<}\,\omega_1)$-proper is a slight adjustment of the proof of this fact, and the proofs of Dee-completeness and ω_2-p.i.c. are routine and duplicate proofs of these properties for similar posets in the literature. To avoid artificially extending the length and complexity of the article by including routine but lengthy details, we only verify total properness and leave the interested reader to explore the other properties using arguments from other sources (specifically, see [13, Ch. V, Th. 6.1] and [8, Sec. 10]).[7]

For the remainder of the proof of Theorem 1, fix normal special Aronszajn trees T and S which are downwards closed subtrees of ${}^{\omega_1}\omega$ with corresponding lexicographical orderings L and J which are each Countryman lines. Applying Lemma 2.2 to S, fix a club $E \subseteq \omega_1$ such that for all $\eta \in E$ and for all $x \in S \restriction \eta$, the set of successors of x in S_η, considered as a suborder of J, is isomorphic to the rationals. Most of the properties of the poset $\mathbb{P}(T, S)$ and its associated objects can be proved without additional hypotheses, but at several key places we assume that L^* and J are not near.

[7]Complete proofs of all results stated in this article are included in the second author's PhD dissertation.

4. Tuples, 1

We work with finite tuples of pairs of the form $\vec{x} = ((a_k, b_k) : k < n)$, where $n < \omega$ and for some $\alpha \in E$, $a_0, \ldots, a_{n-1}$ are in T_α and $b_0, \ldots, b_{n-1}$ are in S_α. For a fixed n, we consider such tuples as elements of the n-dimensional product tree $((T \restriction E) \otimes (S \restriction E))^{\otimes n}$.

Definition 4.1. Let $0 < n < \omega$, $\alpha \in E$, and $\vec{x} = ((a_k, b_k) : k < n) \in (T_\alpha \times S_\alpha)^n$.

(1) We refer to α as the *height* of $\vec{x}$.
(2) $\vec{x}$ is *lex-increasing* if $a_0 <_L \cdots <_L a_{n-1}$ and $b_0 <_J \cdots <_J b_{n-1}$.
(3) For any $\xi \leq \alpha$:
   (a) $\vec{x}$ has *unique drop-downs to* ξ if $(a_k \restriction \xi : k < n)$ and $(b_k \restriction \xi : k < n)$ are each injective;
   (b) Define $\vec{x} \restriction \xi = ((a_k \restriction \xi, b_k \restriction \xi) : k < n)$.

Lemma 4.2. *Let $\xi < \alpha \in E$. Suppose that $\vec{x} \in (T_\alpha \otimes S_\alpha)^n$ and $\vec{x}$ has unique drop-downs to ξ. Then $\vec{x}$ is lex-increasing iff $\vec{x} \restriction \xi$ is lex-increasing.*

Definition 4.3. Consider $\alpha \in E$ and lex-increasing tuples $\vec{x} = ((a_k, b_k) : k < m)$ and $\vec{y} = ((c_k, d_k) : k < n)$ in $(T_\alpha \times S_\alpha)^{<\omega}$.

(1) The tuples $\vec{x}$ and $\vec{y}$ are *disjoint* if $a_i \neq c_j$ and $b_i \neq d_j$ for all $i < m$ and $j < n$.
(2) If $\vec{x}$ and $\vec{y}$ are disjoint, we say that they are *compatible* if for all $i < m$ and $j < n$, $a_i <_L c_j$ iff $b_i <_J d_j$.

An easy fact which is used frequently below is that if $\vec{x}$ and $\vec{y}$ are compatible, then there exists a lex-increasing tuple which lists exactly the pairs occurring in either $\vec{x}$ or $\vec{y}$.

Lemma 4.4. *Let $\xi < \alpha \in E$. Suppose that $\vec{x}$ and $\vec{y}$ are lex-increasing tuples in $(T_\alpha \otimes S_\alpha)^{<\omega}$ each of which has unique drop-downs to ξ and satisfy that $\vec{x} \restriction \xi$ and $\vec{y} \restriction \xi$ are disjoint. Then $\vec{x}$ and $\vec{y}$ are compatible iff $\vec{x} \restriction \xi$ and $\vec{y} \restriction \xi$ are compatible.*

Lemmas 4.2 and 4.4 both follow easily from Lemma 2.1.

Suppose that $\vec{x} = ((a_k, b_k) : k < n)$ and $\vec{y} = ((c_k, d_k) : k < n)$ are in $((T \restriction E) \otimes (S \restriction E))^{\otimes n}$, not necessarily of the same height. We say that $\vec{x}$ and $\vec{y}$ are *componentwise incomparable* if for all $k < n$, a_k and c_k are incomparable in T and b_k and d_k are incomparable in S. We say that $\vec{x}$ and $\vec{y}$ are *fully incomparable* if for all $k, m < n$, a_k and c_m are incomparable in T and b_k and d_m are incomparable in S.

5. Promises, 1

We use a version of the method of promises of Shelah [13, Ch. V, Sec. 6]. The type of promises which we use for the proof of Theorem 1 is described next.

Definition 5.1 (Promises, 1)**.** Define $\mathcal{P}$ to be the set of all P satisfying that for some $0 < n < \omega$ and $\delta \in E$:

(1) P is a non-empty downwards closed subset of the tree $[(T \restriction (E \setminus \delta)) \otimes (S \restriction (E \setminus \delta))]^{\otimes n}$;
(2) for all $\vec{x} \in P$, $\vec{x}$ is lex-increasing;
(3) for all $\vec{x} \in P$, there are uncountably many elements of P above $\vec{x}$.

In the above, we refer to δ as the *base level of* P, which we denote by δ_P. The set of $P \in \mathcal{P}$ with base level δ is denoted by $\mathcal{P}_\delta$.

Following Shelah, we informally refer to the members of $\mathcal{P}$ as *promises*.

Proposition 5.2. *Assume that L^* and J are not near. For all $P \in \mathcal{P}$ and for any $\vec{x} \in P$, there exists a family $\{\vec{x}_n : n < \omega\}$ of elements of P of the same height above $\vec{x}$ which is pairwise disjoint and pairwise compatible.*

Proof. Write $\vec{x} = ((a_k, b_k) : k < n)$. For each $\alpha \in E$ greater than the height of $\vec{x}$, pick some $\vec{x}_\alpha \in P$ with height α above $\vec{x}$. Write each $\vec{x}_\alpha$ as $((a_k^\alpha, b_k^\alpha) : k < n)$. Since $\vec{x}$ is injective and $a_k <_T a_k^\alpha$ and $b_k <_T b_k^\alpha$ for all α and $k < n$, it follows that for all distinct $k, l < n$ and for all $\alpha \leq \beta$, a_k^α and a_l^β are incomparable in T and b_k^α and b_l^β are incomparable in S.

Applying the fact that T and S are special in $2n$-many steps, fix an uncountable set $X \subseteq E$ of ordinals greater than the height of $\vec{x}$ so that for all $\alpha < \beta$ in X, $\vec{x}_\alpha$ and $\vec{x}_\beta$ are componentwise incomparable. In particular, for all $\alpha < \beta$ in X, $\vec{x}_\alpha$ and $\vec{x}_\beta$ are fully incomparable and hence $\vec{x}_\alpha$ and $\vec{x}_\beta \restriction \alpha$ are disjoint. Using the fact that L and J are Countryman lines in two steps, fix an uncountable set $Y \subseteq X$ such that $\{(a_0^\alpha, \ldots, a_{n-1}^\alpha) : \alpha \in Y\}$ is a chain in L^n and $\{(b_0^\alpha, \ldots, b_{n-1}^\alpha) : \alpha \in Y\}$ is a chain in J^n.

Define a function $F : [Y]^2 \to 2$ so that for all $\alpha < \beta$ in Y, $F(\{\alpha, \beta\})$ is equal to 1 if $\vec{x}_\alpha$ and $\vec{x}_\beta \restriction \alpha$ are compatible, and 0 otherwise. By the Dushnik-Miller theorem, there exists a set $Z \subseteq Y$ such that either (a) Z is infinite and for all $\alpha < \beta$ in Z, $F(\{\alpha, \beta\}) = 1$, or else (b) Z is uncountable and for all $\alpha < \beta$ in Z, $F(\{\alpha, \beta\}) = 0$.

Suppose for a contradiction that (b) holds. Consider any $\alpha < \beta$ in Z. Since $F(\{\alpha, \beta\}) = 0$, $\vec{x}_\alpha$ and $\vec{x}_\beta \restriction \alpha$ are not compatible. By Lemma 2.1 and the fact that $\vec{x}$ is lex-increasing, for any distinct $k, l < n$, $a_k^\alpha <_L a_l^\beta$ iff $a_k <_L a_l$ iff $k < l$ iff $b_k <_J b_l$ iff $b_k^\alpha <_J b_l^\beta$. Consequently, there exists some $k < n$ such that either $a_k^\alpha <_L a_k^\beta$ and $b_k^\alpha >_J b_k^\beta$, or $a_k^\alpha >_L a_k^\beta$ and $b_k^\alpha <_J b_k^\beta$. Since $\{(a_0^\xi, \ldots, a_{n-1}^\xi) : \xi \in Z\}$ and $\{(b_0^\xi, \ldots, b_{n-1}^\xi) : \xi \in Z\}$ are each chains, the last sentence is true for 0 in place of k. Therefore, $a_0^\alpha <_L a_0^\beta$ implies $b_0^\alpha >_J b_0^\beta$ and $a_0^\alpha >_L a_0^\beta$ implies $b_0^\alpha <_J b_0^\beta$. So the map $g : \{a_0^\alpha : \alpha \in Z\} \to \{b_0^\alpha : \alpha \in Z\}$ defined by $g(a_0^\alpha) = b_0^\alpha$ for all $\alpha \in Z$ is an order preserving function from an uncountable subset of L^* into J. Hence L^* and J are near, which is a contradiction.

So (a) holds. Pick some $\delta \in E$ greater than the height of $\vec{x}_\alpha$ for all $\alpha \in Z$. For each $\alpha \in Z$, choose some $\vec{x}_\alpha^+ \in P$ with height δ which is above $\vec{x}_\alpha$. Now for all $\alpha < \beta$ in Z, $\vec{x}_\alpha$ and $\vec{x}_\beta \restriction \alpha$ are disjoint and compatible. By Lemma 4.4, $\vec{x}_\alpha^+$ and $\vec{x}_\beta^+$ are disjoint and compatible. $\square$

6. The Working Part, 1

The forcing $\mathbb{P}(T, S)$ adds the desired club and function by countable approximations. The next definition describes these objects.

Definition 6.1. Define $\mathcal{F}$ to be the set of functions of the form $f : T \restriction c_f \to S \restriction c_f$, where c_f is a closed countable subset of E, satisfying:

1. f is level preserving (that is, for all $x \in \mathrm{dom}(f)$, $\mathrm{ht}_T(x) = \mathrm{ht}_S(f(x))$;
2. for all $x, y \in \mathrm{dom}(f)$:
   (a) $x <_T y$ implies $x <_S y$;
   (b) $x <_L y$ implies $f(x) <_J f(y)$.

In the above, let α_f denote the maximum element of c_f.

Definition 6.2. Suppose that $0 < n < \omega$, $\xi < \omega_1$, and $\vec{x} = ((a_k, b_k) : k < n)$ is in $(T_\xi \times S_\xi)^n$. For any $f \in \mathcal{F}$ with $\xi \in c_f$, we say that f *is consistent with* $\vec{x}$ if for all $k < n$, $f(a_k) = b_k$.

Note that if $f \in \mathcal{F}$ is consistent with two disjoint tuples of the same height, then the tuples are compatible.

Lemma 6.3. *Assume the following:*

(1) $f \in \mathcal{F}$*;*
(2) $\alpha_f \leq \delta \in E$*;*
(3) $\vec{x}$ *and* $\vec{y}$ *are lex-increasing tuples in* $(T_\delta \times S_\delta)^{<\omega}$ *each with unique drop-downs to* α_f*;*
(4) $\vec{x} \upharpoonright \alpha_f$ *and* $\vec{y} \upharpoonright \alpha_f$ *are disjoint;*
(5) f *is consistent with* $\vec{x} \upharpoonright \alpha_f$ *and* $\vec{y} \upharpoonright \alpha_f$*.*

Then $\vec{x}$ *and* $\vec{y}$ *are compatible. Hence, there is a lex-increasing tuple in* $(T_\delta \times S_\delta)^{<\omega}$ *which lists the elements of* $\vec{x}$ *and* $\vec{y}$*.*

The proof follows easily from Lemma 4.4.

Definition 6.4. Suppose that $f \in \mathcal{F}$ and $P \in \mathcal{P}$ has dimension n. We say that f *fulfills* P if $\delta_P \leq \alpha_f$ and there exists a pairwise disjoint family $\{\vec{x}_n : n < \omega\}$ of members of P with height α_f such that for all $n < \omega$, f is consistent with $\vec{x}_n$.

The next lemma assists with constructing members of $\mathcal{F}$ in limit processes.

Lemma 6.5. *Assume the following:*

(1) $\langle \alpha_n : n < \omega \rangle$ *is an increasing sequence of ordinals in* E *with supremum* δ*;*
(2) for each n*:*
  (a) $f_n \in \mathcal{F}$ *and* $\alpha_n = \alpha_{f_n}$*;*
  (b) $f_n \subseteq f_{n+1}$*;*
(3) $g : T \upharpoonright c_g \to S \upharpoonright c_g$*, where* $c_g = (\bigcup_n c_{f_n}) \cup \{\delta\}$*;*
(4) $g \upharpoonright (T \upharpoonright \delta) = \bigcup_n f_n$*;*
(5) $g \upharpoonright T_\delta$ *is a lexicographical-order preserving function mapping into* S_δ*;*
(6) for all $x \in T_\delta$*, there exists* n *such that for all* $m \geq n$*,* $f_m(x \upharpoonright \alpha_m) = g(x) \upharpoonright \alpha_m$*.*

Then $g \in \mathcal{F}$*.*

Proof. Clearly, g is level preserving. Suppose that $c <_T d$ are in $\mathrm{dom}(g)$ and we show that $g(c) <_S g(d)$. This is obvious except when $d \in T_\delta$. Fix m large enough so that $\mathrm{ht}_T(c) < \alpha_m$ and $f_m(d \upharpoonright \alpha_m) = g(d) \upharpoonright \alpha_m$. Then $g(c) = f_m(c) <_S f_m(d \upharpoonright \alpha_m) <_S g(d)$.

Now suppose that c and d are in $\mathrm{dom}(g)$ and $c <_L d$. We prove that $g(c) <_J g(d)$. This is immediate if $c, d \in T_\delta$ or if $c, d \in T \upharpoonright \delta$. If $c <_T d$, then by the above, $g(c) <_S g(d)$, so $g(c) <_J g(d)$. Otherwise, c and d are incomparable in T.

Case 1: $c \in T \upharpoonright \delta$ and $d \in T_\delta$. Pick m large enough so that $\alpha_m > \mathrm{ht}_T(c)$ and $f_m(d \upharpoonright \alpha_m) = g(d) \upharpoonright \alpha_m$. By Lemma 2.1, $c <_L d$ implies that $c <_L d \upharpoonright \alpha_m$. So $g(c) = f_m(c) <_J f_m(d \upharpoonright \alpha_m)$, and $f_m(d \upharpoonright \alpha_m) <_S g(d)$ implies that $f_m(d \upharpoonright \alpha_m) <_J g(d)$.

Case 2: $c \in T_\delta$ and $d \in T \upharpoonright \delta$. Fix m large enough so that $\alpha_m > \mathrm{ht}_T(d)$ and $f_m(c \upharpoonright \alpha_m) = g(c) \upharpoonright \alpha_m$. Since c and d are incomparable, by Lemma 2.1 we have that $c \upharpoonright \alpha_m <_L d$. Hence, $f_m(c \upharpoonright \alpha_m) <_J f_m(d)$. In particular, $f_m(d) \not<_S f_m(c \upharpoonright \alpha_m)$, so $f_m(d)$ and $f_m(c \upharpoonright \alpha_m)$ are incomparable in S. By Lemma 2.1 and the fact that $f_m(c \upharpoonright \alpha_m) <_S g(c)$, $g(c) <_J f_m(d) = g(d)$. □

7. Extension, 1

The next lemma allows us to stretch a condition up arbitrarily high in the trees. Since the generic function is only defined on a club of levels, such simple extensions do not require a limit process of the kind described in Lemma 6.5. But we can only avoid the limit case temporarily, and we will confront it in Proposition 8.3 and Theorem 9.5 below.

Lemma 7.1 (Extension, 1). *Assume the following:*

(1) $f \in \mathcal{F}$;
(2) $\Gamma \subseteq \mathcal{P}$ *is countable and* f *fulfills every member of* Γ;
(3) $\alpha_f < \delta \in E$;
(4) $\vec{x}$, $\vec{y}$, *and* $\vec{z}$ *are in* $(T_\delta \times S_\delta)^{<\omega}$, *each with unique drop-downs to* α_f;
(5) either $\vec{y} = \vec{z} = \emptyset$ *or* $\vec{y}$ *and* $\vec{z}$ *are disjoint and compatible;*
(6) $\vec{y} \restriction \alpha_f = \vec{z} \restriction \alpha_f$;
(7) $\vec{x} \restriction \alpha_f$ *is disjoint from* $\vec{y} \restriction \alpha_f$;
(8) f *is consistent with* $\vec{x} \restriction \alpha_f$ *and* $\vec{y} \restriction \alpha_f$.

Then there exist $g \in \mathcal{F}$ *such that:*

(a) $f \subseteq g$;
(b) $c_g = c_f \cup \{\delta\}$;
(b) g *fulfills every member of* Γ;
(c) g *is consistent with* $\vec{x}$, $\vec{y}$, *and* $\vec{z}$.

Proof. Write $\vec{y} = ((a^0_k, b^0_k) : k < n)$ and $\vec{z} = ((a^1_k, b^1_k) : k < n)$. Let $\langle P_m : m < \omega\rangle$ be an enumeration of Γ such that every member appears infinitely often. Since f fulfills every member of Γ, we can fix a pairwise disjoint sequence $\langle \vec{w}_m : m < \omega\rangle$ satisfying that for all $m < \omega$:

(1) $\vec{w}_m$ is in P_m and has height α_f;
(2) $\vec{w}_m$ is disjoint from $\vec{x} \restriction \alpha_f$ and $\vec{y} \restriction \alpha_f$;
(3) f is consistent with $\vec{w}_m$.

For each $m < \omega$, fix $\vec{w}^+_m$ in P_m of height δ which is above $\vec{w}_m$. It suffices to define a function $g \in \mathcal{F}_\delta$ with $c_g = c_f \cup \{\delta\}$ such that $f \subseteq g$ and g is consistent with $\vec{x}$, $\vec{y}$, $\vec{z}$, and $\vec{w}_m$ for all $m < \omega$.

Consider $x \in T_{\alpha_f}$ and we define a function g_x from the set of the successors of x in T_δ into the successors of $f(x)$ in S_δ. Then we let $h = g \cup \{g_x : x \in T_{\alpha_f}\}$. We split into three mutually exclusive cases.

Case 1: There exists a tuple $((a_k, b_k) : k < p)$ in the set $\{\vec{w}^+_m : m < \omega\} \cup \{\vec{x}\}$ and there exists $k < p$ such that $x <_T a_k$. Since f is consistent with this tuple's restriction to α_f, $x <_T a_k$ implies $f(x) <_S b_k$. As $\delta \in E$, the set of successors of $f(x)$ in S_δ, considered as a suborder of J, is isomorphic to the rationals. So we can fix a lexicographical-order preserving function g_x from the set of successors of x in T_δ into the set of successors of $f(x)$ in S_δ such that $g_x(a_k) = b_k$.

Case 2: There exists $l < n$ such that $x <_T a^0_l$. Then $x <_T a^1_l$, $f(x) <_S b^0_l$, and $f(x) <_S b^1_l$. Let Q be the set of successors of $f(x)$ in S_δ. Then again Q is isomorphic to the rationals. Since $\vec{y}$ and $\vec{z}$ are compatible, $a^0_l <_L a^1_l$ iff $b^0_l <_J b^1_l$. So we can fix a lexicographical-order preserving function g_x from the set of successors of x in T_δ into Q such that $g_x(a^0_l) = b^0_l$ and $g_x(a^1_l) = b^1_l$.

Case 3: Neither case 1 nor case 2 hold. Let Q be the set of successors of $f(x)$ in S_δ. Fix a lexicographical-order preserving function g_x from the set the successors of x in T_δ into the set of successors of $f(x)$ in S_δ.

To show that $g \in \mathcal{F}_\delta$, it suffices to show that g is lexicographical-order preserving. Suppose that $c <_L d$ are in the domain of g and we prove that $g(c) <_J g(d)$. This is immediate if either c and d are in $T \restriction (\alpha_f + 1)$, or if c and d are in T_δ and $c \restriction \alpha_f = d \restriction \alpha_f$. We consider the remaining possibilities.

Case a: c and d are in T_δ and $c \restriction \alpha_f \neq d \restriction \alpha_f$. By Lemma 2.1 and the fact that f is lexicographical-order preserving,

$$c <_L d \iff c \restriction \alpha_f <_L d \restriction \alpha_f \iff$$
$$g(c) \restriction \alpha_f = f(c \restriction \alpha_f) <_J f(d \restriction \alpha_f) = g(d) \restriction \alpha_f \iff g(c) <_J g(d).$$

Case b: $c \in T \restriction \delta$ and $d \in T_\delta$. First, assume that $c <_T d$. Then $c \leq_T d \restriction \alpha_f <_T d$. Now $c \leq_T d \restriction \alpha_f$ implies that $c \leq_L d \restriction \alpha_f$, and hence $f(c) \leq_J f(d \restriction \alpha_f)$. And $d \restriction \alpha_f <_T d$ implies that $g(d \restriction \alpha_f) <_S g(d)$, and hence $g(d \restriction \alpha_f) <_J g(d)$. By transitivity, $g(c) <_J g(d)$. Secondly, assume that c and d are incomparable. By Lemma 2.1, $c <_L d$ implies that $c <_L d \restriction \mathrm{ht}_T(c)$ and hence $f(c) <_J f(d \restriction \mathrm{ht}_T(c))$. On the other hand, $d \restriction \mathrm{ht}_T(c) \leq_T d \restriction \alpha_f$ implies that $d \restriction \mathrm{ht}_T(c) \leq_L d \restriction \alpha_f$, and hence $f(d \restriction \mathrm{ht}_T(c)) \leq_J f(d \restriction \alpha_f)$. Finally, $d \restriction \alpha_f <_T d$ implies that $g(d \restriction \alpha_f) <_S g(d)$, so $g(d \restriction \alpha_f) <_J g(d)$. By transitivity, $g(c) <_J g(d)$.

Case c: $c \in T_\delta$ and $d \in T \restriction \delta$. Now $c \restriction \alpha_f <_T c$ implies that $c \restriction \alpha_f <_L c <_L d$. So $f(c \restriction \alpha_f) <_J f(d)$. In particular, $f(d) \not\leq_S f(c \restriction \alpha_f)$, so $f(d)$ and $f(c \restriction \alpha_f)$ are incomparable in S. But $f(c \restriction \alpha_f) <_S g(c)$, so by Lemma 2.1, $g(c) <_J g(d)$ iff $g(c) \restriction \alpha_f = f(c \restriction \alpha_f) <_J f(d)$, which is true. $\square$

8. Adding Promises, 1

The general idea of the promise method is that if properness of the forcing fails, then there exist a condition and a promise which witnesses this failure. On the other hand, this promise can be added to the condition, and the fact that the working part of the extended condition fulfills the promise implies that there could have been no failure to begin with (this vague argument is crystallized in Lemma 9.4 below). For this method to work, there must be a way to add promises to conditions, a challenge which we handle now.

Definition 8.1. For any $f \in \mathcal{F}$ and $P \in \mathcal{P}$, we say that P is *suitable for* f if there exist $\alpha_f < \delta < \omega_1$, $\langle \delta_n : n < \omega \rangle$, and $\langle \vec{x}_s : s \in {}^{\omega}\omega \rangle$ satisfying:

(1) $P \in \mathcal{P}_\delta$;
(2) $\langle \delta_n : n < \omega \rangle$ is a strictly increasing sequence of ordinals cofinal in δ with $\delta_0 = \alpha$ and $\delta_n \in E$ for all n;
(3) for all $n < \omega$, $\{\vec{x}_s : s \in {}^{n}\omega\}$ is a pairwise disjoint and pairwise compatible subset of $(T_{\delta_n} \otimes S_{\delta_n})^{\otimes m}$, where m is the dimension of P;
(4) for all $s \subseteq t$ in ${}^{\omega}\omega$, $\vec{x}_t \restriction \delta_{|s|} = \vec{x}_s$;
(5) for all $s \in {}^{\omega}\omega$, there exists $\vec{x}_s^+ \in P \cap (T_\delta \otimes S_\delta)^{\otimes m}$ such that $\vec{x}_s^+ \restriction \delta_{|s|} = \vec{x}_s$;
(6) f is consistent with $\vec{x}_\emptyset$.

Lemma 8.2 (Existence of Promises, 1)**.** *Assume the following:*

(1) $f \in \mathcal{F}$,
(2) $P \in \mathcal{P}$;
(3) there exists $\vec{x} \in P$ with height greater than or equal to α_f such that $\vec{x}$ has unique drop-downs to α_f and f is consistent with $\vec{x} \restriction \alpha_f$.

Then there exists $Q \subseteq P$ in $\mathcal{P}$ such that Q is suitable for f.

Proof. Using Proposition 5.2, construct by induction sequences $\langle \delta_n : n < \omega \rangle$ and $\langle \vec{x}_s : s \in {}^{\omega}\omega \rangle$ satisfying:

(1) $\langle \delta_n : n < \omega \rangle$ is a strictly increasing sequence of countable ordinals in E with $\delta_0 = \alpha_f$;

(2) for all $n < \omega$, $\{\vec{x}_s : s \in {}^n\omega\}$ is a family of lex-increasing tuples in $T^m_{\delta_n}$, where m is the dimension of P, which is pairwise disjoint and pairwise compatible;
(3) for all $s \subseteq t$ in ${}^{\omega}\omega$, $\vec{x}_t \upharpoonright \delta_{|s|} = \vec{x}_s$;
(4) $\vec{x}_\emptyset = \vec{x} \upharpoonright \alpha_f$.

For the base case, let $\delta_0 = \alpha_f$ and $\vec{x}_\emptyset = \vec{x} \upharpoonright \alpha_f$. Now let $n < \omega$ and assume that δ_n and $\{\vec{x}_s : s \in {}^n\omega\}$ are defined as required. Apply Proposition 5.2 to fix a pairwise disjoint and pairwise compatible family $\{\vec{x}_{s,l} : l < \omega\}$ of members of P above $\vec{x}_s$. Now fix $\delta_{n+1} \in E$ which is greater than δ_n and greater than the height of $\vec{x}_{s,l}$ for all $s \in {}^n\omega$ and $l < \omega$. Then for each such s and l, choose $\vec{x}_{s^\frown l}$ in P above $\vec{x}_{s,l}$ with height δ_{n+1}. This completes the induction. Let $\delta = \sup_n \delta_n$. For each $s \in {}^{\omega}\omega$, pick some $\vec{x}^+_s \in P$ with height δ above $\vec{x}_s$. Now define $Q = \bigcup\{P_{\vec{x}^+_s} : s \in {}^{\omega}\omega\}$. □

Proposition 8.3 (Adding Promises, 1)**.** *Assume the following:*

(1) $f \in \mathcal{F}$;
(2) $\Gamma \subseteq \mathcal{P}$ *is countable and* f *fulfills every member of* Γ;
(3) $P \in \mathcal{P}$ *is suitable for* f *with witnesses* δ, $\langle \delta_n : n < \omega\rangle$ *and* $\langle \vec{x}_s : s \in {}^{\omega}\omega\rangle$;
(4) f *is consistent with* $\vec{x}_\emptyset$.

Then there exists $g \in \mathcal{F}_{\delta_U}$ *such that* $f \subseteq g$ *and* g *fulfills every member of* $\Gamma \cup \{P\}$.

Proof. Let m be the dimension of P. Let $\langle P_n : n < \omega\rangle$ be an enumeration of Γ in which each member appears infinitely many times, and let $\langle c_n : n < \omega\rangle$ be an enumeration of T_δ. We define by induction α_n, $\vec{z}_n$, s_n, and f_n for $n < \omega$, maintaining the following inductive hypotheses:

- $\delta_n \le \alpha_n < \omega_1 \cap \delta$;
- $\vec{z}_n$ is a lex-increasing tuple in $(T_\delta \times T_\delta)^{<\omega}$ with unique drop-downs to α_n;
- $s_n \in {}^{\omega}\omega$, $\vec{x}_{s_n}$ has height α_n, and for $n > 0$, $\vec{x}_{s_n}$ and $\vec{z}_n \upharpoonright \alpha_n$ are disjoint and compatible;
- $k < n$ implies $s_k \subseteq s_n$;
- $f_n \in \mathcal{F}$;
- $\alpha_{f_n} = \alpha_n$;
- f_n is consistent with $\vec{z}_n \upharpoonright \alpha_n$ and $\vec{x}_{s_n}$;
- f_n fulfills every member of Γ.

For the base case, let $\alpha_0 = \alpha_f$, $\vec{z}_0$ is the empty tuple, $s_0 = \emptyset$, and $f_0 = f$. Now assume that $n < \omega$ and for all $m \le n$, α_m, $\vec{z}_m$, s_m, and f_m are defined as required. Write $\vec{z}_n = ((a^n_k, b^n_k) : k < l)$. Let α_{n+1} be the least ordinal satisfying:

- $\alpha_{n+1} > \alpha_n$;
- α_{n+1} is equal to δ_q for some $q \ge n+1$;
- $\{a^n_k : k < l\} \cup \{c_n\}$ has unique drop-downs to α_{n+1}.

Pick $s_{n,0}$ and $s_{n,1}$ in ${}^q\omega$ such that for each $i < 2$, $s_n \subseteq s_{n,i}$, $\vec{x}_{s_{n,i}}$ is disjoint from $\vec{z}_n \upharpoonright \alpha_{n+1}$, and $c_n \upharpoonright \alpha_{n+1}$ is not the first coordinate of any pair in $\vec{x}_{s_{n,i}}$.

Apply Proposition 7.1 to fix $f_{n+1} \in \mathcal{F}$ such that $\alpha_{f_{n+1}} = \alpha_{n+1}$, $f_n \subseteq f_{n+1}$, f_{n+1} fulfills every member of Γ, and f_{n+1} is consistent with $\vec{z}_n \upharpoonright \alpha_{n+1}$, $\vec{x}_{s_{n,0}}$, and $\vec{x}_{s_{n,1}}$. Define $s_{n+1} = s_{n,1}$. Pick some $\vec{x}^+_{s_{n,0}} \in P$ above $\vec{x}_{s_{n,0}}$ with height δ.

We define $\vec{z}_{n+1}$ in three steps. For the first step, apply Lemma 6.3 and let $\vec{z}\,'_n$ be the lex-increasing tuple which lists the elements of $\vec{z}_n$ and $\vec{x}^+_{s_{n,0}}$. For the second step, we first ask whether c_n is equal to a^n_k for some $k < l$. If so, then c_n has already been handled, and we let $\vec{z}\,''_n = \vec{z}\,'_n$ and move on to step three. Suppose not. Since f_{n+1} is consistent with $\vec{z}_n \upharpoonright \alpha_{n+1}$ and is injective,

$f_{n+1}(c_n \restriction \alpha_{n+1})$ is not equal to $b^n_k \restriction \alpha_{n+1}$ for any $k < l$. Similarly, $f_{n+1}(c_n \restriction \alpha_{n+1})$ is not the second coordinate of any pair in $\vec{x}_{s_{n,i}}$, for each $i < 2$. Pick some $d \in S_\delta$ which is above $f_{n+1}(c_n \restriction \alpha_{n+1})$. Applying Lemma 6.3, let $\vec{z}''_n$ be the lex-increasing tuple which lists the elements of $\vec{z}'_n$ together with (c_n, d).

Now we describe step three. Since f_{n+1} fulfills P_n, fix some $\vec{w} \in P_n$ of height α_{n+1} which is consistent with f_{n+1} and disjoint from $\vec{z}''_n \restriction \alpha_{n+1}$ and $\vec{x}_{s_{n,1}}$. Fix $\vec{w}^+ \in P_n$ above $\vec{w}$ of height δ. Applying Lemma 6.3, let $\vec{z}_{n+1}$ be the lex-increasing tuple which lists the elements of $\vec{w}^+$ and $\vec{z}''_n$.

This completes the induction. Define $g \in \mathcal{F}$ by letting $g \restriction (T \restriction \delta) = \bigcup_n f_n$ and $g(c) = d$ iff for some $n < \omega$, the pair (c, d) appears in $\vec{z}_n$. By Lemma 6.5, $g \in \mathcal{F}$, and g fulfills P as witnessed by $\{\vec{x}^+_{s_{n,0}} : n < \omega\}$. □

9. The First Poset

We are now ready to introduce and derive the main properties of the forcing which we use to prove Theorem 1.

Definition 9.1. Define $\mathbb{P}(T, S)$ to be the forcing consisting of conditions which are ordered pairs $p = (f_p, \Gamma_p)$ satisfying:

(1) $f_p \in \mathcal{F}$;
(2) $\Gamma_p \subseteq \mathcal{P}$ is countable;
(3) f_p fulfills every member of Γ_p.

For any $p \in \mathbb{P}(T, S)$, let $c_p = c_{f_p}$ and $\alpha_p = \alpha_{f_p}$. Define $q \leq p$ if $f_p \subseteq f_q$ and $\Gamma_p \subseteq \Gamma_q$.

The next two lemmas follow immediately from Lemmas 7.1 and 8.3 respectively.

Lemma 9.2 (Poset Extension, 1)**.** *Assume the following:*

(1) $p \in \mathbb{P}(T, S)$;
(2) $\alpha_p < \delta \in E$;
(3) $\vec{x}$ is a lex-increasing tuple in $(T_\delta \otimes S_\delta)^{<\omega}$ with unique drop-downs to α_p;
(4) f_p is consistent with $\vec{x} \restriction \alpha_p$.

Then there exists $q \leq p$ such that $\alpha_q = \delta$ and f_q is consistent with $\vec{x}$.

It follows that, assuming $\mathbb{P}(T, S)$ preserves ω_1, any generic filter on $\mathbb{P}(T, S)$ yields a function as described in Section 3.

Lemma 9.3 (Poset Adding Promises, 1)**.** *Assume the following:*

(1) $p \in \mathbb{P}(T, S)$;
(2) $P \in \mathcal{P}$;
(3) P is suitable for f_p.

Then there exists $q \leq p$ such that $P \in \Gamma_q$.

Lemma 9.4. *Assume that L^* and J are not near. Let $\lambda \geq (2^{|\mathbb{P}(T,S)|})^+$ be a regular cardinal and let $N \prec H(\lambda)$ be countable such that T, S, E, and $\mathbb{P}(T, S)$ are in N. Let $\delta = N \cap \omega_1$. Assume the following:*

(1) $p \in N \cap \mathbb{P}(T, S)$;
(2) $D \in N$ is a dense subset of $\mathbb{P}(T, S)$;
(3) $\vec{x}$ is a lex-increasing tuple in $(T_\delta \otimes S_\delta)^{<\omega}$ with unique drop-downs to α_p;
(4) f_p is consistent with $\vec{x} \restriction \alpha_p$.

Then there exists $q \leq p$ in $N \cap D$ such that f_q is consistent with $\vec{x} \restriction \alpha_q$.

Proof. Suppose for a contradiction that the conclusion fails. Let n be the dimension of $\vec{x}$. Define P_0 as the set of all lex-increasing tuples $\vec{y}$ in $(((T \restriction E) \otimes (S \restriction E))^{\otimes n})_{\vec{x} \restriction \alpha_p}$ such that for any $q \leq p$ in D with α_q less than or equal to the height of $\vec{y}$, f_q is not consistent with $\vec{y} \restriction \alpha_q$. Note that $P_0 \in N$ by elementarity, and P_0 is a downwards closed subset of the tree $(((T \restriction E) \otimes (S \restriction E))^{\otimes n})_{\vec{x} \restriction \alpha_p}$. We claim that for all $\alpha_p \leq \gamma < \delta$, $\vec{x} \restriction \gamma \in P_0$. Otherwise, there exists $q \leq p$ in D such that $\alpha_q \leq \gamma$ and f_q is consistent with $(\vec{x} \restriction \gamma) \restriction \alpha_q = \vec{x} \restriction \alpha_q$. Since p, D, γ, and $\vec{x} \restriction \gamma$ are in N, by elementarity we may assume that $q \in N$. But then the conclusion of the lemma holds, which is a contradiction. It follows that P_0 is uncountable. Let $P \in N$ be an uncountable downwards closed subset of P_0 such that every element of P has uncountably many members of P above it. Note that $P \in \mathcal{P}$ and $\vec{x} \restriction \alpha_p \in P$.

By Lemma 8.2, we can fix $Q \in \mathcal{P}$ such that $Q \subseteq P$ and Q is suitable for f_p. Since the root of P_0 is $\vec{x} \restriction \alpha_p$, $\vec{x} \restriction \alpha_p$ is the only element of P of height α_p. Hence, $\vec{x} \restriction \alpha_p$ is equal to $\vec{x}_\emptyset$, where $\{\vec{x}_s : s \in {}^{\omega}\omega\}$ witnesses that Q is suitable for f_p. Applying Lemma 9.3, fix $q \leq p$ such that $Q \in \Gamma_q$. Since D is dense, we can fix $r \leq q$ in D. As f_r fulfills Q, there exists $\vec{y} \in Q \cap (T_{\alpha_r} \otimes S_{\alpha_r})^{\otimes n}$ such that f_r is consistent with $\vec{y}$. Since $Q \subseteq P_0$, $\vec{y} \in P_0$. But $r \leq p$ is in D and f_r is consistent with $\vec{y}$, which contradicts the definition of P_0. □

Theorem 9.5. *The forcing $\mathbb{P}(T, S)$ is totally proper.*

Proof. Let $\lambda \geq (2^{|\mathbb{P}(T,S)|})^+$ be a regular cardinal and let $N \prec H(\lambda)$ be countable such that T, S, E, and $\mathbb{P}(T, S)$ are in N. Let $\delta = N \cap \omega_1$. Consider $p \in N \cap \mathbb{P}(T, S)$. We claim that there exists $q \leq p$ such that q is a total master condition for N and $\mathbb{P}(T, S)$. Specifically, we build by induction a descending sequence of conditions $\langle p_n : n < \omega \rangle$ in $N \cap \mathbb{P}(T, S)$ such that for every dense set $D \in N$, there exists some n with $p_n \in D$, together with a lower bound q of this sequence.

Fix the following objects:

- an enumeration $\langle P_n : n < \omega \rangle$ of $N \cap \mathcal{P}$ in which each element appears infinitely many times;
- an enumeration $\langle c_n : n < \omega \rangle$ of T_δ;
- a sequence $\langle D_n : n < \omega \rangle$ which lists every dense open subset of $\mathbb{P}(T, S)$ which lies in N.

We define by induction a descending sequence $\langle p_n : n < \omega \rangle$ of conditions in $N \cap \mathbb{P}(T, S)$ together with a sequence $\langle \vec{x}_n : n < \omega \rangle$ of lex-increasing tuples in $T_\delta^{<\omega}$ as follows.

For the base case, let $p_0 = p$ and let $\vec{x}_0$ be the empty tuple. Now assume that p_n and $\vec{x}_n$ are defined for some $n < \omega$ such that $\vec{x}_n$ has unique drop-downs to α_{p_n} and f_{p_n} is consistent with $\vec{x}_n \restriction \alpha_{p_n}$. Write $\vec{x}_n = ((a_k^n, b_k^n) : k < l)$. We define p_{n+1} and $\vec{x}_{n+1}$ in three steps. In the first step, apply Lemma 9.4 to fix $p_{n+1} \leq p_n$ in $N \cap D_n$ such that $f_{p_{n+1}}$ is consistent with $\vec{x}_n \restriction \alpha_{p_{n+1}}$. Moreover, using Lemma 9.2 we may assume without loss of generality that $\alpha_{p_{n+1}}$ is high enough that the set $\{a_k^n : k < l\} \cup \{c_n\}$ has unique drop-downs to $\alpha_{p_{n+1}}$.

For the second step, we ask whether c_n is equal to a_k^n for some $k < l$. If yes, then c_n has already been handled, so we let $\vec{x}_n' = \vec{x}_n$ and move on to step three. Suppose not. Since $f_{p_{n+1}}$ is consistent with $\vec{x}_n \restriction \alpha_{p_{n+1}}$ and is injective, $f_{p_{n+1}}(c_n \restriction \alpha_{p_{n+1}})$ is not equal to $b_k^n \restriction \alpha_{p_{n+1}}$ for any $k < l$. Pick some $d \in S_\delta$ which is above $f_{p_{n+1}}(c_n \restriction \alpha_{p_{n+1}})$. By Lemma 2.1 and the fact that f_{n+1} is consistent with $\vec{x}_n \restriction \alpha_{n+1}$, for all $k < l$ we have that

$$\begin{aligned} a_k^n <_L c_n \iff a_k^n \restriction \alpha_{p_{n+1}} <_L c_n \restriction \alpha_{p_{n+1}} \\ \iff b_k^n \restriction \alpha_{p_{n+1}} <_J f_{n+1}(c_n \restriction \alpha_{p_{n+1}}) \iff b_k^n <_J d. \end{aligned}$$

Hence, the tuples $\vec{x}_n$ and $((c_n, d))$ are compatible. Let $\vec{x}'_n$ be the lex-increasing tuple which lists the elements of $\vec{x}_n$ together with (c_n, d).

Now we describe step three. If $P_n \notin \Gamma_{p_{n+1}}$, then let $\vec{x}_{n+1} = \vec{x}'_n$ and we are done. Otherwise, since $f_{p_{n+1}}$ fulfills P_n, fix some $\vec{w} \in P_n$ of height $\alpha_{p_{n+1}}$ which is disjoint from $\vec{x}'_n \upharpoonright \alpha_{p_{n+1}}$ such that $f_{p_{n+1}}$ is consistent with $\vec{w}$. Fix $\vec{w}^+ \in P_n$ above $\vec{w}$ of height δ. By Lemma 6.3, we can let $\vec{x}_{n+1}$ be the lex-increasing tuple which lists the elements of $\vec{w}^+$ and $\vec{x}'_n$.

This completes the induction. Note that by Lemma 9.2 and the fact that the decreasing sequence meets every dense set in N, $\sup_n \alpha_{p_n} = \delta$. Define q as follows. Let $f_q \upharpoonright (T \upharpoonright \delta) = \bigcup_n f_{p_n}$ and for all $c \in T_\delta$, $f_q(c) = d$ iff for some $n < \omega$, (c, d) appears in $\vec{x}_n$. By Lemma 6.5, $f_q \in \mathcal{F}$. Let $\Gamma_q = \bigcup_n \Gamma_{p_n}$. Then $q = (f_q, \Gamma_q)$ is in $\mathbb{P}(T, S)$ and is a lower bound of $\langle p_n : n < \omega \rangle$. □

10. Making $\mathcal{U}(T)$ an Ultrafilter

We now turn to the second main theorem of the article, which is the consistency of the statement that CH holds and for any coherent Aronszajn tree $T \subseteq {}^{\omega_1}\omega$, $\mathcal{U}(T)$ is an ultrafilter. Fix such a tree T and consider a set $X \subseteq \omega_1$. If $\omega_1 \setminus X \in \mathcal{U}(T)$, then X is already handled. Otherwise, every member of $\mathcal{U}(T)$ has non-empty intersection with X, that is, $X \in \mathcal{U}(T)^+$. In this case, our goal is to define a forcing $\mathbb{P}(T, X)$ which is totally proper and adds an uncountable antichain $A \subseteq T$ such that $\Delta(A) \subseteq X$. Rather than adding the antichain explicitly, we design a forcing which adds an uncountable subtree which splits only at levels in X. This approach is justified by the following lemma.

Lemma 10.1. *Suppose that $T \subseteq {}^{\omega_1}\omega$ is a non-Suslin coherent Aronszajn tree. For any set $X \subseteq \omega_1$, $X \in \mathcal{U}(T)$ iff there exists an uncountable downwards closed subtree U of T such that for every $x \in U$, if x splits in U then $ht_T(x) \in X$.*

Proof. Suppose that $X \in \mathcal{U}(T)$. Fix an uncountable antichain $A \subseteq T$ such that $\Delta(A) \subseteq X$. Let $U = \{x \in T : \exists y \in A \ x <_T y\}$. Then U is uncountable. For any splitting element x of U, the height of x is equal to $\Delta(y, z)$ for any $y, z \in A$ which are above two distinct immediate successors of x in U. Hence, the height of any splitting element of U is in X. Conversely, suppose that U is an uncountable downwards closed subtree of T such that the height of any splitting element of U is in X. Since T is non-Suslin, we can fix an uncountable antichain $A \subseteq U$. If y and z are distinct members of A, then $y \upharpoonright \Delta(y, z) = z \upharpoonright \Delta(y, z)$ splits in U, and hence its height, which is $\Delta(y, z)$, is in X. □

We now describe how to obtain the desired consistency result. We iterate with countable support of length ω_2 over a model of $2^\omega = \omega_1$ and $2^{\omega_1} = \omega_2$, alternating between forcings which specialize Aronszajn trees and forcings of the form $\mathbb{P}(T, X)$, where T is as above and $X \in \mathcal{U}(T)^+$. Both types of forcings have size 2^{ω_1} and are $(<\omega_1)$-proper, Dee-complete, and ω_2-p.i.c. Bookkeeping to handle all relevant trees and sets, we arrange that the final iteration forces that all Aronszajn trees are special and that $\mathcal{U}(T)$ is an ultrafilter for any coherent Aronzajn tree $T \subseteq {}^{\omega_1}\omega$. As in the first application, the core result about $\mathbb{P}(T, X)$ is that it is totally proper, and we leave the verification of the other properties to the interested reader (see the comments towards the end of Section 3). The structure of our proof is quite similar to that of Theorem 1.

We make use of the following basic result about coherent Aronszajn trees. For a proof, see the arguments of [17, Lems. 4.2.4, 4.2.5].

Lemma 10.2 (Fundamental Lemma on Coherent Trees)**.** *Suppose that $T \subseteq {}^{\omega_1}\omega$ is a coherent Aronszajn tree which is non-Suslin. Let $\langle \vec{x}_\alpha : \alpha < \omega_1 \rangle$ be a sequence of distinct elements of T^n*

for some $0 < n < \omega$. *Write each* $\vec{x}_\alpha$ *as* $(a_0^\alpha, \ldots, a_{n-1}^\alpha)$. *Then there exists an uncountable set* $Y \subseteq \omega_1$ *such that for all* $\alpha < \beta$ *in* Y, *there exists* $\xi < \omega_1$ *such that for all* $k < n$, a_k^α *and* a_k^β *are incomparable and* $\Delta(a_k^\alpha, a_k^\beta) = \xi$.

For the remainder of this application, fix a coherent Aronszajn tree $T \subseteq {}^{\omega_1}\omega$ which is non-Suslin, and fix a set $X \subseteq \omega_1$.

11. Tuples, 2

We work with injective tuples of the form $\vec{x} = (a_0, \ldots, a_{n-1})$, where for some $\alpha < \omega_1$, $a_0, \ldots, a_{n-1}$ are in T_α. We refer to α as the *height of* $\vec{x}$. For $\xi \leq \alpha$, let $\vec{x} \restriction \xi = (a_0 \restriction \xi, \ldots, a_{n-1} \restriction \xi)$. We say that $\vec{x}$ has *unique drop-downs to* ξ if $\vec{x} \restriction \xi$ is injective.

Definition 11.1. Let $\vec{x} = (a_0, \ldots, a_{n-1})$ be an injective tuple in T_α^n, for some $\alpha < \omega_1$ and $0 < n < \omega$. We say that $\vec{x}$ *has meets in* X if for all $k < l < n$, $\Delta(a_k, a_l) \in X$.

Lemma 11.2. *Let* $\xi < \alpha < \omega_1$ *and suppose that* $\vec{x}$ *is an injective tuple in* $T_\alpha^{<\omega}$ *which has unique drop-downs to* ξ. *Then* $\vec{x}$ *has meets in* X *iff* $\vec{x} \restriction \xi$ *has meets in* X.

Definition 11.3. Let $\vec{x} = (a_0, \ldots, a_{n-1})$ and $\vec{y} = (b_0, \ldots, b_{m-1})$ be injective tuples in $T_\alpha^{<\omega}$ for some $\alpha < \omega_1$.

(1) The tuples $\vec{x}$ and $\vec{y}$ are *disjoint* if $a_i \neq b_j$ for all $i < n$ and $j < m$;
(2) If $\vec{x}$ and $\vec{y}$ are disjoint, then we say that they are *compatible* if for all $i < n$ and $j < m$, $\Delta(a_i, b_j) \in X$.

Note that $\vec{x}$ and $\vec{y}$ are compatible iff there exists an injective tuple in $T_\alpha^{<\omega}$ with meets in X which lists the elements of $\vec{x}$ and $\vec{y}$.

Lemma 11.4. *Let* $\xi < \alpha < \omega_1$. *Suppose that* $\vec{x}$ *and* $\vec{y}$ *are disjoint injective tuples in* $T_\alpha^{<\omega}$, *each with unique drop-downs to* ξ, *such that* $\vec{x} \restriction \xi$ *and* $\vec{y} \restriction \xi$ *are disjoint. Then* $\vec{x}$ *and* $\vec{y}$ *are compatible iff* $\vec{x} \restriction \xi$ *and* $\vec{y} \restriction \xi$ *are compatible.*

Suppose that $\vec{x} = (a_0, \ldots, a_{n-1})$ and $\vec{y} = (b_0, \ldots, b_{n-1})$ are in $T^{\otimes n}$, not necessarily of the same height. We say that $\vec{x}$ and $\vec{y}$ are *componentwise incomparable* if for all $k < n$, a_k and b_k are incomparable in T. Componentwise incomparable tuples $\vec{x}$ and $\vec{y}$ are said to have *uniform meets* if there exists some $\gamma < \omega_1$, which we denote by $\Delta(\vec{x}, \vec{y})$, such that for all $k < n$, $\Delta(a_k, b_k) = \gamma$. The tuples $\vec{x}$ and $\vec{y}$ are *fully incomparable* if for all $k, m < n$, a_k and b_m are incomparable in T.

12. Promises, 2

The type of promises which we use for the proof of Theorem 2 is described next.

Definition 12.1 (Promises, 2)**.** Define $\mathcal{P}$ to be the set of all P satisfying that for some $\gamma < \omega_1$ and for some $0 < n < \omega$:

(1) P is a non-empty downwards closed subset of the tree $[T \restriction (\omega_1 \setminus \gamma)]^{\otimes n}$;
(2) every member of P has uncountably many elements of P above it;
(3) every member of P has meets in X.

In the above, γ is called the *base level of* P and n is called the *dimension of* P. The set of $P \in \mathcal{P}$ such that P has base level γ is denoted by $\mathcal{P}_\gamma$.

Proposition 12.2. *For all* $P \in \mathcal{P}$ *and for any* $\vec{x} \in P$, *there exists a family* $\{\vec{x}_n : n < \omega\}$ *of elements of* P *above* $\vec{x}$ *with the same height which is pairwise disjoint, pairwise compatible, and any two members of the family have uniform meets.*

Proof. Write $\vec{x} = (a_0, \ldots, a_{n-1})$. For each $\alpha < \omega_1$ greater than the height of $\vec{x}$, pick some $\vec{x}_\alpha \in P$ above $\vec{x}$ with height α. Write each $\vec{x}_\alpha$ as $(a_0^\alpha, \ldots, a_{n-1}^\alpha)$. Since $\vec{x}$ is injective and $a_k <_T a_k^\alpha$ for all α and for all $k < n$, it follows that for all distinct $k, l < n$ and for all $\alpha \leq \beta$, a_k^α and a_l^β are incomparable in T. Applying Lemma 10.2, fix an uncountable set $Y \subseteq \omega_1$ so that for all $\alpha < \beta$ in Y, $\vec{x}_\alpha$ and $\vec{x}_\beta$ are componentwise incomparable and $\vec{x}_\alpha$ and $\vec{x}_\beta$ have uniform meets. Then for all $\alpha < \beta$ in Y, $\vec{x}_\alpha$ and $\vec{x}_\beta$ are fully incomparable, and hence $\vec{x}_\alpha$ and $\vec{x}_\beta \restriction \alpha$ are disjoint and have uniform meets.

Define a function $F : [Y]^2 \to 2$ so that for all $\alpha < \beta$ in Y, $F(\{\alpha, \beta\})$ is equal to 1 if $\vec{x}_\alpha$ and $\vec{x}_\beta \restriction \alpha$ are compatible, and 0 otherwise. By the Dushnik-Miller theorem, there exists a set $Z \subseteq Y$ such that either (a) Z is infinite and for all $\alpha < \beta$ in Z, $F(\{\alpha, \beta\}) = 1$, or else (b) Z is uncountable and for all $\alpha < \beta$ in Z, $F(\{\alpha, \beta\}) = 0$.

Suppose for a contradiction that (b) holds. Consider any $\alpha < \beta$ in Z. Since $F(\{\alpha, \beta\}) = 0$, $\vec{x}_\alpha$ and $\vec{x}_\beta \restriction \alpha$ are not compatible. But for all distinct $k, m < n$, $\Delta(a_k^\alpha, a_m^\beta) = \Delta(a_k, a_m) \in X$. So there must exist some $k < n$ such that $\Delta(a_k^\alpha, a_k^\beta) \notin X$. Since $\vec{x}_\alpha$ and $\vec{x}_\beta$ have uniform meets, $\Delta(a_0^\alpha, a_0^\beta) = \Delta(a_k^\alpha, a_k^\beta) \notin X$. To summarize, for all $\alpha < \beta$ in Z, $\Delta(a_0^\alpha, a_0^\beta) \notin X$. So $A = \{a_0^\alpha : \alpha \in Z\}$ is an uncountable antichain of T such that $\Delta(A) \subseteq \omega_1 \setminus X$. By the definition of $\mathcal{U}(T)$, it follows that $\omega_1 \setminus X \in \mathcal{U}(T)$, which contradicts that $X \in \mathcal{U}(T)^+$.

So (a) holds. Pick some $\delta \in E$ greater than the height of $\vec{x}_\alpha$ for all $\alpha \in Z$. For each $\alpha \in Z$, choose some $\vec{x}_\alpha^+ \in P$ with height δ which is above $\vec{x}_\alpha$. Now for all $\alpha < \beta$ in Z, $\vec{x}_\alpha$ and $\vec{x}_\beta \restriction \alpha$ are disjoint, compatible, and have uniform meets. By Lemma 11.4, $\vec{x}_\alpha^+$ and $\vec{x}_\beta^+$ are disjoint, compatible, and have uniform meets. $\square$

13. The Working Part, 2

The forcing $\mathbb{P}(T, X)$ adds the desired subtree by countable approximations. The next definition describes these objects.

Definition 13.1. Define $\mathcal{F}$ to be the set of functions of the form $f : T \restriction (\alpha + 1) \to 2$, for some $\alpha < \omega_1$, satisfying:

(1) $f(\emptyset) = 1$;
(2) if $f(x) = 1$, then there exists $y \in T_\alpha$ such that $x \leq_T y$ and $f(x) = 1$;
(3) for all $x, y \in T \restriction (\alpha + 1)$:
   (a) if $f(y) = 1$ and $x <_T y$, then $f(x) = 1$;
   (b) if x and y are incomparable in T and $f(x) = f(y) = 1$, then $\Delta(x, y) \in X$.

For any $\alpha < \omega_1$, define $\mathcal{F}_\alpha$ to be the set of $f \in \mathcal{F}$ such that $\mathrm{dom}(f) = T \restriction (\alpha + 1)$.

Definition 13.2. Suppose that $\xi \leq \alpha < \omega_1$, $f \in \mathcal{F}_\alpha$, and $\vec{x} = (a_0, \ldots, a_{n-1})$ is an injective tuple in T_ξ^n. We say that f *is consistent with* $\vec{x}$ if for all $i < n$, $f(a_i) = 1$.

Definition 13.3. Suppose that $\gamma \leq \alpha < \omega_1$, $f \in \mathcal{F}_\alpha$, and $P \in \mathcal{P}_\gamma$. We say that f *fulfills* P if there exists a pairwise disjoint family $\{\vec{x}_n : n < \omega\}$ of members of P with height α such that for all n, f is consistent with $\vec{x}_n$.

14. Extension, 2

The next three lemmas allows us to stretch a condition up arbitrarily high in the tree. Lemma 14.1 handles the successor case and Lemma 14.2 handles the general case. We also need Lemma 14.3 in order to add promises to a condition (Proposition 15.3).

Lemma 14.1. *Assume the following:*

1. *$\alpha < \omega_1$ and $f \in \mathcal{F}_\alpha$;*
2. *$\Gamma \subseteq \mathcal{P}$ is countable and f fulfills every member of Γ;*
3. *$\vec{x}$, $\vec{y}$, and $\vec{z}$ are injective tuples in $T^{<\omega}_{\alpha+1}$, each with unique drop-downs to α;*
4. *either $\vec{y} = \vec{z} = \emptyset$, or $\vec{y}$ and $\vec{z}$ are non-empty, disjoint, and compatible;*
5. *$\vec{y} \restriction \alpha = \vec{z} \restriction \alpha$;*
6. *$\vec{x} \restriction \alpha$ and $\vec{y} \restriction \alpha$ are disjoint;*
7. *f is consistent with $\vec{x} \restriction \alpha$ and $\vec{y} \restriction \alpha$.*

Then there exist $g \in \mathcal{F}_{\alpha+1}$ such that:

(a) *$f \subseteq g$;*
(b) *g fulfills every member of Γ;*
(c) *g is consistent with $\vec{x}$, $\vec{y}$, and $\vec{z}$.*

Proof. Write $\vec{y} = (a^0_k : k < n)$ and $\vec{z} = (a^1_k : k < n)$. Let $\langle P_m : m < \omega\rangle$ be an enumeration of Γ so that every member appears infinitely often. Since f fulfills every member of Γ, we can fix a pairwise disjoint sequence $\langle \vec{w}_m : m < \omega\rangle$ satisfying that for all $m < \omega$:

1. $\vec{w}_m$ is in P_m and has height α;
2. $\vec{w}_m$ is disjoint from $\vec{x} \restriction \alpha$ and $\vec{y} \restriction \alpha$;
3. f is consistent with $\vec{w}_m$.

For each $m < \omega$, fix $\vec{w}^+_m$ in P_m of height $\alpha + 1$ which is above $\vec{w}_m$. It suffices to define some $g \in \mathcal{F}_{\alpha+1}$ such that $f \subseteq g$ and g is consistent $\vec{x}$, $\vec{y}$, $\vec{z}$, and $\vec{w}^+_m$ for all $m < \omega$.

To construct g, we consider an arbitrary $x \in T_\alpha$ and define g on the immediate successors of x. If $f(x) = 0$, then define $g(y) = 0$ for every immediate successor y of x. Suppose that $f(x) = 1$. We split into four mutually exclusive cases.

Case 1: There exists $m < \omega$ such that x appears in $\vec{w}_m$. Write $\vec{w}_m = (a_0, \dots, a_{l-1})$ and $\vec{w}^+_m = (a^+_0, \dots, a^+_{l-1})$, and fix $k < l$ such that $a_k = x$. Define $g(a^+_k) = 1$, and $g(y) = 0$ for any other immediate successor y of x.

Case 2: x appears in $\vec{x} \restriction \alpha$. Proceed similarly as in case 1.

Case 3: x appears in $\vec{y} \restriction \alpha$. In this case, $\vec{y}$ and $\vec{z}$ are non-empty, disjoint, and compatible. In particular, since $\vec{y} \restriction \alpha = \vec{z} \restriction \alpha$, $\Delta(a^0_0, a^1_0) = \alpha \in X$. Fix $k < n$ such that $x = a^0_k \restriction \alpha$. Then also $x = a^1_k \restriction \alpha$. Define $g(a^0_k) = 1$, $g(a^1_k) = 1$, and $g(y) = 0$ for any other immediate successor y of x.

Case 4: None of the above. In this case, arbitrarily choose some immediate successor y of x, and define $g(z) = 1$ iff $z = y$ for any immediate successor z of x. □

Proposition 14.2 (Extension, 2)**.** *Assume the following:*

1. *$\alpha \leq \delta < \omega_1$ and $f \in \mathcal{F}_\alpha$;*
2. *$\Gamma \subseteq \mathcal{P}$ is countable and f fulfills every member of Γ;*
3. *$\vec{x}$ is a tuple in $T^{<\omega}_\delta$ with unique drop-downs to α;*
4. *f is consistent with $\vec{x} \restriction \alpha$.*

Then there exists $g \in \mathcal{F}_\delta$ such that:

(a) *$f \subseteq g$;*
(b) *g fulfills every member of Γ;*
(c) *g is consistent with $\vec{x}$.*

Proof. The proof is by induction on δ. The base case $\delta = \alpha$ is immediate.

Successor case: Assume that the statement is true for some $\xi \geq \alpha$, and we prove that it is true for $\delta = \xi + 1$. By the inductive hypothesis, fix $g \in \mathcal{F}_\xi$ such that $f \subseteq g$, g fulfills every member of Γ, and g is consistent with $\vec{x} \upharpoonright \xi$. Now apply Lemma 14.1 (letting $\vec{x}_0 = \vec{x}_1$ be the empty tuple) to fix $h \in \mathcal{F}_{\xi+1}$ such that $g \subseteq h$, h fulfills every member of Γ, and h is consistent with $\vec{x}$.

Limit case: Let $\delta > \alpha$ be a countable limit ordinal and assume that the statement holds for all ξ with $\alpha \leq \xi < \delta$. Fix an enumeration $\langle P_n : n < \omega \rangle$ of Γ in which each member appears infinitely many times, fix an enumeration $\langle c_n : n < \omega \rangle$ of $T \upharpoonright \delta$ in which each member appears infinitely many times, and fix a strictly increasing sequence of ordinals $\langle \delta_n : n < \omega \rangle$ cofinal in δ with $\delta_0 = \alpha$. We define by induction $\vec{x}_n$ and f_n satisfying the following inductive hypotheses:

- $\vec{x}_n$ is an injective tuple in $T_\delta^{<\omega}$ with meets in X;
- $\vec{x}_n$ has unique drop-downs to δ_n;
- $f_n \in \mathcal{F}_{\delta_n}$;
- f_n is consistent with $\vec{x}_n \upharpoonright \delta_n$;
- f_n fulfills every member of Γ.

For the base case, let $\vec{x}_0 = \vec{x}$ and $f_0 = f$. Now assume that $n < \omega$ and for all $m \leq n$, $\vec{x}_m$ and f_m are defined as required. Write $\vec{x}_n = (a_0, \ldots, a_{l-1})$. Applying the inductive hypothesis, fix $f_{n+1} \in \mathcal{F}_{\delta_{n+1}}$ such that $f_n \subseteq f_{n+1}$, f_{n+1} fulfills every member of Γ, and f_{n+1} is consistent with $\vec{x}_n \upharpoonright \delta_{n+1}$.

We define $\vec{x}_{n+1}$ in two steps. First, we ask whether c_n has height less than or equal to δ_{n+1}. If not, then we are not yet ready to handle c_n, so we move on to step two. Suppose that it does. If $f_{n+1}(c_n) = 0$, then c_n has been handled, and we move on to step two. Suppose that $f_{n+1}(c_n) = 1$. Fix $d \in T_{\delta_{n+1}}$ such that $c \leq_T d$ and $f_{n+1}(d) = 1$. If $d <_T a_k$ for some $k < l$, then c_n is already handled and we move on to step two. Otherwise, fix some $e \in T_\delta$ above d, and let $\vec{x}_n'$ enumerate the elements of $\vec{x}_n$ together with e. In any of the previous cases mentioned other than the last case, let $\vec{x}_n' = \vec{x}_n$.

Now we describe step two. Since f_{n+1} fulfills P_n, we can fix some $\vec{w} \in P_n$ of height δ_{n+1} which is consistent with f_{n+1} and disjoint from $\vec{x}_n' \upharpoonright \delta_{n+1}$. Fix $\vec{w}^+ \in P_n$ above $\vec{w}$ of height δ. Since f_{n+1} is consistent with both $\vec{w}$ and $\vec{x}_n' \upharpoonright \delta_{n+1}$, $\vec{w}$ and $\vec{x}_n' \upharpoonright \delta_{n+1}$ are compatible. By Lemma 11.4, $\vec{w}^+$ and $\vec{x}_n'$ are compatible. Let $\vec{x}_{n+1}$ be an injective tuple which lists the elements of $\vec{w}^+$ and $\vec{x}_n'$.

This completes the induction. Define $g \in \mathcal{F}_\delta$ by letting $g \upharpoonright (T \upharpoonright \delta) = \bigcup_n f_n$ and $g(y) = 1$ iff for some $n < \omega$, y appears in $\vec{x}_n$. $\square$

Lemma 14.3 (Splitting Extension, 2)**.** *Assume the following:*

(1) $\alpha \leq \delta < \omega_1$ *and* $f \in \mathcal{F}_\alpha$*;*
(2) $\Gamma \subseteq \mathcal{P}$ *is countable and* f *fulfills every member of* Γ*;*
(3) $\vec{x}$, $\vec{y}$, *and* $\vec{z}$ *are injective tuples in* $T_\delta^{<\omega}$*;*
(4) $\vec{x}$, $\vec{y}$, *and* $\vec{z}$ *are pairwise disjoint and pairwise compatible;*
(5) $\vec{y} \upharpoonright \alpha = \vec{z} \upharpoonright \alpha$*;*
(6) $\vec{x} \upharpoonright \alpha$ *is disjoint from* $\vec{y} \upharpoonright \alpha$*;*
(7) f *is consistent with* $\vec{x} \upharpoonright \alpha$ *and* $\vec{y} \upharpoonright \alpha$*;*
(8) $\vec{y}$ *and* $\vec{z}$ *have uniform meets.*

Then there exists $h \in \mathcal{F}_\delta$ *such that:*

(a) $f \subseteq h$*;*
(b) h *fulfills every member of* Γ*;*
(c) h *is consistent with* $\vec{x}$, $\vec{y}$, *and* $\vec{z}$.

Proof. Let $\gamma = \Delta(\vec{y}, \vec{z})$. Then $\alpha \leq \gamma < \delta$ and $\vec{y} \restriction \gamma = \vec{z} \restriction \gamma$. Since $\vec{x}$ and $\vec{y}$ are compatible, by Lemma 11.4 so are $\vec{x} \restriction \gamma$ and $\vec{y} \restriction \gamma$. Let $\vec{d}$ be an injective tuple which lists the elements of $\vec{x} \restriction \gamma$ and $\vec{y} \restriction \gamma$. As f is consistent with $\vec{x} \restriction \alpha$ and $\vec{y} \restriction \alpha$, it is also consistent with $\vec{d} \restriction \alpha$. By Proposition 14.2, we can fix $g \in \mathcal{F}_\gamma$ such that $f \subseteq g$, g fulfills every member of Γ, and g is consistent with $\vec{d}$. It follows that g is consistent with both $\vec{x} \restriction \gamma$ and $\vec{y} \restriction \gamma$. Because $\vec{y} \restriction \gamma = \vec{z} \restriction \gamma$, g is also consistent with $\vec{z} \restriction \gamma$.

Applying Lemma 14.1 and the fact that $\gamma = \Delta(\vec{y}, \vec{z})$, we can find $g^+ \in \mathcal{F}_{\gamma+1}$ such that $g \subseteq g^+$, g^+ fulfills every member of Γ, and g^+ is consistent with $\vec{x} \restriction (\gamma + 1)$, $\vec{y} \restriction (\gamma + 1)$, and $\vec{z} \restriction (\gamma + 1)$. If $\delta = \gamma + 1$, then we are done. Assume that $\gamma + 1 < \delta$. Let $\vec{e}$ be an injective tuple which lists the elements of $\vec{x}$, $\vec{y}$, and $\vec{z}$. Since $\vec{x}$, $\vec{y}$, and $\vec{z}$ are pairwise compatible, $\vec{e}$ has meets in X. Also, g^+ is consistent with $\vec{e} \restriction (\gamma + 1)$. Applying Proposition 14.2, fix $h \in \mathcal{F}_\delta$ such that $g^+ \subseteq h$, h fulfills every member of Γ, and h is consistent with $\vec{e}$. Then h is consistent with $\vec{x}$, $\vec{y}$, and $\vec{z}$. $\square$

15. Adding Promises, 2

In this section, we prove a result which implies that, under some circumstances, a promise can be added to a condition.

Definition 15.1. For any $f \in \mathcal{F}$ and $P \in \mathcal{P}$, we say that P *is suitable for* f if there exist $\alpha < \delta < \omega_1$, $\langle \delta_n : n < \omega \rangle$, and $\langle \vec{x}_s : s \in {}^{\omega}\omega \rangle$ satisfying:

(1) $f \in \mathcal{F}_\alpha$ and $P \in \mathcal{P}_\delta$;
(2) $\langle \delta_n : n < \omega \rangle$ is strictly increasing and cofinal in δ with $\delta_0 = \alpha$;
(3) for all $n < \omega$, $\{\vec{x}_s : s \in {}^{n}\omega\}$ is a family of injective tuples in $T_{\delta_n}^{\otimes m}$, where m is the dimension of P, which is pairwise disjoint, pairwise compatible, and any two elements of the family have uniform meets;
(4) for all $s \subseteq t$ in ${}^{\omega}\omega$, $\vec{x}_t \restriction \delta_{|s|} = \vec{x}_s$;
(5) for all $s \in {}^{\omega}\omega$, there exists $\vec{x}_s^+ \in P \cap T_\delta^{\otimes m}$ such that $\vec{x}_s^+ \restriction \delta_{|s|} = \vec{x}_s$;
(6) f is consistent with $\vec{x}_\emptyset$.

Lemma 15.2 (Existence of Promises, 2)**.** *Assume the following:*

(1) $\alpha < \omega_1$ and $f \in \mathcal{F}_\alpha$,
(2) $P \in \mathcal{P}$;
(3) there exists $\vec{x} \in P$ such that $\vec{x}$ has unique drop-downs to α and f is consistent with $\vec{x} \restriction \alpha$.

Then there exists $Q \subseteq P$ in $\mathcal{P}$ such that Q is suitable for f.

Proof. Using Proposition 12.2, construct by induction sequences $\langle \delta_n : n < \omega \rangle$ and $\langle \vec{x}_s : s \in {}^{\omega}\omega \rangle$ satisfying:

(1) $\langle \delta_n : n < \omega \rangle$ is a strictly increasing sequence of countable ordinals with $\delta_0 = \alpha$;
(2) for all $n < \omega$, $\{\vec{x}_s : s \in {}^{n}\omega\}$ is a family of injective tuples in $T_{\delta_n}^{\otimes m}$, where m is the dimension of P, which is pairwise disjoint, pairwise compatible, and any two elements of the family have uniform meets;
(3) for all $s \subseteq t$ in ${}^{\omega}\omega$, $\vec{x}_t \restriction \delta_{|s|} = \vec{x}_s$;
(4) $\vec{x}_\emptyset = \vec{x} \restriction \alpha$.

For the base case, let $\delta_0 = \alpha$ and $\vec{x}_\emptyset = \vec{x} \restriction \alpha$. Now let $n < \omega$ and assume that δ_n and $\{\vec{x}_s : s \in {}^{n}\omega\}$ are defined as required. For each $s \in {}^{n}\omega$, apply Proposition 12.2 to fix a pairwise disjoint and pairwise compatible family $\{\vec{x}_{s,l} : l < \omega\}$ of members of P above $\vec{x}_s$ with the same height such that any two elements of the family have uniform meets. Now fix $\delta_{n+1} < \omega_1$ which is greater than the height of $\vec{x}_{s,l}$ for all $s \in {}^n\omega$ and $l < \omega$. Then for each such s and l, choose $\vec{x}_{s^\frown l}$ in P above $\vec{x}_{s,l}$ with height δ_{n+1}. Let $\delta = \sup_n \delta_n$. For each $s \in {}^{\omega}\omega$, pick some $\vec{x}_s^+ \in P$ with height δ above $\vec{x}_s$. Now define $P = \bigcup\{P_{\vec{x}_s^+} : s \in {}^{\omega}\omega\}$. □

Proposition 15.3 (Adding Promises, 2)**.** *Assume the following:*

(1) $\alpha < \delta < \omega_1$*;*
(2) $f \in \mathcal{F}_\alpha$ *and* $P \in \mathcal{P}_\delta$*;*
(3) $\Gamma \subseteq \mathcal{P}$ *is countable and* f *fulfills every member of* Γ*;*
(4) $\vec{x}$ *is an injective tuple in* $T_\delta^{<\omega}$ *with unique drop-downs to* α*;*
(5) P *is suitable for* f *as witnessed by* $\langle \delta_n : n < \omega\rangle$ *and* $\langle \vec{x}_s : s \in {}^{\omega}\omega\rangle$*;*
(6) $\vec{x} \restriction \alpha = \vec{x}_\emptyset$*.*

Then there exists $g \in \mathcal{F}_\delta$ *such that:*

(a) $f \subseteq g$*;*
(b) g *is consistent with* $\vec{x}$*;*
(c) g *fulfills every member of* $\Gamma \cup \{P\}$*.*

Proof. Let m be the dimension of P. Fix an enumeration $\langle P_n : n < \omega\rangle$ of Γ in which each member appears infinitely many times and fix an enumeration $\langle c_n : n < \omega\rangle$ of $T \restriction \delta$ in which every element appears infinitely many times. We define by induction $\vec{z}_n$, s_n, and f_n for all $n < \omega$, maintaining the following inductive hypotheses:

- $\vec{z}_n$ is an injective tuple in $T_\delta^{<\omega}$ with meets in X and unique drop-downs to δ_n;
- $s_n \in {}^n\omega$, $\vec{x}_{s_n}$ has height δ_n, and for $n > 0$, $\vec{x}_{s_n}$ and $\vec{z}_n \restriction \delta_n$ are disjoint;
- $k < n$ implies $s_k \subseteq s_n$;
- $f_n \in \mathcal{F}_{\delta_n}$;
- f_n is consistent with $\vec{z}_n \restriction \delta_n$ and $\vec{x}_{s_n}$;
- f_n fulfills every member of Γ.

For the base case, let $\vec{z}_0 = \vec{x}$, $s_0 = \emptyset$, and $f_0 = f$. Now assume that $n < \omega$ and for all $m \leq n$, $\vec{z}_m$, s_m, and f_m are defined as required. Write $\vec{z}_n = (a_0, \dots, a_{l-1})$. Pick $s_{n,0}$ and $s_{n,1}$ in ${}^{n+1}\omega$ such that for each $i < 2$, $s_n \subseteq s_{n,i}$ and $\vec{x}_{s_{n,i}}$ is disjoint from $\vec{z}_n \restriction \delta_{n+1}$. Define $s_{n+1} = s_{n,1}$.

Apply Lemma 14.3 to find $f_{n+1} \in \mathcal{F}_{\delta_{n+1}}$ such that $f_n \subseteq f_{n+1}$, f_{n+1} fulfills every member of Γ, and f_{n+1} is consistent with $\vec{z}_n \restriction \delta_{n+1}$, $\vec{x}_{s_{n,0}}$, and $\vec{x}_{s_{n,1}}$. Fix $\vec{x}^+_{s_{n,0}}$ in $P \cap T_\delta^{\otimes m}$ such that $\vec{x}^+_{s_{n,0}} \restriction \delta_{n+1} = \vec{x}_{s_{n,0}}$. Since f_{n+1} is consistent with $\vec{z}_n \restriction \delta_{n+1}$ and $\vec{x}_{s_{n,0}}$, by Lemma 11.4 we have that $\vec{z}_n$ and $\vec{x}^+_{s_{n,0}}$ are compatible. Let $\vec{z}_n'$ be an injective tuple which lists the elements of $\vec{z}_n$ and $\vec{x}^+_{s_{n,0}}$.

We define $\vec{z}_{n+1}$ in two steps. First, we ask whether c_n has height less than or equal to δ_{n+1}. If not, then we are not yet ready to handle c_n, so we move on to step two. Assume that it does. If $f_{n+1}(c_n) = 0$, then c_n has been handled, and we move on to step two. Suppose that $f_{n+1}(c_n) = 1$. Fix $d \in T_{\delta_{n+1}}$ with $d \geq_T c_n$ such that $f_{n+1}(d) = 1$. If d is below some member of $\vec{z}_n'$, then c_n is already handled and we move on to step two. Otherwise, fix some $e \in T_\delta$ above d, and let $\vec{z}_n''$ enumerate the elements of $\vec{z}_n'$ together with e. In any of the cases mentioned other than the last case, let $\vec{z}_n'' = \vec{z}_n'$.

Now we describe step two. As f_{n+1} fulfills P_n, we can fix some $\vec{w} \in P_n$ of height δ_{n+1} which is disjoint from $\vec{z}_n'' \restriction \delta_{n+1}$ such that f_{n+1} is consistent with $\vec{w}$. Fix $\vec{w}^+ \in P_n$ above $\vec{w}$ with height δ. Since f_{n+1} is consistent with both $\vec{w}$ and $\vec{z}_n'' \restriction \delta_{n+1}$, $\vec{w}$ and $\vec{z}_n'' \restriction \delta_{n+1}$ are compatible. By

Lemma 11.4, $\vec{w}^+$ and $\vec{z}''_n$ are compatible. Let $\vec{z}_{n+1}$ be an injective tuple which lists the elements of $\vec{w}^+$ and $\vec{z}''_n$.

This completes the induction. Define $g \in \mathcal{F}_\delta$ by letting $g \restriction (T \restriction \delta) = \bigcup_n f_n$ and $g(y) = 1$ iff for some $n < \omega$, y appears in $\vec{z}_n$. $\square$

16. The Second Poset

We are now ready to introduce and derive the main properties of the forcing which we use to prove Theorem 2.

Definition 16.1. Define $\mathbb{P}(T, X)$ to be the forcing consisting of conditions which are ordered pairs $p = (f_p, \Gamma_p)$ satisfying:

(1) $f_p \in \mathcal{F}_{\alpha_p}$ for some $\alpha_p < \omega_1$;
(2) $\Gamma_p \subseteq \mathcal{P}$ is countable;
(3) f_p fulfills every member of Γ_p.

Define $q \leq p$ if $f_p \subseteq f_q$ and $\Gamma_p \subseteq \Gamma_q$.

The next two lemmas follow immediately from Propositions 14.2 and 15.3 respectively.

Lemma 16.2 (Poset Extension, 2)**.** *Assume the following:*

(1) $p \in \mathbb{P}(T, X)$*;*
(2) $\alpha_p < \delta < \omega_1$*;*
(3) $\vec{x}$ *is an injective tuple with height* δ *and unique drop-downs to* α_p*;*
(4) f_p *is consistent with* $\vec{x} \restriction \alpha_p$*.*

Then there exists $q \leq p$ *such that* $\alpha_q = \delta$ *and* f_q *is consistent with* $\vec{x}$*.*

It follows that, assuming $\mathbb{P}(T, X)$ preserves ω_1, any generic filter on $\mathbb{P}(T, X)$ yields a subtree which witnesses that $X \in \mathcal{U}(T)$, as described in Section 10.

Lemma 16.3 (Poset Adding Promises, 2)**.** *Assume the following:*

(1) $p \in \mathbb{P}(T, X)$*;*
(2) $\alpha_p < \delta < \omega_1$*;*
(3) $P \in \mathcal{P}_\delta$*;*
(4) $\vec{x}$ *is an injective tuple of height* δ *with unique drop-downs to* α_p*;*
(5) f_p *is consistent with* $\vec{x} \restriction \alpha_p$*;*
(6) P *is suitable for* f *with witness* $\{\vec{x}_s : s \in {}^{\omega}\omega\}$*;*
(7) $\vec{x} \restriction \alpha_p = \vec{x}_\emptyset$*.*

Then there exists $q \leq p$ *such that* $\alpha_q = \delta$*,* f_q *is consistent with* $\vec{x}$*, and* $P \in \Gamma_q$*.*

Lemma 16.4. *Let* $\lambda \geq (2^{|\mathbb{P}(T,X)|})^+$ *be a regular cardinal and let* $N \prec H(\lambda)$ *be countable such that* T*,* X*, and* $\mathbb{P}(T, X)$ *are in* N*. Let* $\delta = N \cap \omega_1$*. Assume the following:*

(1) $p \in N \cap \mathbb{P}(T, X)$*;*
(2) $D \in N$ *is a dense subset of* $\mathbb{P}(T, X)$*;*
(3) $\vec{x}$ *is an injective tuple of height* δ *with unique drop-downs to* α_p*;*
(4) f_p *is consistent with* $\vec{x} \restriction \alpha_p$*.*

Then there exists $q \leq p$ *such that* $q \in N \cap D$ *and* f_q *is consistent with* $\vec{x} \restriction \alpha_q$*.*

Proof. Suppose for a contradiction that the conclusion of the lemma fails. Let n be the dimension of $\vec{x}$. Define P_0 as the set of all $\vec{y} \in (T^{\otimes n})_{\vec{x} \restriction \alpha_p}$ such that for any $q \leq p$ in D with α_q less than or equal to the height of $\vec{y}$, f_q is not consistent with $\vec{y} \restriction \alpha_q$. Note that $P_0 \in N$ by elementarity, and P_0 is a downwards closed subset of the tree $(T^{\otimes n})_{\vec{x} \restriction \alpha_p}$. We claim that for all $\alpha_p \leq \gamma < \delta$, $\vec{x} \restriction \gamma \in P_0$. Otherwise, there exists $q \leq p$ in D such that $\alpha_q \leq \gamma$ and f_q is consistent with $(\vec{x} \restriction \gamma) \restriction \alpha_q = \vec{x} \restriction \alpha_q$. Since p, D, γ, and $\vec{x} \restriction \gamma$ are in N, by elementarity we may assume that $q \in N$. But then the conclusion of the lemma holds, which is a contradiction. It follows from the claim that P_0 is uncountable. Let $P \in N$ be a non-empty downwards closed subset of P_0 such that every element of P has uncountably many members of P above it. Note that $\vec{x} \restriction \alpha_p \in P$ and $P \in \mathcal{P}$.

By Lemma 15.2, there exists $Q \in \mathcal{P}$ such that $Q \subseteq P$ and Q is suitable for f_p. Let γ be the base level of Q. Since the root of P_0 is $\vec{x} \restriction \alpha_p$, $\vec{x} \restriction \alpha_p$ is the only element of P of height α_p. Hence, $\vec{x} \restriction \alpha_p$ is equal to $\vec{x}_\emptyset$, where $\{\vec{x}_s : s \in {}^{\wp}\omega\}$ witnesses that Q is suitable for f_p. By Lemma 16.3, we can fix $q \leq p$ such that $Q \in \Gamma_q$. As D is dense, we can fix $r \leq q$ in D. Because f_r fulfills Q, there exists some $\vec{y} \in Q \cap (T_{\alpha_r})^n$ such that f_r is consistent with $\vec{y}$. Then $\vec{y} \in P_0$. But $r \leq p$ is in D and f_r is consistent with $\vec{y}$, contradicting the definition of P_0. □

Theorem 16.5. *The forcing poset $\mathbb{P}(T, X)$ is totally proper.*

Proof. Let $\lambda \geq (2^{|\mathbb{P}(T,X)|})^+$ be a regular cardinal and let $N \prec H(\lambda)$ be countable such that T, X, and $\mathbb{P}(T, X)$ are in N. Let $\delta = N \cap \omega_1$. Consider $p \in N \cap \mathbb{P}(T, X)$. We claim that there exists $q \leq p$ such that q is a total master condition for N and $\mathbb{P}(T, X)$. Specifically, we build by induction a descending sequence of conditions $\langle p_n : n < \omega \rangle$ in $N \cap \mathbb{P}(T, X)$ such that for every dense set $D \in N$, there exists some n with $p_n \in D$, and then define a lower bound q of the sequence.

Fix the following objects:

(1) an enumeration $\langle P_n : n < \omega \rangle$ of all members of $\mathcal{P}$ which lie in N such that every element appears infinitely many times;
(2) an enumeration $\langle c_n : n < \omega \rangle$ of $T \restriction \delta$ in which each member appears infinitely many times;
(3) a sequence $\langle D_n : n < \omega \rangle$ which lists all dense subsets of $\mathbb{P}(T, X)$ which lie in N.

We define by induction a descending sequence $\langle p_n : n < \omega \rangle$ of conditions in $N \cap \mathbb{P}(T, X)$ together with a sequence $\langle \vec{x}_n : n < \omega \rangle$ of injective tuples in $T_\delta^{<\omega}$ as follows.

For the base case, let $p_0 = p$ and let $\vec{x}_0$ be the empty tuple. Now assume that p_n and $\vec{x}_n$ are defined such that $\vec{x}_n$ has unique drop-downs to α_{p_n}. We define p_{n+1} and $\vec{x}_{n+1}$ in several steps. In the first step, apply Lemma 16.4 to find $p_{n+1} \leq p_n$ in $N \cap D_n$ such that $f_{p_{n+1}}$ is consistent with $\vec{x}_n \restriction \alpha_{p_{n+1}}$.

In the second step, we ask whether c_n has height less than or equal to $\alpha_{p_{n+1}}$. If not, then we are not yet ready to handle c_n, and we move on to step three. Suppose that it does. If $f_{p_{n+1}}(c_n) = 0$, then c_n has been handled, so we move on to step three. Suppose that $f_{n+1}(c_n) = 1$. Fix $d \in T_{\alpha_{p_{n+1}}}$ such that $c \leq_T d$ and $f_{p_{n+1}}(d) = 1$. If d is below some member of $\vec{x}_n$, then c_n has already handled and we move on to step three. Otherwise, fix some $e \in T_\delta$ above d, and let $\vec{x}_n'$ injectively enumerate the elements of $\vec{x}_n$ together with e. In any of the previous cases mentioned other than the last case, let $\vec{x}_n' = \vec{x}_n$.

Now we describe step three. Consider P_n. If P_n is not in $\Gamma_{p_{n+1}}$, then let $\vec{x}_{n+1} = \vec{x}_n'$. Suppose that P_n is in $\Gamma_{p_{n+1}}$. Then $f_{p_{n+1}}$ fulfills P_n, so we can fix some $\vec{y} \in P_n$ of height $\alpha_{p_{n+1}}$ which is consistent with $f_{p_{n+1}}$ and disjoint from $\vec{x}_n' \restriction \alpha_{p_{n+1}}$. Fix $\vec{y}^+ \in P_n$ above $\vec{y}$ of height δ. Since $f_{p_{n+1}}$ is consistent with both $\vec{y}$ and $\vec{x}_n' \restriction \alpha_{p_{n+1}}$, $\vec{y}$ and $\vec{x}_n' \restriction \alpha_{p_{n+1}}$ are compatible. By Lemma

11.4, $\vec{y}^+$ and $\vec{x}_n'$ are compatible. Let $\vec{x}_{n+1}$ be an injective tuple which lists the elements of $\vec{y}^+$ and $\vec{x}_n'$.

This completes the induction. Now define q as follows. Let $f_q \restriction (T \restriction \delta) = \bigcup_n f_{p_n}$, and for all $z \in T_\delta$, $f_q(z) = 1$ iff for some $n < \omega$, z appears in $\vec{x}_n$. Let $\Gamma_q = \bigcup_n \Gamma_{p_n}$. Then $q = (f_q, \Gamma_q)$ is in $\mathbb{P}(T, X)$ and is a lower bound of $\langle p_n : n < \omega \rangle$. □

17. Further Results and Open Problems

Abraham and Shelah [2] introduced a variation of the promise method which allows for the preservation of Suslin trees. Using this alternative version of promises, we can prove:

Theorem 17.1. *Each of the following statements is consistent with* CH *and the existence of a Suslin tree:*

(1) any two Countryman lines are compatible;
(2) for any coherent Aronszajn tree $T \subseteq {}^{\omega_1}\omega$ *which is non-Suslin,* $\mathcal{U}(T)$ *is an ultrafilter.*

Statement (1) does not need any adjustment in the presence of a Suslin tree, since lexicographically ordered Aronszajn trees which are Countryman lines are already non-Suslin.[8] The proof of Theorem 17.1 is similar to [8, Th. 8.1] and will appear in the second author's PhD disseration.

Recall Problem 1 which is stated in the introduction: Is it consistent with CH that for any Aronszajn lines L and J, there exists a Countryman line C such that each of L and J contain either a copy of C or its converse C^*? This question can be approached in a series of steps, which we outline below. The most difficult to prove affirmatively is almost certainly Problem 2.

Problem 2. Is it consistent with CH that there exists an Aronszajn tree T which satisfies CAT?

In light of [10], a natural candidate for such a tree would be a normal coherent Aronszajn subtree of ${}^{\omega_1}2$. Recall that assuming mild forcing axioms, the existence of an Aronszajn tree satisfying CAT is equivalent to CAT holding for all Aronszajn trees. Using arguments as in [8], it seems likely that the same could be arranged in the CH context.

Problem 3. Is it consistent with CH that CAT holds?

Problem 4. Is it consistent with CH that every Aronszajn line contains a Countryman line?

Each of these problems has a variation involving the existence of a Suslin tree, for example, by restricting to non-Suslin Aronszajn trees. In particular:

Problem 5. Is it consistent with CH and the existence of a Suslin tree that any two Aronszajn lines, neither of which contains a Suslin line, are compatible?

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[8] Since we are unable to find a proof of this fact in the literature, we give a brief sketch. By passing to a subtree, we may assume that our Aronszajn tree T with lexicographical ordering L satisfies that the set of successors of any element at any higher level is isomorphic to the rationals (use Lemma 2.2 and the argument of [3, Ch. 4, Prop. 2]). Given an uncountable set $A \subseteq T$, find a sequence $\langle (a_i, b_i) : i < \omega_1 \rangle$ such that for all i, $a_i \in A$, a_i and b_i have the same height, $a_i <_L b_i$, and a_i has height less than $\Delta(a_j, b_j)$ for all $j > i$. Then for all $i < j$, if $a_i <_T a_j$, then $a_i <_T b_j$, so $b_j <_L b_i$ by Lemma 2.1. Since L is Countryman, fix an uncountable set $X \subseteq \omega_1$ such that $\{(a_i, b_i) : i \in X\}$ is a chain in L^2. Then for all $i < j$, a_i and a_j are incomparable in T, for otherwise $a_i <_L a_j$ but $b_j <_L b_i$.

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